Subscales on the element boundaries in the variational two-scale finite element method

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1. Introduction

The variational multiscale (VMS) framework to approximate boundary value problems starts with the variational formulation of the problem. In particular, in the two-scale version we consider, it consists in splitting the unknown and the test function into a component in a discrete approximating space and another component in its complement, for which an approximation needs to be proposed. This component is called subgrid scale or, simply, subscale. This idea was proposed in the finite element context in [16,17]. The standard Galerkin method accommodates this framework simply by considering the subscales to be negligible.

The main interest of the VMS framework is to develop stabilized finite element methods in a broad sense, meaning that it allows to design discrete variational formulations that do not suffer from the stability problems of the standard Galerkin method. In particular, we are interested here in finite element methods for some model problems arising in fluids mechanics (see [8] for a review of different stabilization methods in flow problems).

The VMS concept as described above is quite general. The way to approximate the subscales is left open. Many questions arise, such as the space for these subscales, the problem to be solved to compute them or their behavior in time dependent problems. In principle, the problem for the subscales is global, that is to say, defined over all the computational domain. In order to simplify it, some sort of localization is necessary, for example by assuming that the subscales vanish on the interelement boundaries, that is to say, they are bubble functions (see for example [4,24] for application of this concept to flow problems).

The treatment of the subscales on the interelement boundaries is precisely the subject of this paper. We propose a way to compute them based on the following ideas:

- We assume the subscales on the element interiors computed, and thus the localization process mentioned consists in computing these subscales without accounting for their boundary values.
- The subscales on the element boundaries are single valued, even if they are discontinuous in the element interiors. This requires a hybrid-type formalism to write the exact variational equations that we develop only in the first problem analyzed.
- The subscales on the element boundaries are computed by imposing that the correct transmission conditions of the problem at hand hold. Obviously, these transmission conditions are problem-dependent.
- The fluxes of the subscales on the interelement boundaries are approximated using a simple finite-difference scheme. This is the only approximation we use, apart from those shared with VMS methods that are required to approximate the subscales in the element interiors.

A completely different approach to compute subscales on the interelement boundaries is proposed in [11], where local problems along these boundaries are set.

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We will not insist on other aspects of the VMS method, such as the problem for the subcales in the element interiors, the space where they belong or their time dependency. Let us only mention that we approximate them using an approximate Fourier analysis, that very often we compute them as $L^2$-orthogonal to the finite element space [9] and that we consider them time dependent in transient problems [10,12]. In order to skip as much as possible this discussion, we will present our formulation without using the explicit expression for the subcales in the element interiors. This approach is, as far as we know, original, and we use it mainly to focus the attention in the expression of the subcales on the interelement boundaries.

The particular transmission conditions between interelement boundaries, that serve us to compute the subcales on these boundaries, are problem dependent. This is why we will treat different problems arising in fluid mechanics, all of them linear and stationary. The first is the convection–diffusion–reaction (CDR) equation considered in Section 2. We show that the subcale on the element boundaries is proportional to the jump of the flux with a negative sign, and also to the average of the subcales computed in the element interiors adjacent to an edge and extended to this edge. In the following, "edge" will be used also in 3D problems, in this case meaning a face. The sign of the subcales on the edges subtracts stability to the problem. However, we show that it is possible to control the new terms added. Neither for this problem nor for the other two discussed in the paper we analyze convergence, since it depends on the particular expression of the subcale on the element interiors. Nevertheless, we provide stability results for all the problems treated.

There is no apparent gain in considering the subcales on the element edges for the CDR equation the way we do. However, this situation is different for the Stokes problem written in velocity–pressure form analyzed in Section 3. We show that the subcales on the edges in this case introduce two terms, one that depends on the velocity gradients and that needs to be controlled with the viscous term and another one that provides pressure stabilization. This second term has the local variant proposed and analyzed in [43]. The result is that very often we compute them as almost negligible, compared to the jump of the stresses, that we approximate them using an approximate Fourier analysis, and also to the average of the subcales computed in the element interiors. To give a variational foundation to the problem for the subcales in the element interiors, the space of traces is single valued, even if they are discontinuous. To this end, we add is similar to the one already introduced in [21] and in [19], which has the local variant proposed and analyzed in [25,20] for the $Q_1/P_0$ (bilinear-constant) and $P_1/P_0$ (linear-constant) velocity–pressure pairs.

Section 4 describes the application of our ideas to Darcy’s problem. We propose a stabilized formulation that includes, with minor modifications, the methods proposed for example in [21] and extended in [22] and in [19]. As in the previous cases, we provide a stability result. In this case, the bilinear form associated to the problem is not coercive, but only an inf–sup condition can be proved.

Let us mention that the ideas presented here can be applied to other problems. In particular, in [11] a method to compute the subcales on the element boundaries for the stress–velocity–pressure formulation of the Stokes problem is proposed and fully analyzed. In this case, subcales on the boundaries are essential to deal with discontinuous pressure and stress interpolations.

The main contributions of our approach can be summarized as follows:

- To provide a consistent VMS justification to some stabilizing terms introduced in previous works to deal with discontinuous pressures.
- To propose a symmetric stabilized problem for the Stokes and the Darcy equations (if subcales in the element interiors can be considered negligible compared to the jump of the stresses, see Remark 5). The sign of the symmetric operator, which subtracts stability from the Galerkin terms, is crucial to achieve this symmetry. The situation is similar to what happens when minus the adjoint of the differential operator applied to the test functions is used instead of the original differential operator in the stabilizing terms. This suggestion was first introduced in [14] and turns out to be completely natural in the VMS framework. Also in this case, the diffusive term subtracts stability in the case of the CDR equation.
- Even though we do not exploit this point in this paper, our approach suggests how to stabilize Neumann boundary conditions, essential for example in some fluid–structure interaction problems (see Remark 6).

Some numerical examples are presented in Section 5. Since the stabilizing effect of the boundary terms introduced for the different problems is well known, we simply check what is particular of our approach, namely, the terms that may deteriorate stability. We show that this is not the case in two cases, namely, a convection–diffusion example and two Stokes problems. As the stability analysis dictates, these terms can be controlled by the rest of the terms appearing in the stabilized formulation. Moreover, in the Stokes problem case, some discontinuous pressure interpolations unstable using the Galerkin method, such as the $P_1/P_0$ pair (see Section 5) can be used.

Finally, some conclusions close the paper in Section 6.
\( T = T \subset T \) and \( F = F \subset F \). Note that the prime in \( V', T' \) and \( F' \) is not used to denote the dual of a space.

If we denote with a subscript \( i \) the restriction of \( u, u', \lambda, \beta, B \) and \( L \) to subdomain \( i (i = 1, 2) \), the problem for the six fields \( u, u', \gamma, \lambda, \beta \) and \( \dot{x} \) can be written as

\[
B_i(\bar{u}, \bar{v}) + B_i(\bar{u}', \bar{v}') = L_i(\bar{v}) \quad \forall \bar{v},
\]

\[
B_i(\bar{u}, \bar{v}') + B_i(\bar{u}', \bar{v}) - (\lambda_i \cdot (k \nabla \bar{u} + a \bar{u})) = L_i(\bar{v}) \quad \forall \bar{v},
\]

\[
B_i(\bar{u}, \bar{v}') + B_i(\bar{u}', \bar{v}) = L_i(\bar{v}) \quad \forall \bar{v},
\]

\[
(\mu_i + \gamma)' - u_i' - u_i = 0 \quad \forall \mu_i,
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In these equations, \( \lambda_i = \gamma_i + \beta_i \in F \) are the fluxes computed from the side of \( \Omega_i \) and \( \mu_i = \mu_i + \gamma_i \in F \) the corresponding test functions \( i = 1, 2 \). The boundary terms in Eqs. (4), (7) correspond to the weak imposition of fluxes on \( \Gamma \), Eqs. (8)-(11) to the weak continuity of \( u_i = u_i + u_i' \) on \( \Gamma (i = 1, 2) \). In other words, we prescribe the flux and the continuity of the flux as in the standard three field formulation.

