An elasto-plastic model for unsaturated soil incorporating the effects of suction and degree of saturation on mechanical behaviour

D. GALLIPOLI,* A. GENS,* R. SHARMA† and J. VAUNAT*

The paper presents an elasto-plastic model for unsaturated soils that takes explicitly into account the mechanisms with which suction affects mechanical behaviour as well as their dependence on degree of saturation. The proposed model is formulated in terms of two constitutive variables directly related to these suction mechanisms: the average skeleton stress, which includes the average fluid pressure acting on the soil pores, and an additional scalar constitutive variable, ξ, related to the magnitude of the bonding effect exerted by meniscus water at the inter-particle contacts. The formulation of the model in terms of variables closely related to specific behaviour mechanisms leads to a remarkable unification of experimental results of tests carried out with different suctions. The analysis of experimental isotropic compression data strongly suggests that the quotient between the void ratio, e, of an unsaturated soil and the void ratio e_s, corresponding to the saturated state at the same average soil skeleton stress, is a unique function of the bonding effect due to water menisci at the inter-particle contacts. The same result is obtained when examining critical states at different suctions. Based on these observations, an elasto-plastic constitutive model is developed using a single yield surface the size of which is controlled by volumetric hardening. In spite of this simplicity, it is shown that the model reproduces correctly many important features of unsaturated soil behaviour. It is especially remarkable that, although only one yield surface is used in the formulation of the model, the irreversible behaviour in wetting–drying cycles is well captured. Because of the behaviour normalisation achieved by the model, the resulting constitutive law is economical in terms of the number of tests required for parameter determination.

KEYWORDS: clays; constitutive relations; partial saturation; plasticity; suction

Cet exposé présente un modèle élasto-plastique pour des sols non saturés, modèle qui prend explicitement en compte les mécanismes par lesquels la succion affecte le comportement mécanique ainsi que leur dépendance vis-à-vis du degré de saturation. Le modèle proposé est formulé en termes de deux variables constitutives directement liées à ces mécanismes de succion ; la contrainte moyenne du squelette, qui inclut la pression de fluide moyenne sur les pores du sol et une autre variante constitutive scalaire supplémentaire, ξ, liée à la magnitude de l'effet d'adhérence exercé par l'eau ménisque aux contacts entre particules. La formulation du modèle en termes de variables étroitement liées à des mécanismes de comportement spécifiques conduit à une unification remarquable des résultats expérimentaux des essais effectués avec différentes succions. L'analyse des données expérimentales de compression isotrope suggère fortement que le quotient entre le taux de pore e d'un sol non saturé et le taux de pore e_s correspondant à l'état saturé pour la même contrainte moyenne de squelette de sol, est une fonction unique de l'effet d'adhérence dû aux ménisques de l'eau aux contacts entre particules. Le même résultat est obtenu lorsque l'on examine les états critiques à différentes succions. En nous basant sur ces observations, nous avons développé un modèle constitutif élasto-plastique utilisant une unique surface d'écoulement dont la dimension est commandée par le durcissement volumétrique. Nous montrons que le modèle, malgré sa simplicité, reproduit correctement bien des caractéristiques importantes du comportement d'un sol non saturé. Il est particulièrement frappant de constater que, bien qu'une seule surface d'écoulement ait été utilisée dans la formulation du modèle, le comportement irréversible dans les cycles mouillage-séchage est bien représenté. En raison de la normalisation de comportement obtenue grâce au modèle, la loi constitutive qui en résulte a besoin de moins d'essais pour la détermination des paramètres.

INTRODUCTION

Suction has long been recognised as a fundamental variable in the understanding of the mechanical behaviour of unsaturated soils. For this reasons many well-known constitutive models (Alonso *et al.*, 1990; Wheeler & Sivakumar, 1995; Cui & Delage, 1996) include suction as a basic stress variable together with the net stress, $\overline{\sigma}$ (defined as total stress minus pore air pressure).

In fact suction influences the mechanical behaviour of an unsaturated soil in two different ways (Karube & Kato, 1994; Wheeler & Karube, 1995):

(*a*) by modifying the skeleton stress through changes in the average fluid pressure acting in the soil pores

(*b*) by providing an additional bonding force at the particle contacts, often attributed to capillary phenomena occurring in the water menisci.

It is important to realise that, for the two mechanisms, the effects of suction are influenced by the state of saturation of the soil. The relative area over which the water and air pressures act depends directly on the degree of saturation (the percentage of pore voids occupied by water), but the same parameter also affects the number and intensity of capillary-induced inter-particle forces.

Therefore models using only suction in their formulation are unlikely to be complete. It is necessary to incorporate, through a parameter such as degree of saturation, information regarding the proportion of the soil over which suction effects are relevant. Therefore it is not surprising that constitutive models that use only suction as the unsaturated

variable face difficulties in describing important features of unsaturated soil behaviour. Another class of elasto-plastic models for unsaturated soils, such as those proposed by Jommi & Di Prisco (1994), Bolzon *et al.* (1996), Loret & Khalili (2000) and Karube & Kawai (2001), are expressed in terms of a different set of constitutive variables that explicitly include the degree of saturation in their definitions. In these models the stress variable has the form of the Bishop (1959) stress:

$$\sigma'_{hk} = \sigma_{hk} - \delta_{hk}[u_a - \chi(u_a - u_w)] \tag{1}$$

where σ'_{hk} is the Bishop (1959) stress, σ_{hk} is the total stress, u_a is the air pressure, u_w is the water pressure, δ_{hk} is Kronecker's delta, and χ is a soil parameter that ranges between 1 (at saturation) and zero (at dry conditions), and which is a function of degree of saturation. The additional scalar variable is given either by suction or by degree of saturation depending on the specific model, with the exception of the model of Karube & Kawai (2001), in which it is a function of both suction and degree of saturation. Although such models explicitly introduce the degree of saturation in the definition of the soil variables, they still present limitations when predicting important aspects of unsaturated soil behaviour unless the additional complexity of multiple yield surfaces is introduced. Two examples of features of behaviour that require adequate modelling are:

(*a*) the irreversible reduction of specific volume that can occur during drying of an unsaturated soil (that is, during increase of suction)
(*b*) the dependence of the soil response during virgin loading at constant suction on the past history of suction variation.

The first type of behaviour has been observed by Alonso *et al.* (1995) and Sharma (1998) in laboratory tests involving wetting–drying cycles (that is, cycles of decrease–increase of suction) on soil samples subjected to oedometric and isotropic conditions respectively. Examples of this type of soil response are shown in Fig. 1. The second type of behaviour has been observed by Sharma (1998) during isotropic loading to virgin states of samples at constant suction. In particular, the results by Sharma (1998) suggest that a compacted unsaturated soil shows a different stiffness during virgin loading at the same constant suction depending on whether the sample undergoes a wetting–drying cycle prior to loading or not. Fig. 2 shows examples of isotropic tests where the dependence of the soil stiffness during virgin loading on the previous history of suction variation can be observed. Moreover, for some of these models the determination of model parameters requires either non-conventional laboratory tests (for example, tests at constant degree of saturation) or a back-analysis for fitting model predictions to conventional laboratory results.

In this paper a model is described that incorporates explicitly in its formulation the two distinct suction effects mentioned previously, including their dependence on degree of saturation. By staying close to these basic behaviour mechanisms, the proposed elasto-plastic model is capable of reproducing the most important patterns of unsaturated soil mechanical behaviour, including those indicated above, in a rather simple manner employing only a single yield surface. The model also provides an effective way of unifying experimental results of tests performed at different suctions. Apart from the conceptual benefits of such unification, this fact results in economical procedures for parameter determination from the point of view of the number of laboratory tests required.
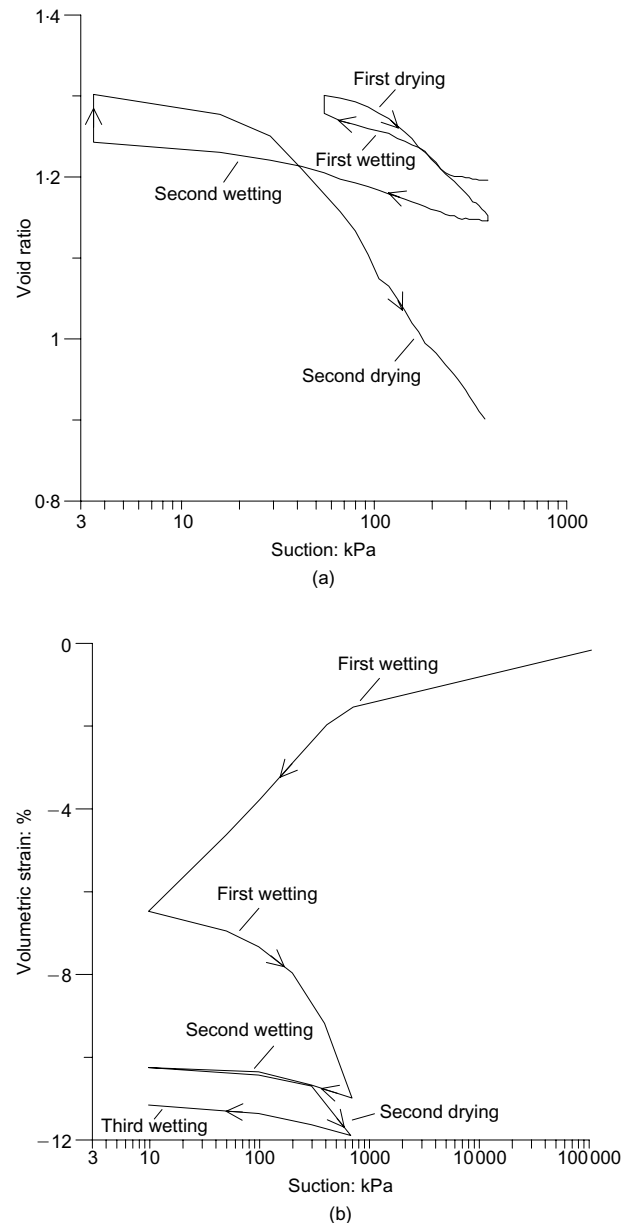

