

**ON THE USE OF DEEP, SHALLOW OR FLAT SHELL  
FINITE ELEMENTS FOR THE ANALYSIS OF THIN SHELL STRUCTURES**

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This paper is confined to the study of thin shells. The aim is to summarize the different theories used and to examine the assumptions upon which each of them is based. The intention is to show when it is more suitable to use a particular approximation and to indicate the errors it introduces. Beginning with the general deep shell theory, some simplifications are introduced to obtain the shallow shell theories. The special implications of this theory for the finite element method are also examined. Finally the particular case of flat elements is discussed.

**1. Deep shell theory**

A shell, by definition, is a body limited by two surfaces separated from each other by a distance  $h$ , which is small in relation to its other dimensions.

This geometrical characteristic allows the three dimensional elasticity theory to be reduced to a two-dimensional theory.

If the correct hypothesis describing the transversal behaviour of the shell is found, all the calculations may be carried out in terms of the middle surface points.

This middle surface is defined by a vector  $r$ , the components of which are the curvilinear coordinates  $x^1, x^2$ .

The two base vectors  $a_\alpha$  tangential to the middle surface are obtained as the derivatives of  $r$  with respect to each curvilinear coordinate. The metric surface tensor  $a_{\alpha\beta}$  and curvature tensor of the middle surface  $b_{\alpha\beta}$  are obtained as indicated in fig. 1. In this paper Greek indexes range from 1 to 2. A single stroke stands for covariant differentiation with respect to the surface metric and a comma denotes partial differentiation with respect to the curvilinear coordinate  $x^\alpha$ .

Excepting the case of thick shells, all shell theories are based on neglecting the effect of normal stresses and assuming that the displacements or the stresses parallel to the middle surface vary linearly across the thickness.

With these assumptions all the displacement fields may be completely defined by the middle surface displacements and the middle surface rotations [1].

The supplementary hypothesis made for thin shell is to identify the rotation components  $\omega_\alpha$  with the normal rotation of the middle surface  $\phi_\alpha$ .

For this particular shell, it has been shown [2], that the strain energy can be explained as the sum of two terms: the stretching energy and the bending energy. The assumption involved in

<p>geometric definitions</p> $\vec{r} = \vec{r}(x^1, x^2)$ $\vec{a}_\alpha = \vec{r}_{ \alpha} \quad \alpha, \beta = 1 \text{ to } 2$ $\vec{a}_3 \cdot \vec{a}_\alpha = 0$ $\vec{a}_3 \cdot \vec{a}_3 = 1$ $a_{\alpha\beta} = \vec{a}_\alpha \cdot \vec{a}_\beta$ $b_{\alpha\beta} = \vec{a}_3 \cdot \vec{a}_{\alpha,\beta}$	<p>displacement field</p> $\vec{u} = \vec{u} + x^3 \vec{\omega}$ <p>covariant components</p> $\vec{u} = u_\alpha \vec{a}^\alpha + w \vec{a}^3$ $\vec{\omega} = \omega_\alpha \vec{a}^\alpha$ <p>Kirchhoff's hypothesis</p> $\omega_\alpha = \phi_\alpha = -w_{ \alpha} - b_\alpha^E u_E$
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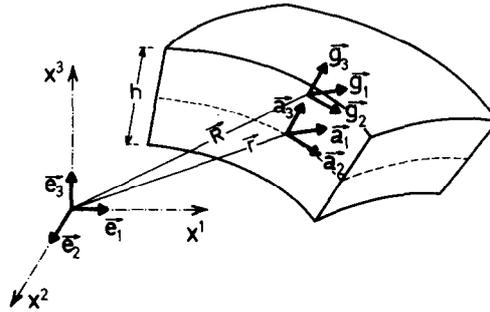


Fig. 1. Shell element.

neglecting the connecting terms is consistent with Kirchhoff's hypothesis. In linear elasticity, the middle surface strain tensor  $\gamma_{\alpha\beta}$  and the curvature change tensor  $\rho_{\alpha\beta}$  are:

$$\gamma_{\alpha\beta} = \bar{a}_{\alpha\beta} - a_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w, \tag{1}$$

$$\rho_{\alpha\beta} = \bar{b}_{\alpha\beta} - b_{\alpha\beta} = \frac{1}{2}(\phi_{\alpha|\beta} + \phi_{\beta|\alpha}) - \frac{1}{2}(b_\alpha^\lambda \omega_{\alpha\lambda} - b_\alpha^\lambda \omega_{\beta\lambda}), \tag{2}$$

with  $2\omega_{\alpha\beta} = u_{\beta|\alpha} - u_{\alpha|\beta}$ .

In these equations the superimposed line stands for a tensor after deformation. All these tensors are defined as a function of the covariant components of the displacement field  $u_\alpha$  and of  $w$ , called from now on the curvilinear components of this field (fig. 2). A family of conforming finite elements based on this formulation has been developed by the author [3].