The previous formulation can be considered a straightforward extension of the classical three field formulation for \( u, \gamma \) and \( \lambda \), obtained by applying a splitting of the spaces where these unknowns belong (see [22] for a three field formulation of the convection–diffusion equation). Our particular formulation is obtained by imposing the fluxes of \( u \) to be \( \lambda_i = n_i \cdot (k \nabla u + a u) \), where \( n_i \) is the normal to \( \Omega_i \) from \( \Omega \), and \( \gamma_i = \gamma_i \) \( (i = 1, 2) \). In other words, we prescribe the flux and the continuity of the flux as in the standard three field formulation.

This approach in particular implies that the test functions \( \mu_i \) must be of the form \( \mu_i = n_i \cdot (k \nabla u + a u) \), for \( \bar{v} \in V_n \), and \( \mu_i \) at \( \Gamma \). Therefore, the previous problem reads

\[
B_i(\bar{u}, \bar{v}) + B_i(\bar{u}', \bar{v}') = L_i(\bar{v}) \quad \forall \bar{v},
\]

\[
B_i(\bar{u}, \bar{v}') + B_i(\bar{u}', \bar{v}) - (\lambda_i \cdot (k \nabla \bar{u} + a \bar{u})) = L_i(\bar{v}) \quad \forall \bar{v},
\]

\[
B_i(\bar{u}, \bar{v}') + B_i(\bar{u}', \bar{v}) = L_i(\bar{v}) \quad \forall \bar{v},
\]

\[
(\mu_i + \gamma)' - u_i' - u_i = 0 \quad \forall \mu_i,
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\[
(\mu_i + \gamma)' - u_i' - u_i = 0 \quad \forall \mu_i.
\]

Adding up (14) and (16) and using (22) yields the original variational equation projected onto \( \bar{V} \), that is to say

\[
B(\bar{u}, \bar{v}) + B(\bar{u}', \bar{v}') = L(\bar{v}) \quad \forall \bar{v}.
\]

Adding up (15) and (17) and integrating the first terms by parts we get

\[
\sum_{i=1}^{2} (B_i(\bar{u}, \bar{v}') + B_i(\bar{u}', \bar{v}) - (\lambda_i \cdot (k \nabla \bar{u} + a \bar{u})) = \sum_{i=1}^{2} L_i(\bar{v}) \quad \forall \bar{v}.
\]

Note that the jumps of the convective fluxes are zero because of the continuity assumed for the finite element functions.

The approximation process consists of different ingredients, all aiming at giving a closed problem for \( \bar{u} \) alone. For that we will propose heuristic approximations for \( \gamma' \) and \( \lambda' \) and then we will perform a stability analysis to check that the resulting formulation...
is stable. Let us insist that, up to this point, problem (30)–(33) is exact. Furthermore, for \( u_0 = 0 \) it could be used as the variational framework to develop discontinuous Galerkin approximations (see Remark 2 below).

### 2.4. Subscales on the element boundaries

Let us consider for simplicity the 2D case and the situation depicted in Fig. 1, where two elements \( K_1 \) and \( K_2 \) share an edge \( E \) (recall that \( E \) stands for “edge” in 2D or face in 3D). Unless otherwise indicated (see Remark 1 below), all the edges are considered interior, that is to say, the element boundaries on \( \partial \Omega \) are excluded.

Let \( \bar{u}_i \) be the subscale approximated in the interior of element \( K_i \), \( i = 1, 2 \). We assume that this approximation is valid up to a distance \( \delta \) to the element boundary. This distance will be taken of the form \( \delta = \delta_b h \), with \( 0 \leq \delta_b \leq 1/2 \).

#### Approximation of \( \bar{u}_i \)

The values of \( \bar{u}_i \) on \( \partial K \) are weak approximations to the fluxes of \( u_i \). Given the trace \( \gamma' \) of this unknown, we delete (33) and propose the following closed form expression for \( \gamma' \):

\[
\gamma'_i \approx \frac{1}{\delta} (\bar{u}_i + \bar{u}_j),
\]

where now \( \bar{u}_i \) has to be understood as the subscale computed in the element interiors and evaluated at edge \( E \). We want to remark that, apart from the assumptions inherent to the VMS framework and the imposition of the transmission conditions (see below), this is the only approximation we really require to compute the subscales on the element boundaries. Obviously, other finite-difference-like approximations to the fluxes of the subscales could be adopted.

#### Approximation of \( \gamma' \)

Eq. (32) states the weak continuity of the total fluxes on the element boundaries. The idea now is to replace this equation by an explicit prescription of this continuity. If \( [n g_{ik}]_{E} = \bar{g}_{ik} + \lambda_{ik}\delta_{ik} \), \( \lambda_{ik} \delta_{ik} \) denotes the jump of a scalar function \( g \) across \( E \), and \( [\hat{n}_i]\delta_{ik} + n_i \cdot \hat{n}_i \) the jump of the normal derivative, the continuity of the total fluxes can be imposed as follows:

\[
0 = [k \bar{c}_i u_i]_E = [k \bar{c}_i u_i]_E + \lambda_{ik}\delta_{ik} + \lambda_{ik}\delta_{ik} \Rightarrow [k \bar{c}_i u_i]_E + k \delta_{ik} = [k \bar{c}_i u_i]_E + \lambda_{ik}\delta_{ik} + \lambda_{ik}\delta_{ik} \approx [k \bar{c}_i u_i]_E + k \frac{\delta}{\delta} \frac{\delta}{\delta} + \frac{\delta}{\delta} \frac{\delta}{\delta},
\]

(35)

From this expression, and for \( k \) constant, we obtain the approximation we were looking for

\[
\gamma'_i \approx \left[ u'_i \right]_E - \frac{\delta}{2} [k \hat{n}_i u_i]_E,
\]

(36)

where \( \left[ u'_i \right]_E := \frac{1}{2} \left( u'_i + u'_j \right) \) is the average of the subscales computed in the element interiors evaluated at edge \( E \). From (36) it is observed that \( \delta_b \) will play the role of an algorithmic parameter for which, following our approach, we have a geometrical interpretation.

From now onwards we will use the symbol \( = \) instead of \( \approx \), understanding that in some places we perform approximation (34) that has led us to (36).

#### Remark 1

(Neumann boundary conditions). Suppose that \( F_\kappa = \partial K \cap \partial \Omega \) and that instead of the Dirichlet condition (2) the Neumann condition \(- k \bar{c}_i u_i = q \) is prescribed. In this case, (35) should be replaced by

\[
q = -k \bar{c}_i u_i |_{F_\kappa} - k \left( \frac{\delta}{\delta} \right) \frac{\delta}{\delta},
\]

so that the contribution to \( L \) in (3) that would appear due to the Neumann condition would be modified by the approximation to the subscale on the boundary, and there would be also a contribution to the bilinear form \( B \). We will come back to this point in the case of the Stokes problem, where this fact has more important consequences.

#### Problem for \( u_h \) and \( u' \)

From (36) we obtain the following approximation for the fluxes of the subscales:

\[
\gamma'_i \approx \frac{1}{\delta} \left[ u'_i \right]_E - \frac{1}{\delta} [k \hat{n}_i u_i]_E + \frac{k}{\delta} [k \hat{n}_i u_i]_E, \quad i = 1, 2.
\]

(37)

Once \( \gamma' \) and \( \gamma' \) are approximated, the problem we are left with reads as follows: find \( u_h \in V_h \) and \( \gamma' \in V' \) such that

\[
B(u_h, v_h) + \int_{K} \left( d_\kappa u_h \right) d \kappa + \frac{k}{\delta} [k \hat{n}_i u_i]_E = L(v_h),
\]

(38)

\[
B(u'_h, v_h') + \int_{K} \left( d_\kappa u'_h \right) d \kappa + \frac{k}{\delta} [k \hat{n}_i u_i]_E + \int_{E} \left( k \hat{n}_i u_i + \{\gamma'_i\} E \right) dE = L(v'_h),
\]

(39)

for all \( v_h \in V_h \) and \( v'_h \in V'_h \).