Fig. 1. Wetting–drying cycle on: (a) compacted mixture of bentonite and kaolin under constant isotropic net stress of 10 kPa (after Sharma, 1998); (b) compacted Boom clay under oedometric conditions at constant vertical stress of 400 kPa (after Alonso *et al.*, 1995)

MODELLING ASSUMPTIONS

In the proposed model the basic stress variable is the *average skeleton stress* (Jommi, 2000), which is equivalent to the Bishop (1959) stress where the parameter χ of equation (1) is equal to the degree of saturation, S_r:

$$\sigma'_{hk} = \sigma_{hk} - \delta_{hk}[u_a - \chi(u_a - u_w)] \tag{2}$$

This variable expresses the average stress acting in the soil skeleton: that is, the difference between the total stress and the average pressure of the two fluid phases (i.e. gas and liquid), with the degree of saturation as a weighting parameter. It therefore incorporates in a direct manner the first of the suction roles noted above. The definition of the average skeleton stress represents a natural extension to the unsaturated domain of the Terzaghi (1936) effective stress for saturated granular materials, and it reduces to the Terzaghi effective stress at saturated condition (that is, degree of saturation equal to unity).
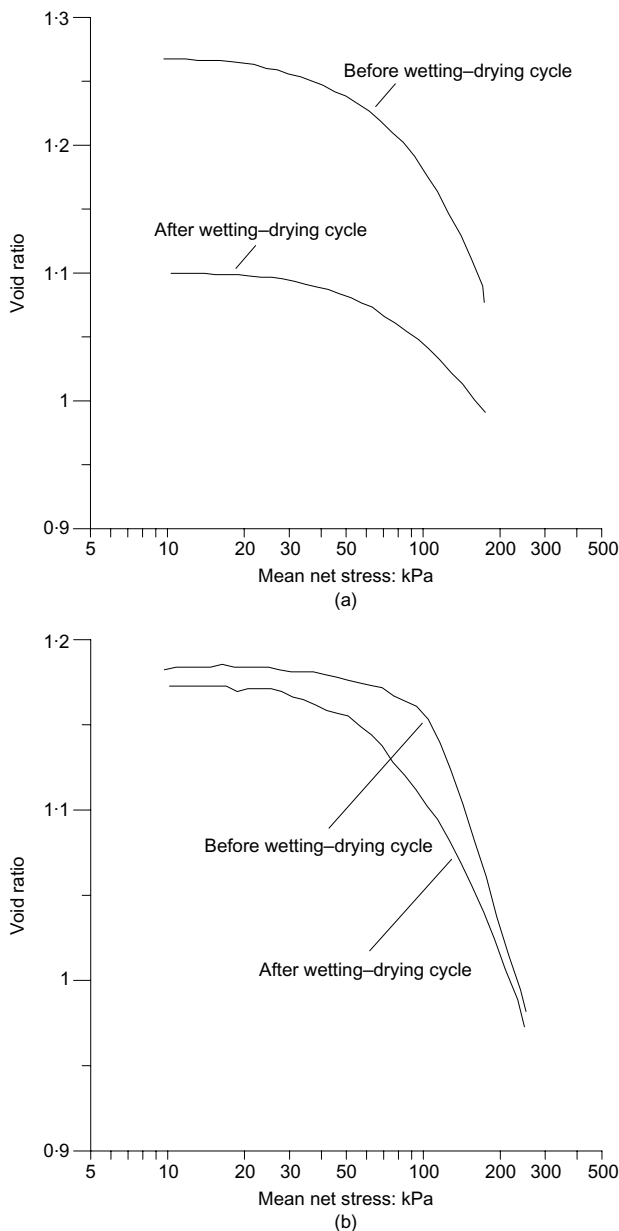
Fig. 2. Isotropic virgin loading of compacted mixture of bentonite and kaolin at: (a) constant suction of 300 kPa (after Sharma, 1998); (b) constant suction of 200 kPa (after Sharma, 1998)

Laboratory tests have shown, however, that it is not possible to explain important features of the behaviour of unsaturated soils, such as the irreversible compression (collapse) during wetting (that is, during a suction reduction) and the increase of the pre-consolidation pressure with increasing suction, by using the average skeleton stress as the only constitutive variable (e.g. Jennings & Burland, 1962). To account for these phenomena it is necessary to consider the second suction mechanism. The irreversible mechanical response of a granular material is associated mainly with the relative slippage taking place at the interface between soil particles. In an unsaturated soil the possibility of such slippage is partially reduced by the stabilising effect of the normal force exerted at the inter-particle contacts by meniscus lenses of water at negative pressure (Wheeler & Karube, 1995). Several features of the elasto-plastic behaviour of unsaturated soil are therefore likely to be the consequence of

bonding and de-bonding phenomena between soil particles due to the formation and vanishing of water menisci at inter-particle contacts, and they cannot be accounted for by using exclusively the average skeleton stress as a constitutive variable.

Consequently an additional constitutive variable, ξ, needs to be introduced as a measure of the magnitude of the inter-particle bonding due to water menisci so that the second type of suction effect is properly accounted for. The magnitude of such inter-particle bonding is expected to be the result of two contributions:

(a) the number of water menisci per unit volume of the solid fraction
(b) the intensity of the stabilising normal force exerted at the inter-particle contact by a single water meniscus.

Hence the variable ξ is defined in the present formulation as the product of two factors: the degree of saturation of the air, $(1 - S_r)$, and the function of suction, $f(s)$:

$$\xi = f(s)(1 - S_r) \tag{3}$$

The factor $(1 - S_r)$ accounts for the number of water menisci per unit volume of solid fraction. The existence of a unique relationship between the value of $(1 - S_r)$ and the number of water menisci per unit volume of solid fraction is a physically reasonable assumption; however, the uniqueness of such relationship is rigorously true only for the ideal case where the soil is rigid (that is, when the dimensions and shapes of voids do not change as a result particle rearrangements), and where each value of degree of saturation corresponds to a given arrangement of water within soil pores. The term $(1 - S_r)$ is equal to zero when the soil is saturated (that is, $S_r = 1$) and water menisci are absent, whereas it assumes positive increasing values when the number of water menisci increases. The number of water menisci per unit volume of solid fraction can therefore be expressed as a monotonic increasing function of the term $(1 - S_r)$. The validity of this definition does not apply to the case of a soil in an extremely dry state, when the water menisci will start to disappear from the particle contacts. Although the extension to the case of extremely dry soils should not present any conceptual difficulty, this is not covered in the present paper because the experimental validation would require experimental results from a test programme conducted on samples at very low degree of saturation, which are currently unavailable.

Clearly, the relationship between the number of water menisci per unit volume of solid fraction and the term $(1 - S_r)$ is dependent on the specific fabric of the soil (that is, on the pore size distribution of the soil). However, for the purposes of this work it is not necessary to characterise such a relationship explicitly, because this information is implicit in the definition of the function, introduced later in the paper, that provides the variation of the ratio e/e_s in terms of the bonding variable ξ.

The function of suction $f(s)$, which multiplies the factor $(1 - S_r)$, varies monotonically between 1 and 1·5 for values of suction ranging between zero and infinity respectively, and it accounts for the increase with increasing suction of the stabilising inter-particle force exerted by a single meniscus. In particular, it expresses the ratio between the value of stabilising force at a given suction, s, and the value of stabilising force at a suction of zero for the ideal case of a water meniscus located at the contact between two identical spheres (the analytical solution of this problem is due to Fisher, 1926). The specific form of the function $f(s)$ depends on the size of the spheres and on the value of the water surface tension, but the range of variation, between 1 and 1·5, is always the same regardless of dimensions and

physical properties. The relationship $f(s)$ used in this work is shown in Fig. 3, corresponding to the case of two spheres having radii of 1 μm and a value of the surface tension of water corresponding to a temperature of 20°C. Haines (1925) suggested that a material with the texture of a compacted kaolin could be represented by spheres having radii equal to 1 μm. Obviously the shapes of the aggregates are far from being spheres of the same size. In addition, for soils with a multi-modal pore size distribution, the dimension of the spherical grains in the solution of Fisher (1926) should be defined as a variable depending on the average size of the soil pores that include water menisci. At this stage of development of the model, however, the assumption of a simplified relationship, such as the one given in Fig. 3, is considered reasonable.

The presence of meniscus water provides a physical explanation for the experimental observation that, at the same value of average skeleton stress, the value of void ratio during virgin loading of unsaturated soil is always greater than the value for the same soil subjected to the same load under saturated conditions. The existence of water in the form of meniscus lenses within an unsaturated soil makes the inter-particle contacts more stable, and therefore restrains the reciprocal slippage of soil particles that causes compressive strains during virgin loading. Consistent with such empirical observations, this work introduces a fundamental modelling assumption specifying that, during virgin loading of an unsaturated soil, the ratio e/e_s between void ratio in unsaturated conditions, e, and void ratio in saturated conditions, e_s, at the same average skeleton stress state is a unique function of the bonding variable, ξ. This assumption not only provides an essential starting point for the development of the model, it also offers a powerful unifying perspective to examine the results of tests performed at different suctions. This assumption is validated in the next section on the basis of published laboratory test data.