Each element of the family is obtained by subdividing the triangular or quadrangular element into three or four regions, in each of which a complete cubic polynomial field is assumed for the transversal displacement component and a complete  $n$ -degree polynomial for

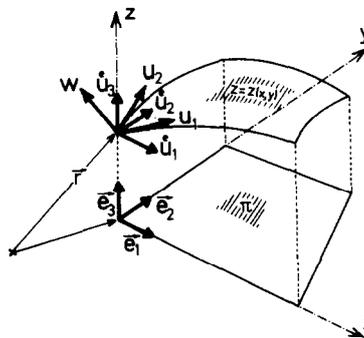


Fig. 2. Covariant and cartesian components of the displacement field.

the tangential displacement field. The strain energy is obtained as a function of the complete strain tensor  $\gamma_{\alpha\beta}$  and the curvature change tensor  $\rho_{\alpha\beta}$  of eqs. (1) and (2). Numerical integration is carried out to achieve the stiffness matrix. After assembling the stretching and the bending part, each element contains three displacement components and two rotational components at each corner node. Along the interfaces, a tangential rotation is always present, while there can be zero, one or two tangential displacements depending on the degree of the polynomial approximation. This general deep shell element is named GDS*i*, where *i* is the degree of the tangential displacement field (fig. 3a).

Another way of stating the shell theory is to calculate the strain tensor as a function of the cartesian displacement components (which are parallel to the cartesian base vector  $e$ ) denoted here  $\hat{u}_1$ ,  $\hat{u}_2$  and  $\hat{u}_3$  (fig. 2). The advantage of this formulation is a proper representation of the rigid-body motions when a finite element approach is used. A deep shell element based on this theory has been developed in [4].

## 2. The shallow shell theory in curvilinear components

In some cases it is possible to define a plane nearly parallel to the middle surface of the shell. If this plane is used as a reference plane and two of the cartesian components of the vector  $r$  are chosen to be parallel to the plane and the third to be perpendicular, the shell equation becomes

$$Z = Z(X, Y). \quad (3)$$

The base vector and the fundamental tensors become

$$\begin{aligned} a_\alpha &= r_{,\alpha} = e_\alpha + Z_{,\alpha}e_3, \\ a_{\alpha\beta} &= \delta_{\alpha\beta} + Z_{,\alpha}Z_{,\beta}, \\ b_{\alpha\beta} &= Z_{,\alpha\beta}/\sqrt{a} \quad \text{with } a = 1/Z_{,\alpha}^2 + Z_{,\beta}^2, \\ b_\alpha^\beta &= (Z_{,\alpha\beta} + Z_{,\gamma\alpha}Z_{,\alpha}Z_{,\beta}(-1)^{\gamma+\beta})/a^{3/2}. \end{aligned} \quad (4)$$

The strain tensor and curvature change tensor in curvilinear components can be obtained from (1), (2) and (4). Developing the covariant differentiation, they can be written as the sum of two parts:

$$\gamma_{\alpha\beta} = \overset{S}{\gamma}_{\alpha\beta} + \overset{D}{\gamma}_{\alpha\beta}, \quad (5)$$

$$\rho_{\alpha\beta} = \overset{S}{\rho}_{\alpha\beta} + \overset{D}{\rho}_{\alpha\beta}, \quad (6)$$

with

$$\overset{S}{\gamma}_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) - b_{\alpha\beta}w, \quad (7)$$

$$\overset{D}{\gamma}_{\alpha\beta} = -\frac{1}{2}[u_{\lambda} a^{\varepsilon\lambda} (a_{\varepsilon\alpha,\beta} - a_{\alpha\beta,\varepsilon} + a_{\beta\varepsilon,\alpha})], \quad (8)$$

$$\overset{S}{\rho}_{\alpha\beta} = -w_{,\alpha\beta}, \quad (9)$$

$$\begin{aligned} \overset{D}{\rho}_{\alpha\beta} = \frac{1}{2}[-b_{\alpha}^{\lambda} u_{\lambda,\beta} - b_{\alpha,\beta}^{\lambda} u_{\lambda} - b_{\beta}^{\lambda} u_{\lambda,\alpha} - b_{\beta,\alpha}^{\lambda} u_{\lambda} \\ - \phi_{\lambda} a^{\varepsilon\lambda} (a_{\varepsilon\alpha,\beta} - a_{\alpha\beta,\varepsilon} + a_{\beta\varepsilon,\alpha}) - b_{\beta}^{\lambda} (u_{\lambda,\beta} - u_{\beta,\lambda}) b_{\alpha}^{\lambda} (u_{\lambda,\alpha} - u_{\alpha,\lambda})]. \end{aligned} \quad (10)$$

The shallow shell theory assumes that

$$\overset{D}{\gamma}_{\alpha\beta} \cong 0 \quad \text{and} \quad \overset{D}{\rho}_{\alpha\beta} \cong 0. \quad (11)$$

In fact, there are two simplifying hypotheses in the shallow shell theory. The first is a geometric assumption:

$$Z_{,\alpha} \ll 1, \quad (12)$$

i.e., the surface slopes with respect to the reference plane are small.