#### Remark 2

Observe that this system of variational equations can be understood as a general framework to approximate unknowns with a continuous part (\( u_h \)) and an approximated discontinuous part (\( u'_h \)). Furthermore, if the continuous part is zero, we are left with (39) with \( u_h = 0 \), which corresponds to the classical Galerkin method enforcing continuity across interelement boundaries through Nitsche’s method, although with approximation (37) for the fluxes, so that the classical terms involving \( d_\kappa u_h \) and \( d_\kappa u'_h \) are missing (see [2,15]). For piecewise constant approximations these terms would not appear, and we would obtain a classical piecewise-constant discontinuous Galerkin approximation.

### 2.5. Subscales in the element interiors

Up to now we have replaced variational equations for the fluxes of the subscales and their traces by approximated closed form expressions. It can be seen from problem (38) and (39) that the resulting formulation is symmetric for symmetric problems. However, now we will use an additional approximation that will make the problem lose its symmetry, but that will greatly simplify the implementation of the formulation. This approximation is inherent to all VMS formulations to yield a closed form expression for the subscales in the element interiors.

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**Fig. 1.** Notation for the approximation of the subscales on the element boundaries.
If we integrate the second term in the left-hand-side (LHS) of (31) by parts we get

\[ B(u', v') = \sum_k \langle \mathcal{A} u', v' \rangle_k + \sum_k \langle k\partial_n u', v' \rangle_k. \]

If instead of using (37) we assume that \( x' \) approximates \( k\partial_n u' \), the second term in this last expression cancels with the third one in the LHS of (31). Therefore, the final problem is: find \( u_h \in V_h \) and \( u' \in \mathcal{V} \) such that

\[ B(u_h, v_h) + \sum_k \langle u', \mathcal{A} u_h \rangle_k + \sum_k \langle \partial_n u', k\partial_n v_h \rangle_k = L(v_h), \]

\[ \sum_k \langle \mathcal{A} u_h, u' \rangle_k + \sum_k \langle \mathcal{A} u', v' \rangle_k = L(v'), \]

for all \( v_h \in V_h \) and \( u' \in \mathcal{V} \). The last term in the LHS of (40) is the main novelty with respect to classical stabilized finite element methods designed in the variational multiscale framework.

Remark 3. Note that if in (39) \( u' \) is considered continuous we obtain (41) with no additional approximation. In other words, if the subscale is approximated with a Petrov–Galerkin method (leading to a non-symmetric formulation) in which the space of test functions is continuous, we recover (41). This is not however the approach we will adopt.

It only remains to approximate \( u' \) in the element interiors. To this end, in (41) the approximation

\[ u' = \tau P_v \mathcal{A} u', \quad \tau = \left( C_1 h^{-2} + C_2 |a|^{-1} \right)^{-1}, \]

may be adopted, where \( C_1 \) and \( C_2 \) are constants. This can be motivated by a Fourier analysis of the problem for the subscales [10]. In particular, it implies that the subscales in the element interiors are not affected by artificial oscillations when the stabilization terms make the formulation feasible from the implementation standpoint. Let us stress once again that this approximation is not original of this work, but common to all VMS methods that compute locally the subscales in the element interiors.

Once all the approximations are made, the final problem is to find \( u_h \in V_h \) and \( u' \in \mathcal{V} \) such that

\[ B(u_h, v_h) + \sum_k \langle u', \mathcal{A} u_h \rangle_k + \sum_k \langle \partial_n u', k\partial_n v_h \rangle_k = L(v_h), \]

\[ \sum_k \langle \mathcal{A} u_h, u' \rangle_k + \sum_k \tau^{-1} \langle u', v' \rangle_k = L(v'), \]

for all \( v_h \in V_h \) and \( u' \in \mathcal{V} \).

The variational Eq. (44) automatically yields and expression for the subscales in the element interiors in terms of the finite element component, provided \( \mathcal{V} \) is approximated by a space of discontinuous functions. For constant \( \tau \) it implies that

\[ u' = \tau P_v \mathcal{A} u', \]

where \( P_v \) is the projection onto \( \mathcal{V} \). However, it will be convenient for the following analysis to keep \( u' \) as unknown of the problem. Particular cases of projection that fit into the present framework are the orthogonal subscales stabilization (OSS) proposed in [9] and the algebraic version of the subgrid–scale stabilization (AGSS) (see [8,16]), where \( P_v \) is the identity (at least when applied to \( \mathcal{A} u_h \)). The expression of \( \tau \) (42) is in fact not important, except for a condition on constant \( C_1 \) indicated later.

2.6. Stability analysis

Let us consider the bilinear form of the problem in \((V_h \times \mathcal{V}) \times (V_h \times \mathcal{V})\):

\[ B_{exp}(u_h, u'; v_h, v') := B(u_h, v_h) + \sum_k \langle u', \mathcal{A} u_h \rangle_k + \sum_k \langle \partial_n u', k\partial_n v_h \rangle_k \]

\[ - \frac{\delta}{2} \sum_k \langle \partial_{\alpha} u_h, k\partial_{\alpha} v_h \rangle_k + \sum_k \langle \mathcal{A} u_h, u' \rangle_k + \sum_k \partial^{-1} \langle u', v' \rangle_k. \]

Let us prove stability of the problem by showing that \( B_{exp} \) is coercive in a certain norm. We have that

\[ B_{exp}(u_h, u'; v_h, v') = B(u_h, u_h) + \sum_k \langle u', \mathcal{A} u_h \rangle_k + \sum_k \langle \partial_n u', k\partial_n u_h \rangle_k \]

\[ - \frac{\delta}{2} \sum_k \langle \partial_{\alpha} u_h, k\partial_{\alpha} u_h \rangle_k + \sum_k \langle \mathcal{A} u_h, u' \rangle_k + \sum_k \partial^{-1} \langle u', v' \rangle_k. \]

We assume now that the classical inverse estimates

\[ \|u_h\|^2 \leq C_{inv} \|\mathcal{A} u_h\|^2, \quad \|v_h\|^2 \leq C_{inv} \|\mathcal{A} v_h\|^2 \quad \forall v_h \in V_h, \]

hold true (see [13,5]). In particular, the second, which also holds for derivatives of finite element functions, implies the trace inequality

\[ \|\partial_{\alpha} u_h\|^2 \leq C_{inv} h^{-1} \|\mathcal{A} u_h\|^2, \]

which applied to \( u_h \) yields

\[ \|\partial_{\alpha} u_h\|^2 \leq C_{inv} h^{-1} \|\mathcal{A} u_h\|^2. \]

Using these inverse estimates, we have (see Fig. 1 for the notation)

\[ \frac{\delta}{2} \sum_k \langle \partial_{\alpha} u_h, k\partial_{\alpha} u_h \rangle_k \geq \frac{\delta}{2} \sum_k \|\partial_{\alpha} u_h\|^2 \geq \frac{\delta}{2} \sum_k \|\mathcal{A} u_h\|^2 \]

\[ \geq \frac{\delta h}{2} \sum_k 2k C_{inv} \|\mathcal{A} u_h\|^2 \geq -\delta h C_{inv} \|\mathcal{A} u_h\|^2. \]

Let us obtain a working inequality. Let \( a \) and \( b \) be discontinuous positive functions defined on the finite element partition. Using the notation \( a_k := a|_{\mathcal{K} = k} \), for any \( \beta > 0 \) we have that

\[ \sum_k (a_k + b_k)(b_1 + b_2) \leq \sum_k \frac{h}{2\beta} (a_k + b_k)^2 + \sum_k \frac{\beta}{2h} (b_1 + b_2)^2 \]

\[ \leq \sum_k \frac{h}{\beta} (a_k^2 + b_k^2) + \sum_k \frac{\beta}{h} (b_1^2 + b_2^2) \]

\[ \leq \sum_k \frac{h}{\beta} a_k^2 + \sum_k \frac{\beta}{h} b_k^2. \]