EXPERIMENTAL VALIDATION OF MODELLING ASSUMPTIONS

The validation of the assumption introduced in the previous section has involved the analysis of different sets of
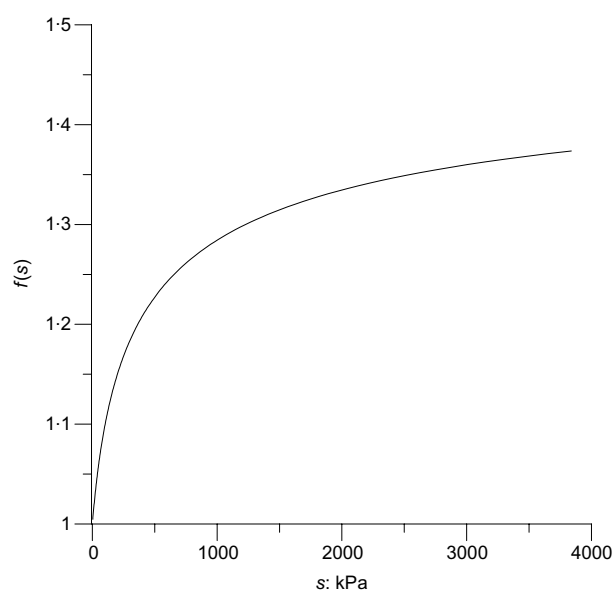

Fig. 3. Ratio between inter-particle forces at suction s and at null suction due to a water meniscus located at the contact between two identical spheres (analytical solution by Fisher, 1926)

data from laboratory tests performed on compacted Speswhite kaolin (Sivakumar, 1993; Wheeler & Sivakumar, 2000), on a compacted mixture of bentonite and kaolin (Sharma, 1998), and on compacted Kiunyu gravel (Toll, 1990). The first part of this section analyses the data from isotropic virgin compression tests at constant suction (Sivakumar, 1993; Sharma, 1998). At the end of the section, the analysis of further experimental data from triaxial shear tests on compacted Speswhite kaolin (Sivakumar, 1993; Wheeler & Sivakumar, 2000) and on compacted Kiunyu gravel (Toll, 1990) demonstrates that the conclusion achieved for isotropic stress states can also be extended to non-isotropic stress states.

Sivakumar (1993) and Sharma (1998) performed isotropic virgin compression of soil samples at different values of suction—100 kPa, 200 kPa and 300 kPa—as well as of saturated samples. During these tests the corresponding changes of void ratio, e, and water ratio, e_w (that is, the volume of water in a volume of soil containing unit volume of solids), were measured. The analysis of the experimental results indicates that, for the range of stresses considered, the normal compression lines at constant suction follow a linear relationship in the semi-logarithmic planes $e - \ln \bar{p}$ and $e_w - \ln \bar{p}$ (where $\bar{p}$ is the isotropic net stress). Each normal compression line is therefore identified by the values of the two parameters that correspond to the slope and to the intercept at a given value of $\bar{p}$. As for the data set by Sharma (1998) there were no virgin loading tests on saturated samples; the slope and intercept of the saturated normal compression line were estimated from the drying branch of a wetting–drying test under isotropic constant load. In this type of test, after an initial wetting that brought the soil to saturation, the sample was subjected to drying that caused significant irreversible changes of void ratio. The principle of effective stress holds during most of such drying because the sample remained saturated for a large increase of suction owing to its high air-entry value. Under saturated conditions the imposed change of suction corresponds to an equivalent change of the effective stress. Hence, by plotting the void ratio against the isotropic effective stress, it was possible to estimate the slope and the intercept of the saturated normal compression line.

The values of slopes and intercepts of normal compression lines of e and e_w at constant suction were used to re-plot the normal compression lines in terms of the isotropic average skeleton stress, p''. Figs 4 and 5 show the normal compression lines at constant suctions of zero (saturated), 100 kPa, 200 kPa and 300 kPa, in the semi-logarithmic plane $e - \ln p''$ for each set of data respectively.

Inspection of Figs 4 and 5 reveals that the normal compression lines at non-zero values of suction are not straight lines in the semi-logarithmic plane $e - \ln p''$, but they are curves with slopes that decrease as they approach the saturated line (zero suction). This is consistent with the experimental observation that the degree of saturation increases during isotropic loading to virgin states at constant suction. Indeed, if a soil sample attains saturation during compression at a positive value of suction, the isotropic average skeleton stress coincides with the saturated effective stress, and the corresponding value of void ratio should lie on the saturated normal compression line. After saturation, the normal compression line at non-zero suction should therefore have the same slope as the saturated normal compression line. It is therefore to be expected that the slope of the normal compression lines at non-zero suction progressively reduces as they converge towards the saturated line.

From the normal compression lines in the semi-logarithmic plane $e - \ln p''$ shown in Figs 4 and 5 it is possible to calculate the ratio between the e value of the unsaturated
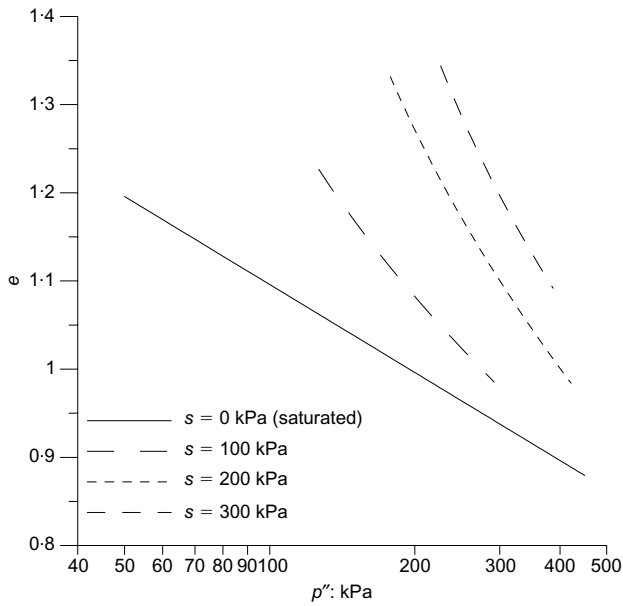
Fig. 4. Normal compression lines at constant suction in the plane $e - \ln p''$ (data by Sharma, 1998)
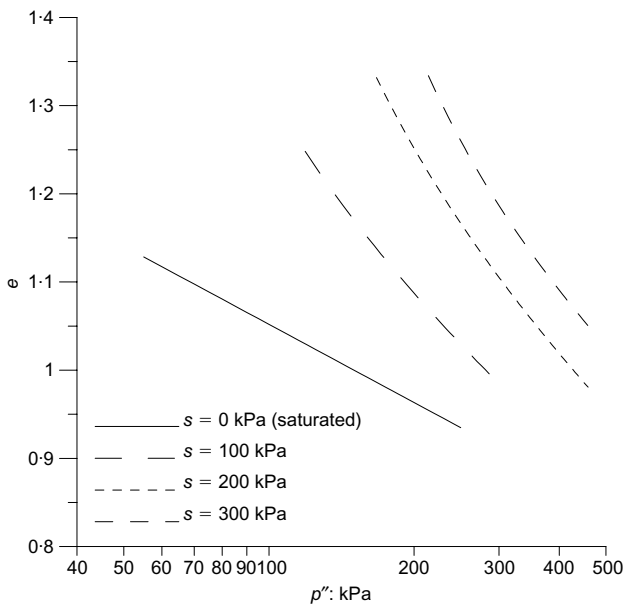

Fig. 6. Relationship between ratio e/e_s and bonding factor ξ during isotropic virgin loading at constant suction (data by Sharma, 1998)

Fig. 5. Normal compression lines at constant suction in the plane $e - \ln p''$ (data by Sivakumar, 1993)

Fig. 7. Relationship between ratio e/e_s and bonding factor ξ during isotropic virgin loading at constant suction (data by Sivakumar, 1993)

soil and that corresponding to the saturated state, e_s, at the same average skeleton stress. Figs 6 and 7 show, for the data sets of Sivakumar (1993) and Sharma (1998) respectively, the value of the ratio e/e_s plotted against the value of the bonding variable, ξ, defined by equation (3) (corresponding to the values of S_r and $f(s)$ of the unsaturated soil). The value of the function of suction, $f(s)$, has been calculated according to the relationship shown in Fig. 3, and it is equal to 1·10, 1·15 and 1·18 for suction values of 100 kPa, 200 kPa and 300 kPa respectively. The relationship shown in Fig. 3 refers to the Fisher (1926) solution where the spherical grains have radii equal to 1 μm. This is the order of magnitude of the macrostructural voids of a clay soil compacted dry of optimum, such as the soils investigated by Sivakumar (1993) and Sharma (1998). Porosimetry studies have shown that clay materials compacted dry of optimum
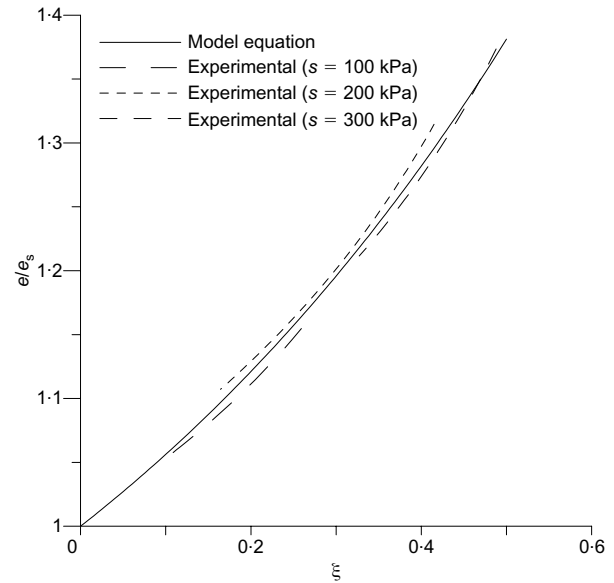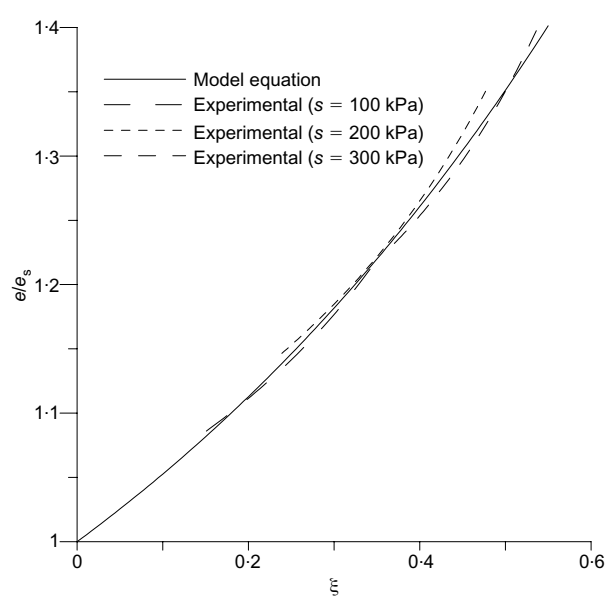
present a marked bimodal pore size distribution (e.g. Gens & Alonso, 1992) with macrostructural and microstructural voids of the order of magnitude of 1 μm and 0·01 μm respectively. For the range of suctions investigated by Sivakumar (1993) and Sharma (1998) it is reasonable to expect that only macro-voids are affected by desaturation (and hence by the formation of water menisci) while the micro-voids stay saturated.