Now we make the assumption that the subscales are such that the inverse estimates also hold for them. Using the previous inequality we obtain, for any \( \beta \) > 0:

\[ -\sum_k \|u_k\|^2 k \partial_{\alpha} u_k \| \geq - \sum_k \frac{1}{2} \|\partial_{\alpha} u_k\|^2 + \sum_k \frac{1}{2} \beta \|\mathcal{A} u_k\|^2 \]

\[ \geq \sum_k (\beta h)^2 \|u_k\|^2 - \sum_k \frac{1}{2} \beta C_{inv} k \|\mathcal{A} u_k\|^2. \]
Using the bounds obtained, it follows that
\[
B_{\text{exp}}(u_h, u'; u, u') \\
\geq k\|\nabla u_h\|^2 + s\|u_h\|^2 - \sum_k \|u'_k\|^2 2k_1^{1/2} h \|\nabla u_h\|_k \\
- \sum_k \|u'_k\|^2 2s_h \|u_h\|^2 - \delta_0 C_k h \|\nabla u_h\|^2 - \sum_k \|u'_k\|_k \|\nabla u_h\|_k \\
+ \sum_k \tau^{-1} \|u'_k\|^2 \geq k\|\nabla u_h\|^2 + s\|u_h\|^2 \\
- \sum_k \left( \beta_1 k_1^{1/2} h \|u'_k\|^2 + \frac{1}{\beta_1} k\|\nabla u_h\|^2 \right) \\
- \sum_k \left( \beta_2 k_1^{1/2} h \|u'_k\|^2 + \frac{1}{\beta_2} s_h \|u_h\|^2 \right) \\
- \sum_k \frac{\beta_3}{2} k_1^{1/2} h \|u'_k\|^2 - \sum_k \frac{1}{2\beta_3} C_k h \|\nabla u_h\|^2 \\
- \delta_0 C_k h \|\nabla u_h\|^2 + \sum_k \tau^{-1} \|u'_k\|^2 \\
= \sum_k \left( 1 - \frac{1}{\beta_1} \right) \left( \alpha C_k - C_k \right) k\|\nabla u_h\|^2 \\
+ \sum_k \left( 1 - \frac{1}{\beta_2} \right) s_h \|u_h\|^2 \\
+ \sum_k \left( 1 - \beta_3 k_1^{1/2} h \right) \left( \frac{1}{2\beta_3} C_k h \|\nabla u_h\|^2 \right),
\]
where \( \beta_i \) are constants, \( i = 1, 2, 3 \). Taking these constants sufficiently large, \( \delta_0 \) sufficiently small and \( C_k \) in the definition of \( \delta_0 \) large enough, the following result follows:

**Theorem 1.** There are constants \( C_k \) and \( \delta_0 \) in the definition of the stabilization parameters such that

\[
B_{\text{exp}}(u_h, u'; u, u') \geq C \left( k\|\nabla u_h\|^2 + s\|u_h\|^2 + \sum_k \tau^{-1} \|u'_k\|^2 \right).
\]

**Remark 4.** Let us enumerate the essential ideas and highlight the original aspects of the analysis presented in this section:

- The driving idea is that the subdomains on the boundary are determined by the transmission condition. In the case of the CDR equation and using continuous interpolations, this is the continuity of the diffusive fluxes.
- The essential approximation to make the problem computationally viable is to compute the subscales in the element interiors without taking into account their values on the boundaries.
- In the stability analysis presented, the subdomains have their own “personality”. They appear explicitly in the stability estimate. The final stability estimate for the finite element unknowns depends on the way the subscales are approximated (that is to say, how \( V' \) is chosen).
- It is observed that the expression of \( \tau \) in terms of \( a \) is not used in the stability analysis. However, it is required in the convergence analysis.
- The only thing we have shown is that the terms introduced by the boundary contribution from the subdomains can be controlled, but there seems to be no gain in considering the subdomains on the boundaries. The stability estimate (47) is the same that would be obtained without the last term in the LHS of (40) which is, as it has been said, the main novelty of our proposal.

As stated in the last item, subdomains on the boundary do not improve stability for the CDR equation. This is not so for the Stokes problem analyzed next.

### 3. Stokes problem

#### 3.1. Problem statement and finite element approximation

In this section, we turn our attention to the Stokes problem, which consists of finding a velocity \( u : \Omega \to \mathbb{R}^d \) and a pressure \( p : \Omega \to \mathbb{R} \) such that

\[
- \nabla u + \nabla p = f \quad \text{in} \quad \Omega \subset \mathbb{R}^d, \\
\nabla \cdot u = 0 \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \partial \Omega.
\]

The purpose is to extend the ideas of the previous section to this problem. Let now \( V = H_0^1(\Omega)^d, Q = L^2(\Omega)/\mathbb{R} \). The variational problem consists of finding \((u, p) \in V \times Q \) such that

\[
B((u, p), (v, q)) := \int (\nabla u \cdot \nabla v - (p, \nabla \cdot v) + (q, \nabla \cdot u)) = \int (f, v) \quad \forall (v, q) \in V \times Q.
\]

For the sake of simplicity, we will consider subdomains only for the velocity, not for the pressure. Pressure subdomains can be easily introduced (see [10]), but they do not contribute to the present discussion. It is also possible to derive a general framework as in the previous section, using the trace of the velocity subdomains and their fluxes as additional variables, leading to a five field formulation, the five fields being velocity, velocity subdomain, trace of velocity subdomain, flux of velocity subdomain and pressure. However, we may directly work with velocity, velocity subdomain and pressure, understanding that the velocity subdomain on the interelement boundaries (and the test function) will be approximated independently, being single valued on these boundaries. If \( V_s \times Q_s + V \times Q \) is a conforming finite element approximation and \( V' \) is the space for the velocity subdomains, the discrete variational problem to be considered is to find \((u_h, p_h) \in V_s \times Q_s \) and \( u' \in V' \) such that

\[
B((u_h, p_h), (v_h, q_h)) + \sum_k \left( (u'_h - \nabla v_h - \nabla q_h)_k + (u'_h, v'_h + q_h n^h)_k \right)
\]

\[
= \sum_k \left( \nabla (\nabla u_h + \partial_0 u'_h) - p_h n, v'_h \right)_k + \sum_k \left( \nabla u_h - \nabla q_h, v'_h \right)_k
\]

\[
+ \sum_k \left( \nabla u'_h, v'_h \right)_k = (f, v'_h),
\]

which must hold for all \((v_h, q_h) \in V_s \times Q_s \) and all \( v'_h \in V' \). The first term in the second discrete variational equation must be zero because of the (weak) continuity of the stress normal to the element boundaries (recall that \( v' \) has to be considered single valued when evaluated at the interelement boundaries).

As for the CDR equation, the approximation

\[
\langle \nabla u'_h, v'_h \rangle_k = \tau^{-1} \langle u'_h, v'_h \rangle_k, \quad \tau^{-1} = C_1 \frac{\hbar}{h^2}
\]

is adopted. Likewise, the subdomain on the boundary will be approximated by an expression \( u'_h \) to be determined, so that the problem to be solved is to find \((u_h, p_h) \in V_s \times Q_s \) and \( u' \in V' \) such that

\[
B((u_h, p_h), (v_h, q_h)) + \sum_k \left( (u'_h - \nabla v_h - \nabla q_h)_k + (u'_h, v'_h + q_h n^h)_k \right)
\]

\[
= \sum_k \left( \nabla u_h + \nabla p_h, v'_h \right)_k + \sum_k \tau^{-1} \langle u'_h, v'_h \rangle_k = (f, v'_h),
\]

which must hold for all \((v_h, q_h) \in V_s \times Q_s \) and all \( v'_h \in V' \). The expression of \( \tau \) is given in (48), but \( u'_h \) is required to close the problem.
3.2. Subscales on the element boundaries

The condition to determine the expression of the subscale velocity on the boundary is that the normal component of the stress be continuous across interelement boundaries. Using the same notation as in the previous section, this can be written as follows:

\[
0 = [-p n + \mathcal{V}_b \mathbf{u}]_E = [-p n + \mathcal{V}_b \mathbf{u}]_E + \mathcal{V}_b (\mathbf{u}')_E
\]

from where the approximation we propose is

\[
(\mathbf{u}')_E = \{\mathbf{u}'\}_E - \frac{\delta}{2\nu} [\mathcal{V}_b \mathbf{u} - p n]_E,
\]

which is the counterpart of (36) for the Stokes problem. Inserting (51) into the discrete variational problem (49) and (50) results in

\[
B(\mathbf{u}_E, \mathbf{p}_E; \{\mathbf{v}_h, q_h\}) + \sum_k (\mathbf{u}'_E - \nu \Delta \mathbf{u}_h - \nabla q_h)_E
\]

\[
+ \sum_k \frac{\delta}{2\nu} (\{\mathcal{V}_b \mathbf{u}_h - p n\}_E, \{\mathcal{V}_b \mathbf{v}_h + q n\}_E) = \{f, \mathbf{p}\}_E
\]

which must hold for all \(\{\mathbf{v}_h, q_h\} \subseteq \mathcal{V}_h \times Q_h\) and all \(\mathbf{u}' \in \mathcal{V}'.\) This is the numerical approximation of the Stokes problem we propose and whose stability is analyzed next.