Inspection of Figs 6 and 7 suggests remarkably that, for all the three values of suction investigated, the data from normal compression are consistent with a unique relationship linking the value of the proportion e/e_s and the bonding variable, ξ. Such a bonding variable therefore appears to be uniquely related to the ability of the skeleton to sustain

higher void ratios when the soil is under suction. For each of the three curves at constant suction shown in Figs 6 and 7, the value of the proportion e/e_s is expected to attain a value of 1 when ξ is equal to zero (that is, when the sample achieves saturation) because in this case the normal compression lines at non-zero suctions coincide with the saturated line in the semi-logarithmic plane $e - \ln p''$. The model equation that fits the three curves of e/e_s against ξ at constant suction in Figs 6 and 7 has the following form:

$$\frac{e}{e_s} = 1 - a \cdot [1 - \exp(b \cdot \xi)] \qquad (4)$$

where a and b are fitting parameters. Equation (4) predicts a value of e/e_s equal to 1 when ξ is equal to zero, consistent with the physical explanation given above.

For the range of suction investigated, the value of the function $f(s)$ varies relatively little in comparison with the variation of the value of the bonding variable ξ. It is therefore reasonable to expect that experimental data might be similarly consistent with a relationship linking the value of the proportion e/e_s during isotropic virgin loading to the value of the degree of saturation of the gas phase, $(1 - S_r)$.

Sivakumar (1993) and Wheeler & Sivakumar (2000) presented further experimental data from shearing tests to critical state on compacted Speswhite kaolin under various suction values. These data have been used to investigate whether the relationship between the ratio e/e_s and the bonding variable ξ could be extended to non-isotropic stress states. The model equation in Fig. 7, which had been defined on the basis of isotropic normal compression tests, was therefore used to predict values of void ratio at critical states. The predicted values of void ratio at critical state were computed following the same procedure as in the isotropic case. First the saturated critical-state line in the semi-logarithmic plane ($e - \ln p''$) was defined (by fixing its slope and intercept) on the basis of shearing tests performed by Sivakumar (1993) and Wheeler & Sivakumar (2000) on saturated samples. Then, for each unsaturated sample sheared to critical state, the corresponding experimental values of isotropic net stress, degree of saturation and suction at critical state were used to calculate the isotropic average skeleton stress, p'', and the bonding variable ξ. These values of p'' and ξ were then employed to compute the void ratio, e_s, from the saturated critical-state line and the ratio e/e_s from the model equation of Fig. 7 respectively. Fig. 8 shows the comparison between predicted and experimental values of void ratio at critical state corresponding to different suction levels.

Inspection of Fig. 8 indicates remarkably that the relationship established between the ratio e/e_s and the bonding factor ξ for isotropic virgin compression and given by equation (4) can also accurately predict the void ratio values at critical state. This implies that such a relationship might be unique for the elasto-plastic loading of an unsaturated soil regardless of the specific stress ratio applied to the sample, and that the selected bonding variable closely represents the real effect of suction on inter-granular stress. The different series of Fig. 8 correspond to different procedures adopted by Wheeler & Sivakumar (2000) for the compaction of Speswhite kaolin at the same dry of optimum water content (that is, for the test series II and III the compaction pressure and method of compaction were different from those employed for preparing the samples from series I shown in Figs 5 and 7). On the basis of their empirical results, Wheeler & Sivakumar (2000) concluded that the behaviour at critical state of a soil compacted at the same dry of optimum water content is not affected by the procedure adopted for compaction. This result is also confirmed by the comparison shown in Fig. 8.
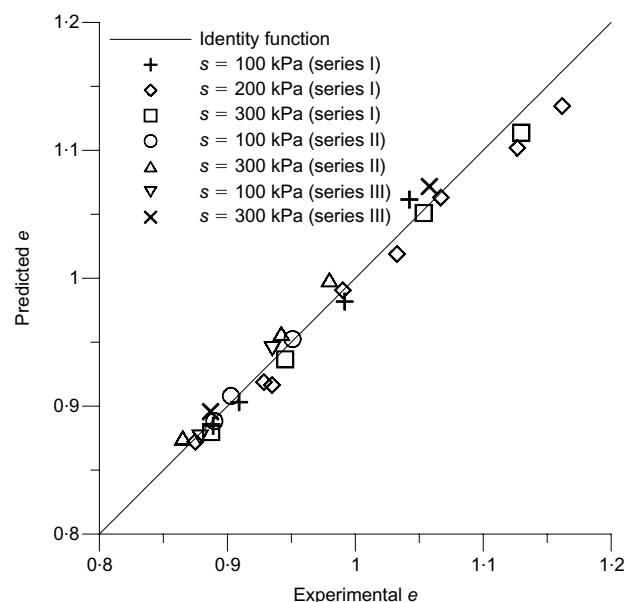

Fig. 8. Comparison between experimental and predicted void ratio at critical state (data by Sivakumar, 1993 and Wheeler & Sivakumar, 2000)

Finally, the analysis of the data from undrained (with respect to the water phase) triaxial shear tests performed by Toll (1990) on compacted samples of a lateritic gravel from Kenya (Kiunyu gravel) is presented. Samples were compacted by Toll (1990) at different values of water content ranging from 17·0 to 27·7 and then sheared in axial compression to critical state while preventing the flow of water. The author reports the measured values of the void ratio, degree of saturation, suction and net stress state at the end of the tests when the unsaturated samples have attained critical state conditions. Together with the results from unsaturated soil samples, the author presents a smaller set of data from undrained triaxial shear tests to critical state performed on saturated samples. For the saturated tests, the soil samples were compacted at water content ranging between 18·7 and 31·0 and were then saturated prior to testing. On the basis of the saturated shear tests, Toll (1990) suggests the values of the slope and intercept of saturated critical-state line in the semi-logarithmic plane ($e - \ln p''$).

The data presented by Toll (1990) are limited to the critical-state values measured at the end of the undrained shearing of each sample. Such data were used in this work to validate the proposed assumption of a unique relationship between the ratio e/e_s (corresponding to a given value of the isotropic average skeleton stress) and the bonding variable ξ at the critical state. The soil tested by Toll (1990) is likely to exhibit a grading different from that of the soils investigated by Sivakumar (1993) and Sharma (1998). However, owing to the absence of precise information on the fabric of the soil tested by Toll (1990), the radii of the spheres in the Fisher (1926) solution (that is, in the function $f(s)$ of equation (3)) were taken equal to 1 μm, the same value as in the previous two analyses. For each experimental data point the value of the function $f(s)$ was then calculated according to the suction measured by Toll (1990) at the critical state.

For the purposes of the study presented here, the unsaturated samples tested by Toll (1990) were classified in two different groups, each one including samples compacted at similar values of water content. The data shown in Fig. 9 refer to unsaturated samples whose compaction water con-
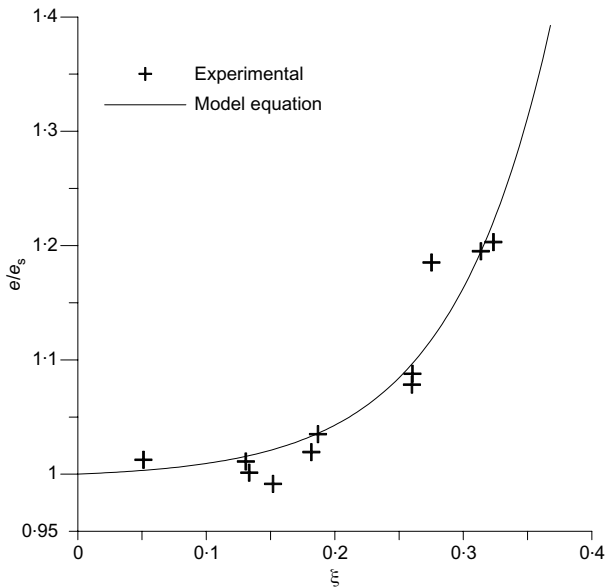
Fig. 9. Relationship between ratio e/e_s and bonding factor ξ at critical state for soil samples compacted at water content between 24·9% and 27·7%. The suctions at critical state range from 2 kPa to 73 kPa (data by Toll, 1990)

tent ranged between 24·9% and 27·7%, whereas for the samples shown in Fig. 10 the compaction water content ranged between 19·6% and 21·9%. The value of the compaction water content significantly affects the ensuing fabric of the unsaturated sample, and this in turn has an effect on the mechanical response of the sample, including the response at critical state (as discussed by Wheeler & Sivakumar, 2000). Hence the purpose of such a distinction was to select two homogeneous groups of samples that presented a similar soil fabric, and whose mechanical response during laboratory testing could therefore be compared. From examination of Figs 9 and 10 it can be observed that all critical-state data points (each of them corresponding to different values of
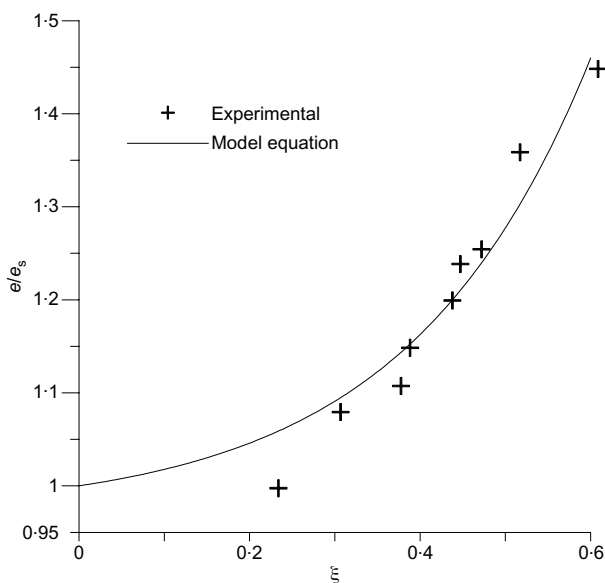

Fig. 10. Relationship between ratio e/e_s and bonding factor ξ at critical state for soil samples compacted at water content between 19·6% and 21·9%. The suctions at critical state range from 22 kPa to 537 kPa (data by Toll, 1990)

suction, net stress state and degree of saturation at critical state) follow a unique trend when plotted in the plane $(e/e_s, \xi)$. In Figs 9 and 10 the expression of the curve interpolating the experimental data is given by equation (4) (the same expression as used for the experimental data of Sharma (1998) and Sivakumar (1993) shown in Figs 6 and 7).

The samples compacted at water contents between 24·9% and 27·7% (Fig. 9) had suction values ranging from 2 kPa to 73 kPa on reaching the critical state, whereas the suction values at critical state for the samples compacted at water contents between 19·6% and 21·9% (Fig. 10) varied between 22 kPa and 537 kPa. The proposed relationship, therefore, is shown to be capable of bringing together results from a very wide range of suction values.

FORMULATION OF THE ELASTO-PLASTIC STRESS–STRAIN MODEL

An elasto-plastic isotropic stress–strain model for unsaturated soils incorporating volumetric hardening is described in this section. The success of the modelling ideas proposed here will be demonstrated in the next section by comparing the predictions with the experimental results from various types of laboratory test, all performed under isotropic loading. The development of the model is hence limited in this section to isotropic stress states in order to concentrate attention on the basic features of the model. Extension to more general stress states is quite straightforward following standard procedures (Gens, 1995).

The formulation of a constitutive model including volumetric hardening requires the definition of:

(a) a normal compression state surface, which relates the values of void ratio, e, isotropic average skeleton stress, p'', and bonding variable, ξ, during the irreversible behaviour of the soil

(b) an incremental expression that relates the elastic part of the change of void ratio, e, to the changes of the isotropic average skeleton stress, p'', and bonding variable, ξ.

The normal compression state surface is defined here as the product of two factors. The first factor is the equation of the saturated normal compression line relating the variation of the void ratio, e_s, to the change of the isotropic average skeleton stress, p'', and the second factor is the equation that links the variation of the ratio e/e_s to the change of the bonding variable, ξ. For the materials studied here, the analytical form of the normal compression state surface is therefore expressed as

$$e(p'', \xi) = \frac{e}{e_s}(\xi)e_s(p'') \tag{5}$$

where $e(p'', \xi)$ is the normal compression state surface, $(e/e_s)(\xi)$ is given by equation (4), and $e_s(p'')$ is the saturated normal compression line (a straight line in the semi-logarithmic plane $e - \ln p''$) having the form

$$e_s(p'') = N - \lambda \ln p'' \tag{6}$$

N and λ in equation (6) are the intercept (at $p'' = 1$ kPa) and the slope of the saturated normal compression line respectively. Note that, for saturated conditions, the isotropic average skeleton stress, p'', coincides with the isotropic effective stress, $\bar{p}$, and therefore the parameters N and λ are equal to those that identify the saturated normal compression line in the semi-logarithmic plane $e - \ln \bar{p}$. Fig. 11(a) shows three examples of normal compression lines that lie on the normal compression state surface and correspond to constant values of the bonding variable, ξ. Equations (5) and (6)
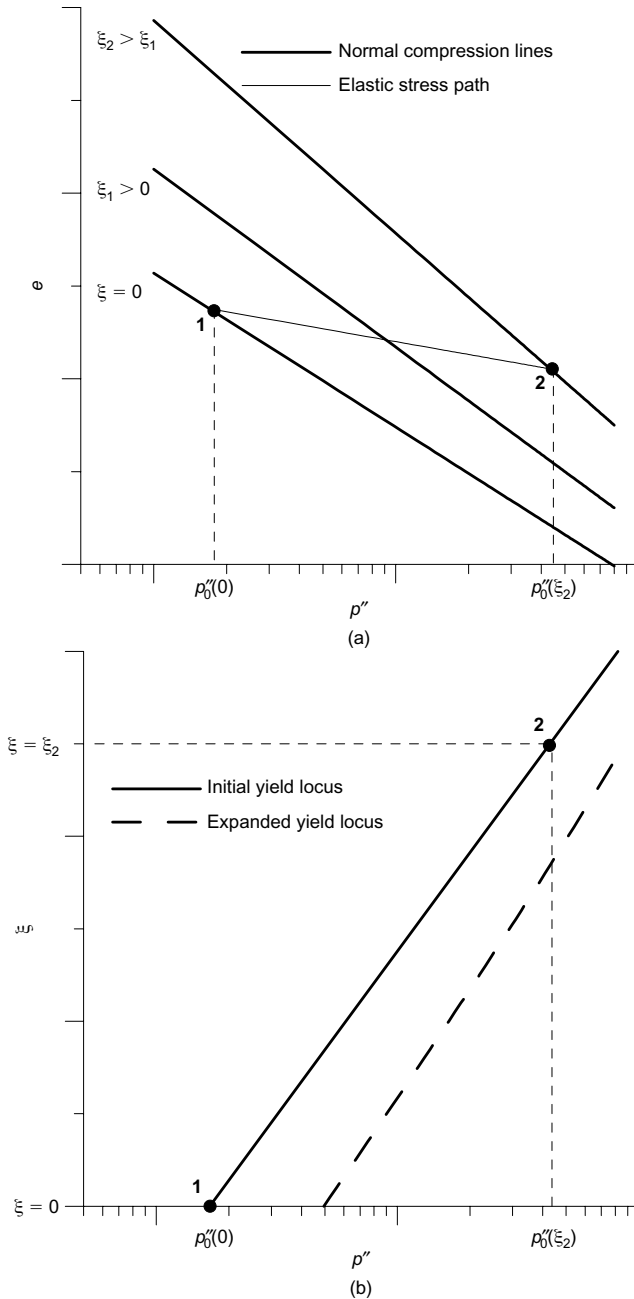
Fig. 11. Derivation of the yield locus in the isotropic plane: (a) change of void ratio; (b) stress path

indicate that the normal compression lines at constant ξ are straight lines in the semi-logarithmic plane $e - \ln p''$ and that their extrapolations at high values of p'' intersect each other at the same point, coinciding with a value of the void ratio e_s on the saturated normal compression line equal to zero.

The elastic change of void ratio, Δe^e, is assumed to be given by

$$\Delta e^e = -\kappa \ln \frac{p_f''}{p_i'} \qquad (7)$$

where κ is the elastic swelling index and p_i'' to p_f'' are the initial and final value of the isotropic average soil skeleton stress respectively. Equation (7) implies that the elastic change of void ratio depends exclusively on the change of the isotropic average skeleton stress, p'' (that is, it is

independent of the variation of the bonding variable ξ). This is equivalent to assuming that the elastic deformation of the soil skeleton is not affected by the bonding action that the water menisci exert at the inter-particle contacts. It will be shown in the next section that this assumption fits well the elastic behaviour of the laboratory tests considered in this work.

The normal compression state surface defined by equation (5) acts as a limiting surface in (e, p'', ξ) space, where it separates the region of attainable soil states from the region of non-attainable soil states. The soil response is elastic while the soil follows a path inside the space of attainable soil states. When the soil path reaches the normal compression state surface, this surface imposes a constraint on further changes of e, p'' and ξ, and the soil state can therefore either move back inside the space of the attainable soil states or follow a path lying on the normal compression state surface. When the latter possibility occurs, irreversible (elasto-plastic) changes of void ratio develop.

Thus the normal compression state surface and the elastic law introduced above (equations (5) and (7) respectively) implicitly define a yield locus that incorporates a volumetric hardening rule. To obtain the analytical form of such a yield locus consider the elastic stress path in Fig. 11(a) starting from the soil state denoted by 1, at an isotropic average skeleton stress $p_o''(0)$ on the saturated normal compression line, and moving to the soil state denoted by 2, at an isotropic average skeleton stress $p_o''(\xi_2)$ on the unsaturated normal compression line corresponding to $\xi = \xi_2$. The change of void ratio during the path from state 1 to state 2 is computed according to the elastic equation (7):

$$\Delta e = -\kappa \ln \frac{p_o''(\xi_1)}{p_o''(0)} \qquad (8)$$

As the soil states 1 and 2 also belong to the normal compression state surface they must lie on the same yield locus, and an alternative expression for the variation of void ratio during the path from state 1 to state 2 can therefore be obtained by using the normal consolidation state surface of equation (5):

$$\Delta e = e[p''(0), 0] - e[p''(\xi_1), \xi_1] = N - \lambda \ln p_o''(0)$$
$$- \frac{e}{e_s}(\xi_1)[N - \lambda \ln p_o''(\xi_1)] \quad (9)$$

By equating equation (8) and equation (9) and then rearranging, the following equation of the yield locus in the isotropic plane $\xi - \ln p''$ is obtained:

$$\ln p_o''(\xi_1) = \frac{\lambda - \kappa}{\frac{e}{e_s}(\xi)\lambda - \kappa} \ln p_o''(0) + \frac{\left[\frac{e}{e_s}(\xi) - 1\right](1 + N)}{\frac{e}{e_s}(\xi)\lambda - \kappa}$$
$$(10)$$

Figure 11(b) shows the yield locus of equation (10) in the isotropic plane $\xi - \ln p''$ together with the two yield points corresponding to the soil states 1 and 2, which are identified by the coordinates $(p_o''(0), 0)$ and $(p_o''(\xi_2), \xi_2)$ respectively.