Remark 6. (Neumann boundary conditions). Suppose again that \(F_E = \partial K \cap \partial \Omega\) and that the Neumann condition \(-p n + \mathcal{V}_b \mathbf{u} = \mathbf{t}\) is prescribed. The subscale \(\mathbf{u}'_E\) should be computed from

\[
\mathbf{t} = -p n + \mathcal{V}_b \mathbf{u} = \mathbf{t}
\]

In this case, the terms

\[
\sum_k (\{\mathbf{u}'_E, \mathcal{V}_b \mathbf{v}_h + q n\}_E - \frac{\delta}{2\nu} (\mathcal{V}_b \mathbf{u}_h - p n, \mathcal{V}_b \mathbf{v}_h + q n) - f)_E
\]

and

\[
\frac{\delta}{2\nu} \sum_k (\mathbf{t}, \mathcal{V}_b \mathbf{v}_h + q n) \]

should be added to the LHS and right-hand-side (RHS) of (52), respectively. Stability on these boundaries will be enhanced by the term \(\sum_k \frac{\beta}{2} (p n, q n)_E\). This approach might be important as well in fluid–structure interaction problems, where one of the problems (the structure for example) is computed using the normal stresses \(\mathbf{t}\) computed in the other domain. It is known that that in some situa-

tions staggered coupled algorithms may suffer from the so called artificial mass effect due to the lack of stability in the imposition of the Neumann condition.

3.3. Stability analysis

As for the CDR equation, it is convenient to define the expanded bilinear form of problem (52) and (53), including the subscales as unknowns, which is

\[
B_{\text{exp}}(\mathbf{u}_h, \mathbf{p}_h, \mathbf{u}'_h; \{\mathbf{v}_h, q_h\}, \{\mathbf{v}', q'\})
\]

\[
= \mathcal{B}(\mathbf{u}_h, \mathbf{p}_h, \{\mathbf{v}_h, q_h\}) + \sum_k \langle \mathbf{u}', -\nu \Delta \mathbf{u}_h - \nabla q_h \rangle_k
\]

\[
+ \sum_k \sum_s \frac{\delta}{2\nu} (\mathcal{V}_b \mathbf{u}_h - p n, \mathcal{V}_b \mathbf{v}_h + q n)_E + \sum_k \tau^{-1} (\mathbf{u}'_h, \mathbf{v}')_E
\]

Taking \(\{\mathbf{v}', q'\} = \{\mathbf{u}_h, \mathbf{p}_h\}\) and \(\mathbf{v}' = \mathbf{u}'\) it follows that

\[
B_{\text{exp}}(\mathbf{u}_h, \mathbf{p}_h, \mathbf{u}'_h; \{\mathbf{v}_h, q_h\}, \{\mathbf{v}', q'\})
\]

\[
= \sum_k \frac{\delta}{2\nu} (\mathcal{V}_b \mathbf{u}_h - p n, \mathcal{V}_b \mathbf{v}_h + q n)_E + \sum_k \tau^{-1} (\mathbf{u}'_h, \mathbf{v}')_E
\]

We may deal with the terms

\[
\sum_k \langle \mathbf{u}', -\nu \Delta \mathbf{u}_h \rangle_k
\]

\[
= -\sum_k \frac{\delta}{2\nu} (\mathcal{V}_b \mathbf{u}_h - p n, \mathcal{V}_b \mathbf{v}_h + q n)_E
\]

\[
- \sum_k \tau^{-1} (\mathbf{u}'_h, \mathbf{v}')_E
\]

exactly as for the CDR equation. It only remains the following bound:

\[
\sum_k \frac{\delta}{2\nu} (\mathcal{V}_b \mathbf{u}_h - p n, \mathcal{V}_b \mathbf{v}_h + q n)_E
\]

\[
- \sum_k \frac{\nu}{2\tau} \mathcal{V}_b (\mathbf{u}'_h, \mathbf{v}')_E
\]

\[
= -\frac{\delta}{2\nu} (\mathcal{V}_b \mathbf{u}_h - p n, \mathcal{V}_b \mathbf{v}_h + q n)_E
\]

\[
- \sum_k \frac{\nu}{2\tau} \mathcal{V}_b (\mathbf{u}'_h, \mathbf{v}')_E
\]

which holds for all \(\beta > 0\). Taking it sufficiently large \((\beta > 1)\) and proceeding exactly as for the CDR equation we obtain

Theorem 2. There are constants \(C_1\) and \(\delta_0\) in the definition of the stabilization parameters such that

\[
B_{\text{exp}}(\mathbf{u}_h, \mathbf{p}_h, \mathbf{u}'_h; \{\mathbf{v}_h, q_h\}, \{\mathbf{v}', q'\})
\]

\[
> C \left( \nu \|\nabla \mathbf{u}_h\|^2 + \sum_k \frac{\delta}{2\nu} (\|\mathbf{p}_n\|)^2 + \sum_k \tau^{-1} (\mathbf{u}'_h, \mathbf{v}')_E \right)
\]

Remark 7. In the previous estimate, it is important to note that

- Contrary to the CDR equation, now there is a clear gain by accounting for the subscales on the boundary: we have control on the pressure jumps over interelement boundaries. This in particular stabilizes elements with discontinuous pressures.
- Control over \(\|\mathbf{p}_n\|\) can be transformed into \(L^2\) control over \(p_n\). This can be proved for example using the strategy presented in [11] and in references therein.

The stability estimate obtained is clearly optimal.
4. Darcy flow

4.1. Problem statement and finite element approximation

We will consider here the simplest situation of Darcy’s problem in which the permeability is isotropic and uniform. The problem to be solved consists in finding a velocity \( u \) and a pressure \( p \) such that

\[
\begin{align*}
\kappa^{-1}u + \nabla p &= 0 \quad \text{in } \Omega, \\
\nabla \cdot u &= f \quad \text{in } \Omega, \\
u \cdot n &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \kappa \) is the permeability coefficient. The functional spaces where the problem can be posed are

\[
V = H_0(\text{div}, \Omega), Q = L^2(\Omega)/\mathbb{R},
\]

for the velocity and the pressure, respectively. In this case, \( f \in L^2(\Omega) \).

The classical variational formulation of the Darcy problem is well posed in these spaces. However, it is observed from the momentum equation that in fact the pressure will belong to \( H^1(\Omega)/\mathbb{R} \).

The weak form of the problem is

\[
\begin{align*}
(\kappa^{-1}u, v) - (p, \nabla \cdot v) = 0, \\
(q, \nabla \cdot u) = (q, f),
\end{align*}
\]

which must hold for all \( (v, q) \in V \times Q \).