Figure 11(b) also shows an expanded yield locus, indicated by the broken line, which refers to a soil sample that has experienced additional plastic volumetric strains and whose yield locus has therefore undergone volumetric hardening. The current size of the yield locus is identified by the value of its intercept $p_o''(0)$ with the horizontal axis, which is the yield value of the isotropic average skeleton stress during isotropic compression of a saturated sample. The saturated yield stress $p_o''(0)$ can therefore be assumed as the

hardening parameter of the present elasto-plastic model. The irreversible change of void ratio, Δe^{p}, associated with the expansion of the yield locus from an initial position identified by $p_o''(0) = p_o''(0)_i$ to a final position identified by $p_o''(0) = p_o''(0)_f$ coincides then with the irreversible change of void ratio calculated by the saturated normal compression line for a variation of the isotropic average skeleton stress from $p_o''(0)_i$ to $p_o''(0)_f$:

$$\Delta e^{\mathrm{p}} = -(\lambda - \kappa)\ln \frac{p_o''(0)_f}{p_o''(0)_i} \qquad (11)$$

and equation (11) thus represents the volumetric hardening rule of the proposed elasto-plastic model.

The complete model includes a formulation to compute the degree of saturation that must incorporate the effect of hydraulic hysteresis and stress-induced changes of soil fabric. The relationships proposed by Vaunat *et al.* (2000) and Gallipoli *et al.* (2003) can be used for this purpose, but a detailed description of this component of the model is outside the scope of the paper. For the model computations presented in the next section, the experimentally observed degrees of saturation have been used. In this way the differences between predictions and observations must be attributed exclusively to the mechanical elasto-plastic model.

MODEL PREDICTIONS

The good performance of the proposed elasto-plastic model is demonstrated here by comparing the results from a selection of experiments performed by Sharma (1998) on a compacted mixture of bentonite and kaolin with the corresponding model predictions. In particular, the comparison will show the potential of the proposed model for correctly predicting:

(a) The initial yield locus of the soil corresponding to the after-compaction state
(b) The irreversible change of void ratio occurring during wetting (collapse)
(c) The irreversible change of void ratio during drying
(d) the dependence of the soil response during isotropic virgin loading at constant suction on the previous history of suction variation.

Points (c) and (d) refer to typical features of unsaturated soil behaviour that are not taken into account by existing elasto-plastic constitutive frameworks formulated in terms of a single yield surface.

The selection process of the model parameter values used for the predictions (see Table 1) has been described earlier, except for the value of the elastic swelling index, κ, which was selected on the basis of elastic isotropic loading–unloading cycles at constant suction.

Figures 12–14 show the comparison between experimental and predicted behaviour for three isotropic loading tests at constant suction (100 kPa, 200 kPa and 300 kPa respectively) that involve elasto-plastic yielding. Inspection of Figs 12–14 reveals that the proposed model correctly calculates the

Table 1. Parameter values for the proposed elasto-plastic model

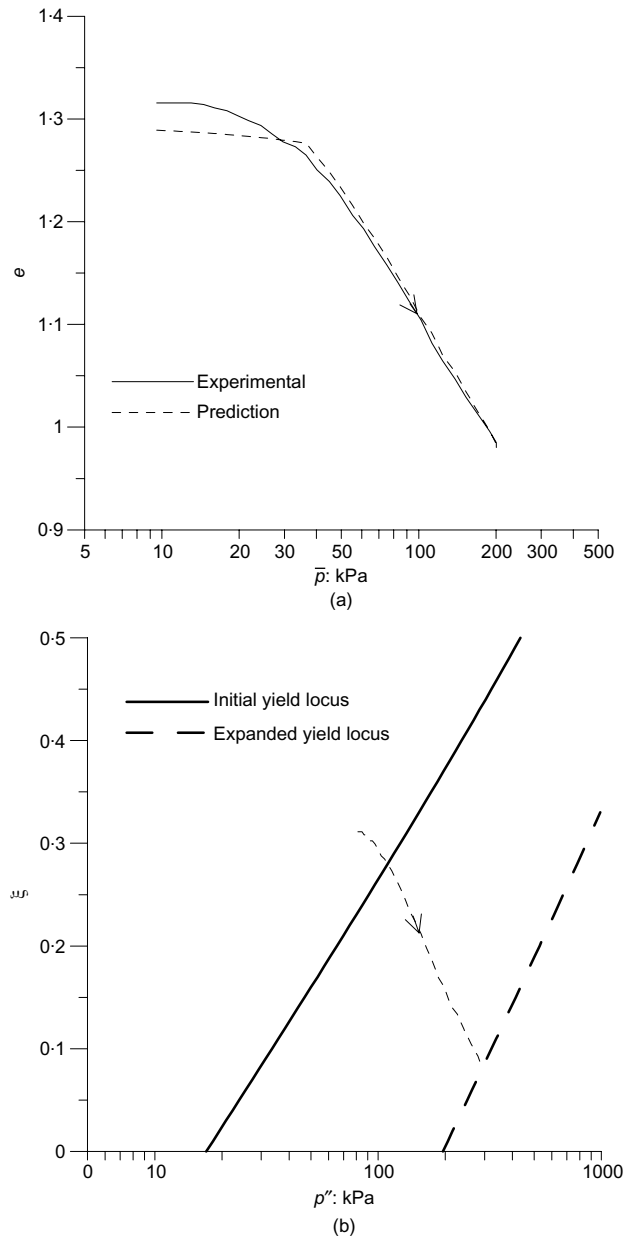| Parameter	Value
λ, slope of NCL at $s = 0$	0·144
N, e on NCL at $s = 0$ for $p'' = 1$ kPa	1·759
κ, swelling index for changes of p''	0·040
a, parameter of model equation (4)	0·369
b, parameter of model equation (4)	1·419

Fig. 12. Model prediction for isotropic virgin loading at constant suction of 100 kPa (experimental data by Sharma, 1998): (a) change of void ratio; (b) stress path

respective yield points by assuming for all three test simulations the same initial yield locus associated with a value of the hardening parameter, $p_o''(0) = 17$ kPa. Such a model prediction is corroborated by the soil response observed by Sharma (1998) during the equalisation stage prior to loading, when the suction of the three samples was decreased from the value after compaction to 100 kPa, 200 kPa and 300 kPa respectively. During this stage all three samples experienced exclusively elastic swelling, which indicates that the initial yield curve after compaction had not undergone further expansion associated with plastic volumetric compression (collapse). It is then expected that all three samples would yield on the same locus during isotropic loading, and the proposed model indeed correctly predicts this. Therefore, for the test simulations presented in the remainder of this section, the value of the hardening parameter corresponding to the soil after compaction was assumed to be equal to 17 kPa.
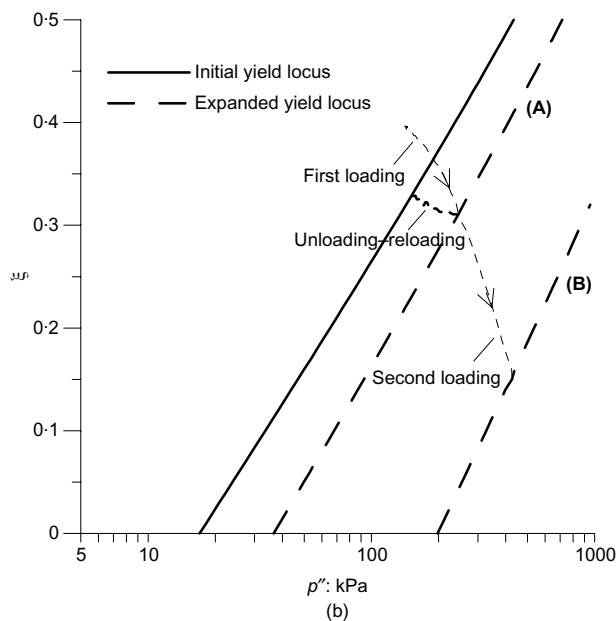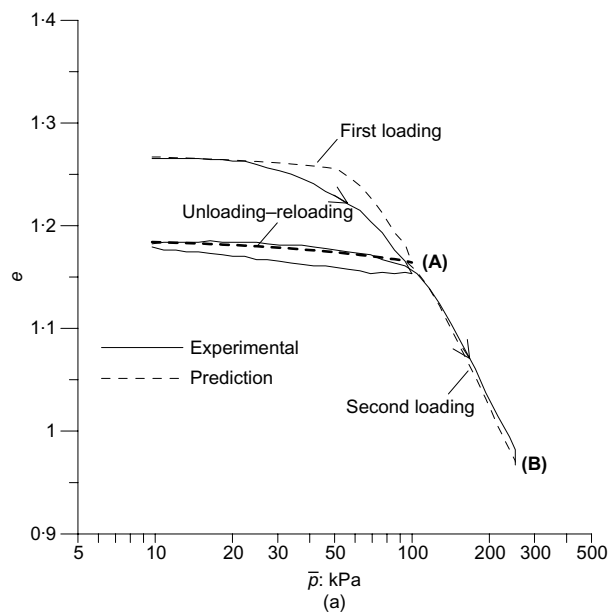
Fig. 13. Model prediction for isotropic virgin loading at constant suction of 200 kPa (experimental data by Sharma, 1998): (a) change of void ratio; (b) stress path
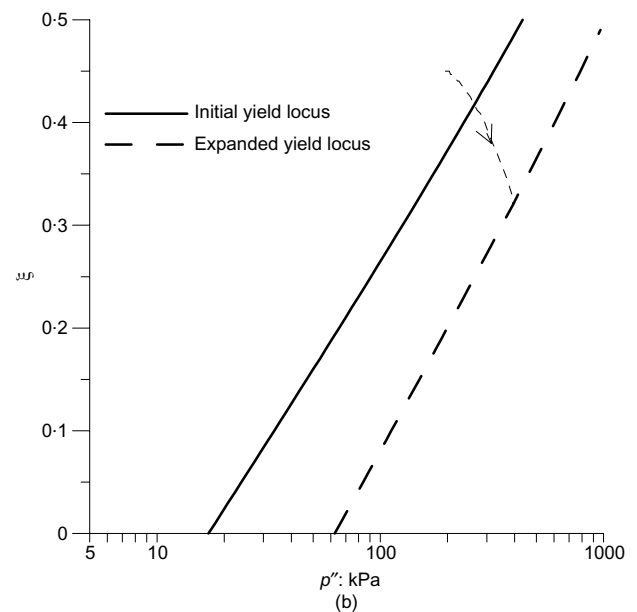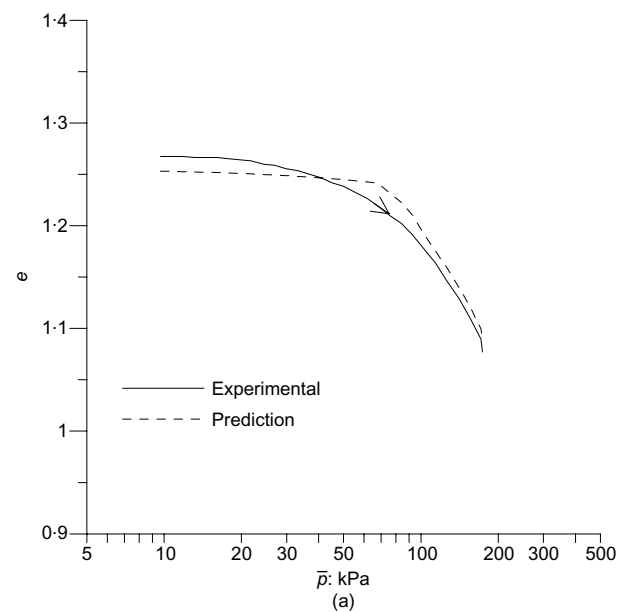
Fig. 14. Model prediction for isotropic virgin loading at constant suction of 300 kPa (experimental data by Sharma, 1998): (a) change of void ratio; (b) stress path

Figure 13 also shows the comparison between model predictions and experimental results for an elastic loading–unloading cycle, which confirms the adequacy of the elastic law given by equation (7). The very good match between experimental and predicted values of void ratio at the beginning of the loading in Figs 12–14 provides a further proof of the validity of equation (7) because the predicted initial values of void ratio are computed by means of an elastic path starting from the saturated yield stress state ($p_o''(0) = 17$ kPa) according to the following expression:

$$e_i = e_o - \kappa \ln \frac{p_i''}{17 \text{ kPa}} \tag{12}$$

where e_o is the void ratio predicted by the saturated normal compression line for $p'' = 17$ kPa and p_i'' is the isotropic average skeleton stress of the unsaturated sample at the beginning of loading.

Figure 15 shows the comparison between experimental and predicted behaviour for a wetting–drying cycle per-

formed at a constant isotropic net stress of 50 kPa. Inspection of the stress path followed by the soil during the test in Fig. 15(b) reveals that the model predicts irreversible changes of void ratio during both the wetting and the drying branch of the test. Yielding of the soil occurs initially during wetting, and the consequent development of elasto-plastic strains produces the first expansion of the yield locus from its initial position to the position indicated by (A) (corresponding to the end of wetting). After the reversal of suction the model continues to predict elasto-plastic deformations during the whole drying, and this corresponds to a further expansion of the yield locus from position (A) to the final position (B). This test simulation clearly demonstrates the potential of the present framework to interpret the elasto-plastic volumetric strains that occur during both the wetting and the drying phases as a single mechanical phenomenon that can be modelled by employing only one yield locus. Part of the discrepancy between experimental results and model prediction in Fig. 15(a) is due to the incomplete
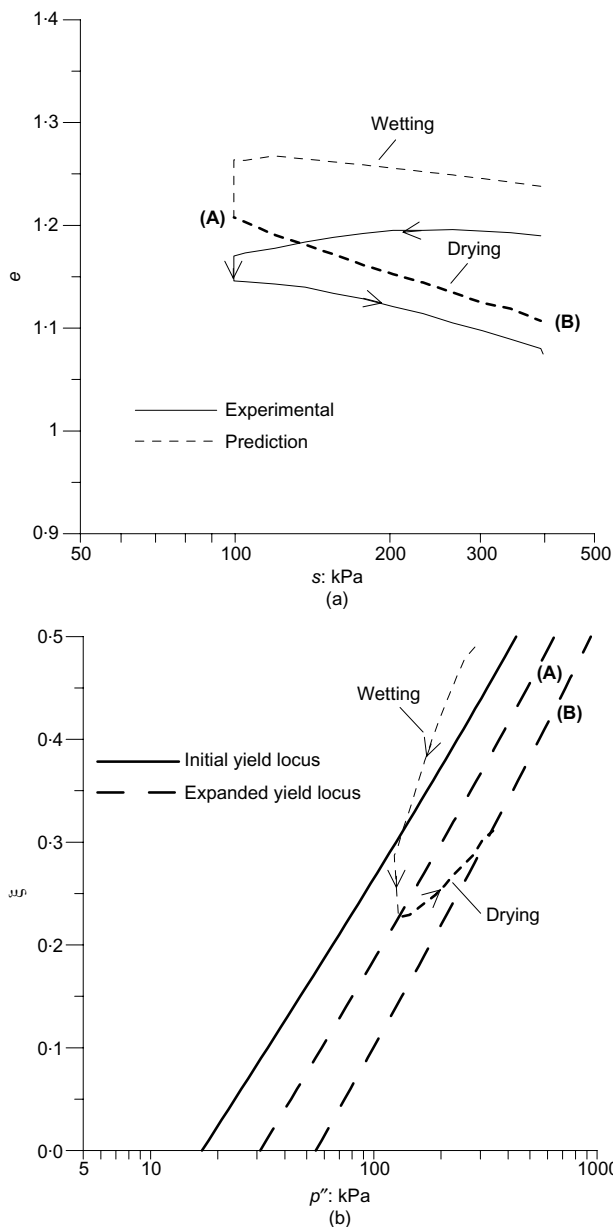
Fig. 15. Model prediction for a wetting–drying cycle of a sample subjected to a constant isotropic net stress of 50 kPa (experimental data by Sharma, 1998): (a) change of void ratio; (b) stress path
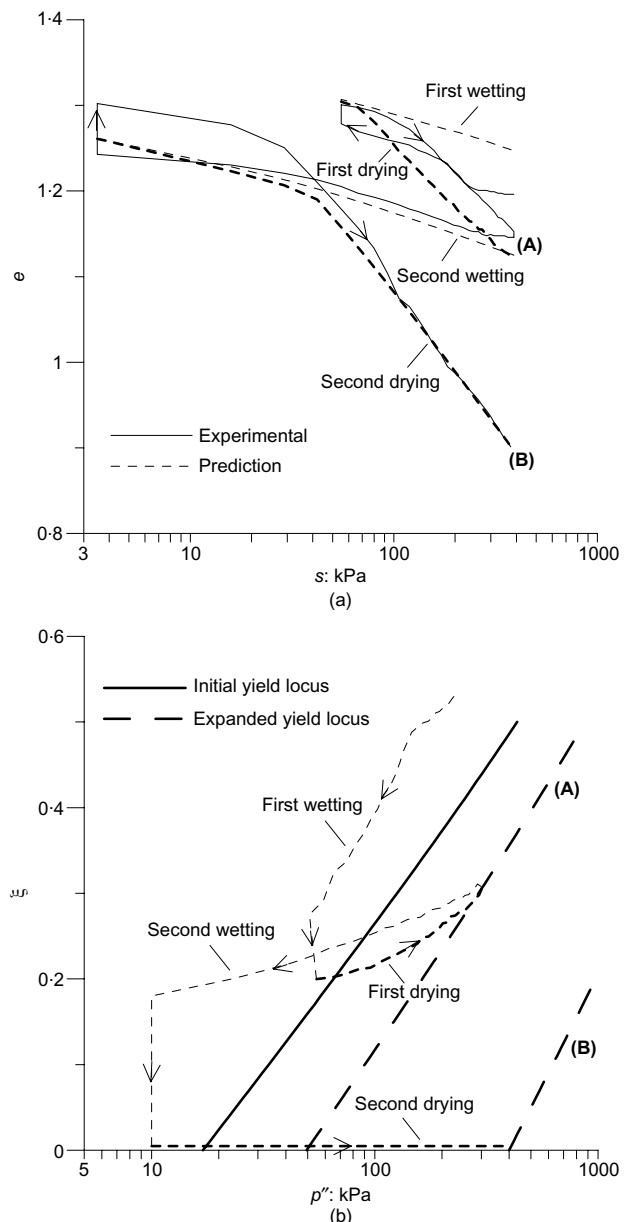
Fig. 16. Model prediction for wetting–drying cycles of a sample subjected to a constant isotropic net stress of 10 kPa (experimental data by Sharma, 1998): (a) change of void ratio; (b) stress path

equalisation of suction within the sample during the test. The occurrence of incomplete equalisation is proven by the experimental observation, reported by Sharma (1998), that a significant amount of water flowed into the sample during the stabilisation period at the end of the wetting stage, when the sample was kept at constant suction of 100 kPa for a period of time necessary to equalise suction before subsequent drying (the vertical change of void ratio at constant suction of 100 kPa shown in Fig. 15(a) corresponds to this stabilisation period). However, despite such experimental limitations, inspection of Fig. 15(a) still indicates a satisfactory agreement between predicted and computed results.

Now the more complex stress paths involving wetting–drying cycles are considered. Fig. 16 shows the comparison between the experimental and predicted behaviour during wetting–drying cycles performed at a constant isotropic net stress of 10 kPa. Inspection of Fig. 16(b) indicates that elasto-plastic strains occur exclusively during the drying phases whereas elastic swelling takes place during the

wetting phases. The irreversible strains generated by the first drying produce an expansion of the yield locus from the initial position to position (A) whereas the second drying originates a further expansion from position (A) to position (B). Note that, as explained above, the discontinuity in the slope of the wetting paths shown in Fig. 16(b) is due to the stabilisation phase following incomplete suction equalisation during previous wetting.

The model predictions in Figs 15 and 16 represent a significant improvement over existing elasto-plastic models based on a single yield locus, which would incorrectly predict elastic compression during all the drying phases of the above tests.