As in the previous section, the finite element spaces for velocity and pressure will be respectively denoted by \( V_h \subset V \), \( Q_h \subset Q \) (conforming approximations will be considered). If we consider as before the scale splitting

\[
\begin{align*}
u &= u_h + u', \quad u_h \in V_h, \\
u' &= u', \quad \text{and } u' \in V', \\
p &= p_h + p', \quad p_h \in Q_h, \quad p' \in Q',
\end{align*}
\]

with spaces \( V' \) and \( Q' \) for the moment undefined, the problem to be solved becomes

\[
\begin{align*}
(\kappa^{-1}u_h, v_h) + (\kappa^{-1}u', v_h) - (p_h, \nabla \cdot v_h) - (p', \nabla \cdot v_h) = 0 \quad \forall v_h \in V_h, \\
(q_h, \nabla \cdot u_h) - \sum_k (u'_h, \nabla q'_h)_h + \sum_k (q_h, n \cdot u'_h)_h = (q_h, f) \quad \forall q_h \in Q_h.
\end{align*}
\]

(55)

It is convenient to write the previous approximation in ‘weak’ form as follows:

\[
\begin{align*}
(k^{-1}u', v') + (\kappa^{-1}u_h, v_h) + \sum_k (\nabla p_h, v') = 0 \quad \forall v' \in V', \\
(q_h, \nabla \cdot u_h) + \sum_k \tau_p (q'_h, q'_h) = (q'_h, f) \quad \forall q'_h \in Q'.
\end{align*}
\]

4.2. Subscales on the element boundaries

The transmission conditions for this problem are different from those of the Stokes problem of the previous section. First of all, observe that

- Only the velocity subscale is needed on the boundary of the elements (see (54) and (55)).
- For each element, this velocity subscale can be computed from the pressure subscale on the boundary by projecting the momentum equation.
- Since in fact \( p \in H^1(\Omega) \), \( p \) must be such that

\[
[np]_{Eh} = 0, \quad [\partial_n p]_{Eh} = 0.
\]

Eq. (57) are the transmission conditions that have to allow us to compute the subscales on the element boundaries. Since the pressure is allowed to be discontinuous across these interelement boundaries, the pressure subscale must also be allowed to be discontinuous. Let us denote by \( p_{Eh} \) the pressure finite element function on an edge \( E \) from the side of \( K \) (see again Fig. 1) and \( p_{Eh} \) the corresponding subscale. Pressure continuity across \( E \) implies

\[
[np]_{Eh} = (p_{Eh} + p_{Eh})_1 + (p_{Eh} - p_{Eh})_2 = 0,
\]

from where

\[
p_{Eh} + p_{Eh} = -p_{Eh} + p_{Eh} = -n_{Eh} \cdot [np]_{Eh}.
\]

Using an approximation for the derivatives of the subscales similar to that of the previous sections, continuity of the pressure normal derivative implies:

\[
[p_{Eh}]_h = \frac{1}{2} [n_{Eh} \cdot [np]_{Eh}]_h = -\frac{1}{2} \tau_p (n_{Eh} \cdot [np]_{Eh})_h.
\]

The solution of system (58) and (59) yields

\[
P_{Eh} = \{p_{Eh}\}_h - \frac{1}{2} \tau_p (n_{Eh} \cdot [np]_{Eh})_h - \frac{1}{2} [np]_{Eh} K \cdot n.
\]

(60)

Eq. (60) is the expression of the pressure subscale on the element edges (now discontinuous), obtained from the application of our ideas to the Darcy problem. However, as mentioned earlier, this expression is only required to compute the velocity subscales on the edges, again considering them discontinuous. Projecting the momentum equation on the element boundaries we have:

\[
n \cdot u'E_h K = -n \cdot u_h K - \kappa \partial_n p_h K - \kappa \partial_n p' K
\]

\[
= -n \cdot u_h K - \kappa \partial_n p_h K - \frac{K}{\delta} (p_k K - p_k)
\]

\[
= -n \cdot u_h K - \kappa \partial_n p_h K
\]

\[
- \frac{K}{\delta} \left( p'_k K - \frac{\delta}{2} \partial_n p_h K - \frac{1}{2} [np]_{Eh} K \cdot n - p_k \right)
\]

\[
= -n \cdot u_h K - \kappa \partial_n p_h K + \frac{K}{\delta} [np]_{Eh} K + \frac{K}{\delta} [np]_{Eh} K \cdot n.
\]
from where we obtain the expression for the velocity subscale on $\partial K$:

$$
\mathbf{n} \cdot \mathbf{u}|_{\partial K} = \mathbf{n} \cdot \mathbf{u}|_{\partial K} - \kappa \{ \partial_a p_h \}|_{\partial K} + \frac{\kappa}{2\delta} \{ \mathbf{n} p_h + p_k \}|_{\partial K} \cdot \mathbf{n}.
$$

(61)

Since no velocity derivatives appear in the transmission conditions for this problem, the velocity subscale on $\partial K$ turns out to be independent from the velocity subscale on $K$.

Note now that all the terms on the RHS of (61) are vectors whose normal component is continuous across interelement boundaries (the first because we assume $V_h \subset \mathcal{V}$). If $\mathbf{w}$ is a vector defined on $E$, with continuous normal component, it holds that

$$
\sum_k (q_h, \mathbf{n} \cdot \mathbf{w})|_{\partial K} = \sum_k (\{n q_h\}, \mathbf{w})|_{\partial K}.
$$

Using this in the finite element approximation for the continuity equation we obtain the final problem to be solved, which consists of finding $\mathbf{u}, p_h \in Q_h, \mathbf{u}' \in \mathcal{V}$ and $q' \in Q'$ such that

$$
(k^{-1} \mathbf{u}, \mathbf{v}) + (k^{-1} \mathbf{u}, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) - (p', \nabla \cdot \mathbf{v}) = 0 \quad \forall \mathbf{v} \in \mathcal{V},
$$

(62)

$$(q, \nabla \cdot \mathbf{u}) - \sum_k (\mathbf{u}', \nabla q_h)|_K - \sum_k (\{n q_h\}, \mathbf{u} + \{k \partial_a p_h\} - \frac{\kappa}{2\delta} \{np\})|_K = \sum_k (\{n q_h\}, \{np\})|_K = (q, f), \quad \forall q \in Q,
$$

(63)

$$(\mathbf{k}^{-1} \mathbf{u}', \mathbf{v}') + (\mathbf{k}^{-1} \mathbf{u}_h, \mathbf{v}') + \sum_k (\nabla p_h, \mathbf{v}') = 0 \quad \forall \mathbf{v}' \in \mathcal{V}',
$$

(64)

$$(q', \nabla \cdot \mathbf{u}_h) + \sum_k \tau_p^{-1} (q', q') = (q', f) \quad \forall q' \in Q',
$$

(65)

with $\tau_p$ given by (56).

4.3. Stability analysis

The previous problem can be written as

$$
\mathcal{B}_{\text{exp}}(\mathbf{u}_h, p_h, \mathbf{u}', p_h, q_h, \mathbf{v}', q') = (q, f) + (q', f),
$$

with the obvious definition for the bilinear form $\mathcal{B}_{\text{exp}}$. The stability analysis in this case is a bit more delicate than for the CDR equation and for the Stokes problem. The problem is that $\mathcal{B}_{\text{exp}}$ is not coercive, but satisfies an inf–sup condition in a norm to be introduced in the following.

We assume that the decomposition $V_h \equiv \mathcal{V}$ is $L^2$-stable, in the sense that for any functions $v_h \in V_h$ and $\mathbf{v}' \in \mathcal{V}$ we have

$$
||v_h + \mathbf{v}'||^2 \geq C_{\text{dec}} (||v_h||^2 + ||\mathbf{v}'||^2),
$$

(66)

for a constant $C_{\text{dec}}$ independent of the equation parameters and of the mesh size. In general, $C_{\text{dec}} \leq 1$ and if $\mathcal{V}$ is taken $L^2$-orthogonal to $V_h$, $C_{\text{dec}} = 1$.

Let $\mathbf{U}_h = [\mathbf{u}_h, p_h, \mathbf{u}', p_h]$ be the unknown of the problem and $\mathbf{v}_h = [v_h, q_h, \mathbf{v}', q']$ the corresponding vector of test functions. Let also

$$
\begin{align*}
||\mathbf{U}_h||^2 &= \mathbf{k}^{-1} ||\mathbf{u}_h||^2 + \sum_k ||n p_h||_E^2 + \sum_K ||n q_h||_E^2 + \sum_k ||\mathbf{u}'||^2_E + \\
&\quad + \sum_k \tau_p^{-1} ||p_k||^2 + \sum_k \mathbf{k}^{-1} ||\nabla p_k||^2 + \sum_k ||n q_h||^2.
\end{align*}
$$

(67)

Fig. 2. Elevations for the diffusion dominated problem without (left) and considering (right) $\mathbf{u}_E$. Cut along $y = 0$ (bottom).
where $P_h$ is the $L^2$-projection onto $V_h$. However, later on we will introduce another norm in which stability holds and that clearly displays the stability enhancement we obtain with respect to the classical Galerkin method.

Let us start writing

$$u_h + \kappa \nabla p_h = P_h(u_h + \kappa \nabla p_h) + P_h(u_h + \kappa \nabla p_h) := m_h - u'. $$

which allows us to write

$$B_{\text{exp}}(U_h, U_h) = \kappa^{-1} \|u_h + u'\|^2 + \sum_k \frac{\kappa}{2 \beta_k} \left( \|np_h\|_k^2 + \sum_k \tau^h_k \|p'\|^2_k \right)$$

$$- \sum_k \langle [np_h], [m_h] \rangle_E + \sum_k \langle [np_h], [u'] \rangle_E$$

$$+ \sum_k \frac{\kappa}{2 \beta_k} \left( \|np_h\|_k^2 + \sum_k \tau^h_k \|p'\|^2_k \right).$$

(67)

The objective now is to bound the last three terms in the RHS of this equality. Let us start with the last one. Using (46) we have that

$$\sum_k \frac{\kappa}{2 \beta_k} \left( \|np_h\|_k^2 + \sum_k \tau^h_k \|p'\|^2_k \right) \geq \sum_k \frac{\kappa}{2 \beta_k} \left( \|np_h\|_k^2 + \sum_k \tau^h_k \|p'\|^2_k \right)$$

$$- \frac{\kappa}{2 \beta_k} \sum_k \|np_h\|_k^2 - \frac{1}{2 \beta_k} \|p'\|^2_k$$

$$\geq \frac{\kappa}{2 \beta_k} \sum_k \|np_h\|_k^2 - \frac{1}{2 \beta_k} \|p'\|^2_k$$

$$\geq \frac{\kappa}{2 \beta_k} \sum_k \|np_h\|_k^2 - \frac{1}{2 \beta_k} \|p'\|^2_k.$$  

(68)

We also have that

$$\sum_k \|np_h\|_k^2 \geq \sum_k \left( \frac{\delta}{2 \beta_k} \|p'\|^2_k + \frac{\beta_k}{2 \beta_k} \|np_h\|^2_k \right)$$

$$\geq \sum_k \frac{\delta}{2 \beta_k} \|p'\|^2_k - \sum_k \frac{\beta_k}{2 \beta_k} \|np_h\|^2_k$$

$$\geq \frac{\delta}{2 \beta_k} \|p'\|^2_k - \sum_k \frac{\beta_k}{2 \beta_k} \|np_h\|^2_k.$$  

(69)

Using (66), (68) and (69) in (67) we have

$$B_{\text{exp}}(U_h, U_h) \geq \kappa^{-1} C_{\text{dec}} \|u_h\|^2 + \kappa^{-1} \left( C_{\text{dec}} - \frac{\delta}{2 \beta_k} C_{\text{tr}} \right) \sum_k \|u'\|^2_k$$

$$+ \sum_k \frac{\kappa}{2 \beta_k} \left( 1 - \beta_1 \beta_2 \right) \|np_h\|_k^2$$

$$+ \sum_k \frac{\kappa}{2 \beta_k} \left( 1 - \frac{\beta_1}{\beta_2} \right) \|p'\|^2_k - \sum_k \langle [np_h], [m_h] \rangle_E.$$  

(70)

It remains to control the last term. It is responsible for the fact that the bilinear form $B_{\text{exp}}$ is not coercive, but it only satisfies an inf–sup condition. By the definition of $m_h$ and using (45) we have that

$$B_{\text{exp}}(U_h, [m_h, 0, 0]) = \kappa^{-1} \|u_h + \nabla p_h - m_h\|_k - \sum_k \langle p_h, n - m_h \rangle_k$$

$$\geq \kappa^{-1} \|m_h\|^2 - \sum_k \langle [np_h], [m_h] \rangle_k$$

$$\geq \kappa^{-1} \|m_h\|^2 - \sum_k \langle [np_h], [m_h] \rangle_k.$$  

Fig. 3. Elevations for the convection dominated problem without (left) and considering (right) $u'_E$. Cut along $y = 0$ (bottom).
which combined with (70) yields

\[ B_{\text{exp}}(\mathbf{U}_h, \mathbf{U}_h + [\mathbf{m}_h, 0, 0, 0]) \geq \kappa^{-1} C_{\text{dec}} \| \mathbf{u}_h \|^2 + \kappa^{-1} \left( C_{\text{dec}} - \frac{\delta_0}{2\beta_2} C_{\text{tr}} \right) \| \mathbf{u}_h \|^2 \]

\[ + \sum_k \frac{K}{2\delta} \left( 1 - \frac{\beta_1}{2} - \beta_2 \right) \| [\mathbf{n}_{p_h}] \|_E^2 \]

\[ + \sum_k \frac{K}{\beta_4} \left( \frac{1}{C_p} \frac{C_{\text{inv}}}{4\delta_0} \left( \frac{C_{\text{tr}}}{2\beta_3} \right) \| \mathbf{p} \|_E^2 \right) \]

\[ + \kappa^{-1} \left( 1 - \frac{\beta_3}{2} \right) \| \mathbf{m}_h \|^2 - 2 \sum_k \langle [\mathbf{n}_{p_h}], \mathbf{m}_h \rangle_E. \]  

(71)

On the other hand

\[ -2 \sum_k \langle [\mathbf{n}_{p_h}], \mathbf{m}_h \rangle_E \geq - \sum_k \beta_4 \frac{K}{2\delta} \| [\mathbf{n}_{p_h}] \|_E^2 - \sum_k \frac{1}{\beta_4} \frac{2\delta}{K} \| \mathbf{m}_h \|_E^2 \]

\[ \geq - \sum_k \beta_4 \frac{K}{2\delta} \| [\mathbf{n}_{p_h}] \|_E^2 - \sum_k \frac{1}{\beta_4} \kappa^{-1} \delta_0 C_{\text{tr}} \| \mathbf{m}_h \|_E^2, \]

which used in (71) gives

\[ B_{\text{exp}}(\mathbf{U}_h, \mathbf{U}_h + [\mathbf{m}_h, 0, 0, 0]) \geq \kappa^{-1} C_{\text{dec}} \| \mathbf{u}_h \|^2 + \kappa^{-1} \left( C_{\text{dec}} - \frac{\delta_0}{2\beta_2} C_{\text{tr}} \right) \| \mathbf{u}_h \|^2 \]

\[ + \sum_k \frac{K}{2\delta} \left( 1 - \frac{\beta_1}{2} - \beta_2 \right) \| [\mathbf{n}_{p_h}] \|_E^2 \]

\[ + \sum_k \frac{K}{\beta_4} \left( \frac{1}{C_p} \frac{C_{\text{inv}}}{4\delta_0} \left( \frac{C_{\text{tr}}}{2\beta_3} \right) \| \mathbf{p} \|_E^2 \right) \]

\[ + \kappa^{-1} \left( 1 - \frac{\beta_3}{2} \right) \| \mathbf{m}_h \|^2. \]

From this expression we see that if we take \( \beta_i, i = 1, 2, 3, 4 \), sufficiently small, then there exists a constant \( C \) for which

\[ B_{\text{exp}}(\mathbf{U}_h, \mathbf{U}_h + [\mathbf{m}_h, 0, 0, 0]) \geq C \| \mathbf{U}_h \|^2, \]  

(72)

provided the constants \( \delta_0 \) and \( C_p \) are small enough. On the other hand, \( \| [\mathbf{n}_{m}, 0, 0, 0] \| \leq C \| \mathbf{U}_h \| \), from where we obtain the result we wished to prove:

**Theorem 3.** There are constants \( C_p \) and \( \delta_0 \) in the definition of the stabilization parameters such that for all \( \mathbf{U}_h \) there exists \( \mathbf{V}_h \) such that

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**Fig. 4.** Results for the flow in a cavity. Left: \( P_1/P_1 \) interpolation (without \( u_i \)). Right: \( P_1/P_0 \) interpolation (with \( u_i \)). From top to bottom: streamlines and pressure contours.
Remark 8. Since \( \mathbf{u}_0 = \mathbf{P}_0 \mathbf{u}_h + \kappa \nabla \mathbf{p}_h \) and in view of (66) our result also applies with the norm

\[ \| \mathbf{U}_h \| := \kappa^{-1} \| \mathbf{u}_h \|^2 + \kappa \sum_k \| \nabla \mathbf{p}_h \|_k^2 + \sum_k \| \mathbf{n}_h \|^2_k + \sum_k \mathbf{r}_p^{-1} \| \mathbf{p}' \|^2_k, \]

which allows us to see that the stability result of Theorem 3 is optimal. Moreover, from the expression of \( \mathbf{p}' \) in the element interiors, usually proportional to the velocity divergence, it is possible to control \( \| \nabla \cdot \mathbf{u}_h \| \) which, together with the stability obtained on \( \| \mathbf{u}_h \| \), leads to full control of \( \mathbf{u}_h \) in \( H_0(\text{div}, \Omega) \) (see [3] for further details).

5. Numerical examples

In this section, we present the results of some numerical examples in order to study the performance of the presented method. We compare the results obtained using the approximation of the subscales on the interelement boundaries \( \mathbf{u}_0 \) given by (36) (or (51) in the case of the Stokes problem) with those obtained considering \( \mathbf{u}_0 = 0 \). A parameter \( \delta_0 = 0.2 \) has been adopted for the computation of the terms corresponding to the subscales on the element boundaries, as it has proved to be suitable for these numerical examples, even though for the Stokes problem the effect of the choice of \( \delta_0 \) has also been analyzed.

No results for the Darcy problem have been included, since in the case of interest, that is to say, for discontinuous pressure interpolations, the accuracy heavily relies on the expression of the subscales in the element interiors. A thorough discussion and a full convergence analysis can be found in [3].

5.1. Convection–diffusion equation

Let us start solving the convection–diffusion equation. We consider a domain \( \Omega \) enclosed in a circle of radius \( R = 1 \), which we discretize in a triangular finite element mesh, and we prescribe \( \mathbf{u} = 0 \) on \( \partial \Omega \).

We now study two different cases: in the first one diffusion dominates over convection (\( \kappa = 0.1, \mathbf{a} = (1,0), s = 0, f = 1 \) in (1)), while the second one is convection dominated (\( \kappa = 10^{-12}, \mathbf{a} = (1,0), s = 0, f = 1 \) in (1)). In both the diffusion and the convection dominated cases, no difference between the solution obtained considering \( \mathbf{u}_0 \) and the one obtained without considering it can be appreciated. Fig. 2 shows and compares the obtained solution \( \mathbf{u} \) for the considered methods in the diffusion dominated case, while Fig. 3 does...
Fig. 6. Results for the flow over a cylinder. Left: $P_1/P_1$ (without $u_0$). Right: $P_1/P_0$ (with $u_0$). From top to bottom: streamlines and pressure contours.

Fig. 7. Results for the flow over a cylinder, pressure in a cut along $y = 4$. Top: $P_1/P_0$ element with $d_0 = 0.2$ compared to the $P_1/P_1$ element. Bottom: $P_1/P_0$ element results for different values of $d_0$, global cut (left) and detail (right).
so when convection dominates over diffusion. In any case, there is no noticeable influence of the value of $\delta_0$ on the results.

5.2. Stokes problem

In this section, we study the performance of the method proposed for the Stokes problem. As stated in Section 3, considering the contribution of the subscales in the element boundaries $u_i^s$ stabilizes elements with discontinuous pressures. In particular it allows the use of $P_1/P_0$ (linear-constant) velocity-pressure pairs. Results using $P_1/P_0$ interpolation and considering the contribution of the subscales on the boundary will be compared with those obtained using $P_1/P_1$ (linear–linear) velocity-pressure pairs, in which no subscales on the boundaries are considered.

5.2.1. Flow in a cavity

In this example, the motion of a fluid enclosed in a square cavity $\Omega = [0,1] \times [0,1]$ is analyzed. The velocity is set to $(1,0)$ at the top horizontal wall ($y = 1$), while it is prescribed to 0 on the other walls ($y = 0$, $x = 0$ and $x = 1$). Pressure is fixed to 0 at an arbitrary point of the domain.

As Fig. 4 shows, little difference can be observed between results obtained using $P_1/P_1$ interpolation and those obtained using $P_1/P_0$ and taking into account the contribution of the subscales on the element boundaries with $\delta_0 = 0.2$. The slight differences which can be observed between both results are due to the fact that a poorer interpolation space for the pressure is used in the second case.

In order to check the behavior of the solution in terms of $\delta_0$, Fig. 5 shows a comparison between the pressure along $y = 1$ for $\delta_0 = 0.05$, 0.2 and 0.5. Note that this last value would be the maximum allowed by our way to motivate the subscales on the element boundaries (see Fig. 1). It is observed that $\delta_0 = 0.05$ allows for pressure oscillations, whereas no much difference is observed for $\delta_0 = 0.2$ and $\delta_0 = 0.5$ (in fact, similar results are obtained for any $\delta_0$ greater than 0.1). Of course, results are more diffusive the greater the value of $\delta_0$.

5.2.2. Flow over a cylinder

In this example, we study the Stokes flow past a cylinder. The computational domain is $\Omega = [0,16] \times [0,8]$; $D$, with the cylinder $D$ of diameter 2 and centered at $(4,4)$. The velocity at $x = 0$ is prescribed at $(1,0)$, whereas at $y = 0$ and $y = 8$, the $y$-velocity component is prescribed to 0 and the $x$-component is left free. The outflow, where both the $x$- and $y$- component are free, is $x = 16$. Tractions are set to 0 on the outflow.

As in the previous example, little difference can be appreciated between the solutions obtained with the $P_1/P_1$ pair with no subscales on the boundaries and the $P_1/P_0$ element with subscales on the boundaries (see Fig. 6).

Once again, the behavior of the solution in terms of $\delta_0$ has been checked. A comparison between the pressure in a cut along $y = 4$ is shown in Fig. 7 for $\delta_0 = 0.05$, 0.2 and 0.5. The same conclusions as for the cavity flow example can be drawn in this case, namely, $\delta_0 = 0.05$ allows for pressure oscillations which do not appear using $\delta_0 = 0.2$ and $\delta_0 = 0.5$, the latter being more diffusive than the former.

6. Conclusions

In this paper, we have extended the two-scale approximation of variational problems with an additional ingredient in the approximation of the subscales, which is an approximation for their values on the interelement boundaries.

The key idea is to assume that the subscales are already computed in the element interiors and to compute the boundary values by imposing the correct transmission conditions of the problem under consideration. Three examples of how to undertake this process have been presented, namely, the CDR equation, the Stokes problem and Darcy’s equations.

In order to be as general as possible, examples of how to compute the subscale on the element interiors have been proposed, but not used, in the sense that our developments are applicable to any approximation of these unknowns (provided they satisfy some conditions on the algorithmic constants on which they depend). In fact, we have proved stability estimates for the three problems considered which are valid for any choice of subscales in the interior of the elements. However, convergence analyses, not presented here, require the expressions of these subscales.

For the case of the CDR equation, the new terms introduced by accounting for the subscales on the interelement boundaries do not contribute to stability. However, our analysis and the numerical example presented show that they do not spoil it, and also that accuracy seems also to be unaffected. However, for the Stokes problem and for Darcy flow the terms introduced by the subscales on the boundaries are crucial to provide stability when discontinuous pressure interpolations are used. The stabilizing terms introduced are shared with other formulations that can be found in the literature. However, some non-standard terms also appear. Again, our analysis, and the numerical examples in the case of the Stokes problems, show that these terms do not harm stability.

References