Finally, Fig. 17 shows the comparison between experimental and predicted behaviour for two constant suction isotropic tests that show different mechanical responses depending on whether or not the sample has undergone a wetting–drying cycle prior to loading. Inspection of Fig. 17 reveals that the model is capable of capturing the different stiffnesses shown by the soil during virgin loading in the two
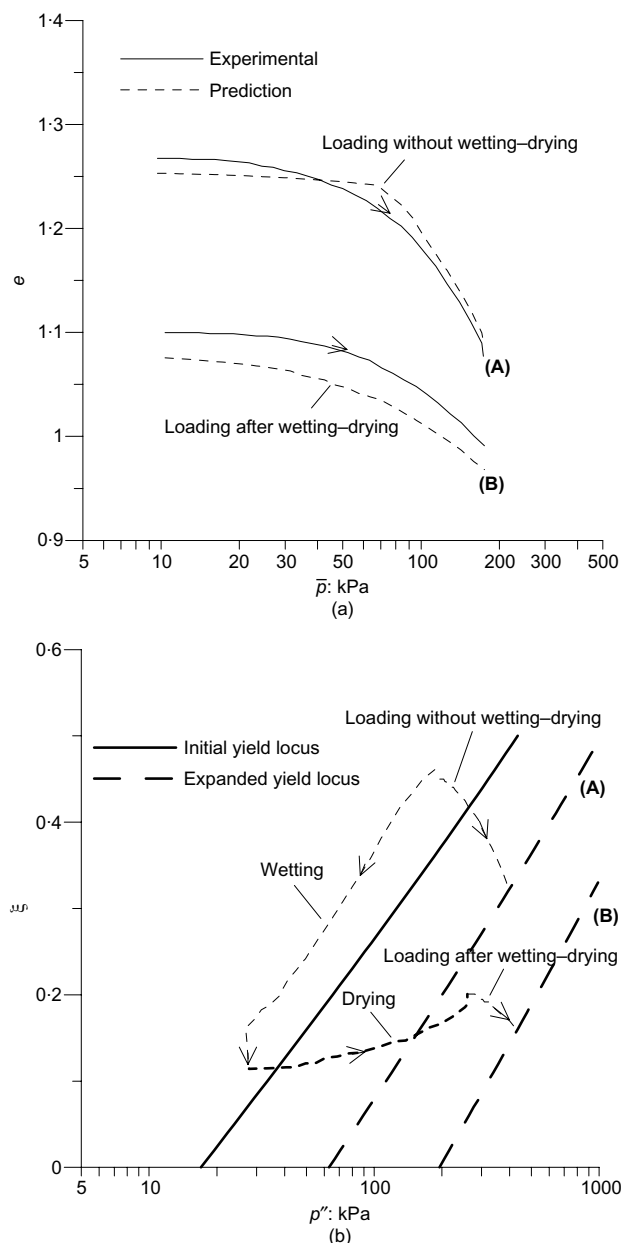
Fig. 17. Model prediction for isotropic virgin loading at constant suction of 300 kPa (experimental data by Sharma, 1998): (a) change of void ratio; (b) stress path

cases. This is a significant advance with respect to existing models that would instead predict the same slope of the normal compression lines for both cases regardless of the previous history of suction variation. A further improvement with respect to existing models is that the present framework correctly predicts different values of void ratio at the beginning of loading in the two cases of Fig. 15(a). This is due to the irreversible change of void ratio that occurs during drying of the sample subjected to the wetting–drying cycle (see Fig. 15(b)), which is also reflected in the different degree of expansion of the yield locus achieved at the end of test in the two cases (position (A) and position (B) respectively).

CONCLUSIONS

The paper proposes an innovative constitutive framework for unsaturated soil that is able to explain the various mech-

anical features of this material by resorting to a physical description of the different effects of suction on soil straining. In the assumed mechanism, the relative slippage of soil particles is governed by two counteracting actions exerted on the assemblage of soil particles:

(a) the perturbing action of the average stress state acting on the soil skeleton
(b) the stabilising action of the normal force exerted at the inter-particle contacts by water menisci.

The variables controlling each one of these actions (that is, the average skeleton stress variable, σ'', and the bonding variable, ξ, respectively) are defined on the basis of the current values of the net stress state, suction and degree of saturation. The introduction of degree of saturation in the definition of the soil constitutive variables is essential to represent properly the contribution of soil suction to the two effects described above.

Based on a physical argument, the present proposal assumes that, during the elasto-plastic loading of a soil element, the proportion e/e_s between the void ratio, e, under unsaturated conditions and the void ratio, e_s, under saturated conditions at the same average skeleton stress state is a unique function of the bonding variable, ξ. This fundamental assumption is successfully validated in this work by the analysis of several published sets of experimental data for different materials. The analysis of one set of data (for which both isotropic and shearing tests are available) also suggests that the relationship between e/e_s and ξ is unique for a given soil, and that it is independent of the applied stress ratio.

On the basis of this assumption a full elasto-plastic stress–strain model for isotropic stress states is formulated, and its good performance is demonstrated by the comparison between predicted and laboratory tests results from a comprehensive experimental programme including a wide variety of different stress paths. This comparison confirms the potential of the proposed model for correctly predicting the most important features of the mechanical behaviour of unsaturated soils by retaining at the same time the simplicity of a model formulated in terms of a single yield curve. In particular it is able to predict correctly the following two typical responses of unsaturated soils that are not modelled by existing elasto-plastic constitutive frameworks based on a single yield surface:

(a) the irreversible change of void ratio during drying
(b) the dependence of the response during virgin compression at constant suction on the previous history of suction variation.

An additional significant advantage is that a reduced number of laboratory tests are necessary for calibrating the proposed model. In particular, the relationship between e/e_s and ξ is the only additional information required for the unsaturated soil behaviour (apart from the parameter values for the saturated model). To define the relationship between e/e_s and ξ, it is possible to choose among alternative testing options that involve irreversible straining of the soil such as virgin loading at constant suction, undrained virgin loading and wetting–drying at constant applied stress.

REFERENCES

Alonso, E. E., Gens, A. & Josa, A. (1990). A constitutive model for partially saturated soils. *Géotechnique* **40**, No. 3, 405–430.

Alonso, E. E., Lloret, A., Gens, A. & Yang, D. Q. (1995). Experimental behaviour of highly expansive double-structure clay. *Proc. 1st Int. Conf. Unsaturated Soils, Paris* **1**, 11–16.

Bishop, A. W. (1959). The principle of effective stress. *Teknisk Ukeblad* **106**, No. 39, 859–863.

Bolzon, G., Schrefler, B. A. & Zienkiewicz, O. C. (1996). Elasto-plastic soil constitutive laws generalized to partially saturated states. *Géotechnique* **46**, No. 2, 279–289.

Cui, Y. J. & Delage, P. (1996). Yielding and plastic behaviour of an unsaturated compacted silt. *Géotechnique* **46**, No. 3, 405–430.

Fisher, R. A. (1926). On the capillary forces in an ideal soil: correction of formulae given by W. B. Haines. *J. Agric. Sc.* **16**, No. 3, 492–505.

Gallipoli, D., Wheeler, S. J. & Karstunen, M. (2003). Modelling the variation of degree of saturation in a deformable unsaturated soil. *Géotechnique* **53**, No. 1, 105–112.

Gens, A. (1995). *Constitutive laws*. Modern issues in non-saturated soils, pp. 129–158. Wien: Springer-Verlag.

Gens, A. & Alonso, E. E. (1992). A framework for the behaviour of unsaturated expansive clays. *Can. Geotech. J.* **29**, 1013–1032.

Haines, W. B. (1925). A note on the cohesion developed by capillary forces in an ideal soil. *J. Agric. Sci.* **15**, No. 4, 529–535.

Jennings, J. E. & Burland, J. B. (1962). Limitation to the use of effective stress in partly saturated soils. *Géotechnique* **12**, No. 2, 125–144.

Jommi, C. (2000). Remarks on the constitutive modelling of unsaturated soils. *Proceedings of the international workshop on unsaturated soils*, Trento, pp. 139–153.

Jommi, C. & Di Prisco, C. (1994). Un semplice approccio teorico per la modellazione del comportamento meccanico dei terreni granulari parzialmente saturi (in Italian). In *Atti Convegno sul Tema: Il Ruolo deiFluidi nei Problemi di Ingegneria Geotecnica, Mondoví*, pp. 167–188.

Karube, D. & Kato, S. (1994). An ideal unsaturated soil and the Bishop's soil. *Proc. 13th Int. Conf. Soil Mech. Found. Engng, New Delhi* **1**, 43–46.

Karube, D. & Kawai, K. (2001). The role of pore water in the mechanical behavior of unsaturated soils. *Geotech. Geol. Engng* **19**, 211–241.

Loret, B. & Khalili, N. (2000). A three phase model for unsaturated soils. *Int. J. Numer. Anal. Methods Geomech.* **24**, 893–927.

Sharma, R. S. (1998). *Mechanical behaviour of unsaturated highly expansive clays*. PhD thesis, University of Oxford, UK.

Sivakumar, V. (1993). *A critical state framework for unsaturated soil*. PhD thesis, University of Sheffield, UK.

Terzaghi, K. (1936). The shearing resistance of saturated soils and the angle between the planes of shear. *Proc. 1st Int. Conf. Soil Mech. Found. Engng, Cambridge, MA* **1**, 54–56.

Toll, D. G. (1990). A framework for unsaturated soil behaviour. *Géotechnique* **40**, No. 1, 31–44.

Vaunat, J., Romero, E. & Jommi, C. (2000). An elastoplastic hydro-mechanical model for unsaturated soils. In *Proceedings of the international workshop on unsaturated soils*, Trento, pp. 121–138.

Wheeler, S. J. & Karube, D. (1995). State of the art report: Constitutive modeling. *Proc. 1st Int. Conf. Unsaturated Soils, Paris*, 1323–1356.

Wheeler, S. J. & Sivakumar, V. (1995). An elasto-plastic critical state framework for unsaturated soil. *Géotechnique* **45**, No. 1, 35–53.

Wheeler, S. J. & Sivakumar, V. (2000). Influence of compaction procedure on the mechanical behaviour of an unsaturated compacted clay. Part 2: shearing and constitutive modelling. *Géotechnique* **50**, No. 4, 369–376.