The second is a displacement approximation which assumes that all the tangential displacements are small in relation to the transversal displacements. Thus, the rotation becomes

$$\phi_{\alpha} = -w_{,\alpha} \quad (13)$$

and the strain tensors read

$$\gamma_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) - Z_{,\alpha\beta} w \quad (14)$$

$$\rho_{\alpha\beta} = -w_{,\alpha\beta}. \quad (15)$$

### 3. The shallow shell theory in cartesian components

On the other hand, using the cartesian displacement components and the assumptions (3) and (4), the strain tensors become [5]:

$$\gamma_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta}^e + u_{\beta,\alpha}^e + Z_{,\alpha} u_{3,\beta}^e + Z_{,\beta} u_{3,\alpha}^e), \quad (16)$$

$$\rho_{\alpha\beta} = \overset{M}{\rho}_{\alpha\beta} + \overset{N}{\rho}_{\alpha\beta}, \quad (17)$$

with

$$\rho_{\alpha\beta}^M = -u_{3,\alpha\beta}^e / \sqrt{a} \quad (18)$$

$$\rho_{\alpha\beta}^N = [Z_{,\alpha\beta}(Z_{,\nu}Z_{,\lambda}u_{\nu,\lambda}^e - Z_{,\nu}u_{3,\nu}^e) / a - Z_{,\nu}u_{\nu,\alpha\beta}^e] / \sqrt{a} . \quad (19)$$

Here again, the shallow shell assumption is [6]:

$$\rho_{\alpha\beta}^M \approx 0 \quad (20)$$

and the strain tensors become

$$\gamma_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta}^e + u_{\beta,\alpha}^e + Z_{,\alpha}u_{3,\beta}^e + Z_{,\beta}u_{3,\alpha}^e), \quad (21)$$

$$\rho_{\alpha\beta} = -u_{3,\alpha\beta}^e . \quad (22)$$

#### 4. Special implications of the shallow shell theory in the finite element method

It is important to emphasize the fundamental aspects of the two shallow shell assumptions. First, the geometric assumption may be introduced when the structure is almost planar. The second assumption is only true if the whole structure is almost planar, but is not necessary valid if only a part of the structure is planar.

In the finite element method it is sometimes possible to calculate deep shells using a shallow shell theory. In this method, the whole structure is subdivided into small regions each of which can be treated as a shallow shell. However, the displacement hypotheses of shallow shells are frequently improperly applied.

In fact, when the number of elements in the model increases the two surface slopes  $Z_{,\alpha}$  tend to zero. However, the second derivatives  $Z_{,\alpha\beta}$  tend to a constant value proportional to the inverse curvature ratio  $R_{\alpha\beta}$ . Taking into account eqs. (4), this means that

$$\begin{aligned} a_{\alpha\beta} &\text{ tends to } \delta_{\alpha\beta}, \\ a &\text{ tends to } 1, \\ b_{\alpha\beta} \text{ and } b_{\alpha}^{\beta} &\text{ tends to } Z_{,\alpha\beta}, \text{ and} \\ a_{\alpha\beta,\gamma} = (Z_{,\alpha\gamma}Z_{,\beta} + Z_{,\alpha}Z_{,\beta\gamma}) &\text{ tends to } 0. \end{aligned} \quad (23)$$

##### 4.1. Curvilinear components

Inspection of eq. (8) reveals that all the terms of  $\gamma_{\alpha\beta}^D$  are multiplied by  $a_{\alpha\beta,\gamma}$  thus they vanish when the number of elements increases. These considerations justify the shallow shell assumption of neglecting those terms. Nevertheless, eq. (10) shows that it is not always possible to neglect  $\rho_{\alpha\beta}^D$ . Several of the terms of eq. (10) are a function of  $b_{\alpha}^{\beta}$  and the tangential displacements thus do not tend to zero when the number of element increases.

When a deep shell structure is analysed with a shallow shell element idealization, the results

converge towards a different solution. The difference between the deep shell energy and the shallow shell energy can be evaluated as a function of the neglected terms that do not tend to zero

$$W_{\text{error}} = \frac{h^3}{24} \int_V E^{\alpha\beta\mu\nu} (\rho_{\alpha\beta}^D \rho_{\mu\nu}^D + 2\rho_{\alpha\beta}^D \rho_{\mu\nu}^S) dv, \tag{24}$$

where  $h$  is the thickness of the shell and  $E^{\alpha\beta\mu\nu}$  the elastic constants in curvilinear coordinates.

However, as it is shown in [7], a deep shell structure can be analysed by shallow shell element in several cases. For instance when the rotations of the shell around the normal

$$\omega_{\alpha\beta} = \frac{1}{2}(\mu_{\beta|\alpha} - \mu_{\alpha|\beta}) \tag{25}$$

are of the same order as the strain tensor  $\gamma_{\alpha\beta}$ .

To test this energy gap, a doubly curved family of finite elements has been developed by the author using the shallow shell theory [7]. The characteristics of the elements are the same as for the deep shell elements described above, but the strain energy is a function of the shallow strain tensor equations (14) and (15). Those elements are named GSSi in [7].

A typical shallow shell element based on this theory, was developed by Olson [10] and is shown in fig. 3b.

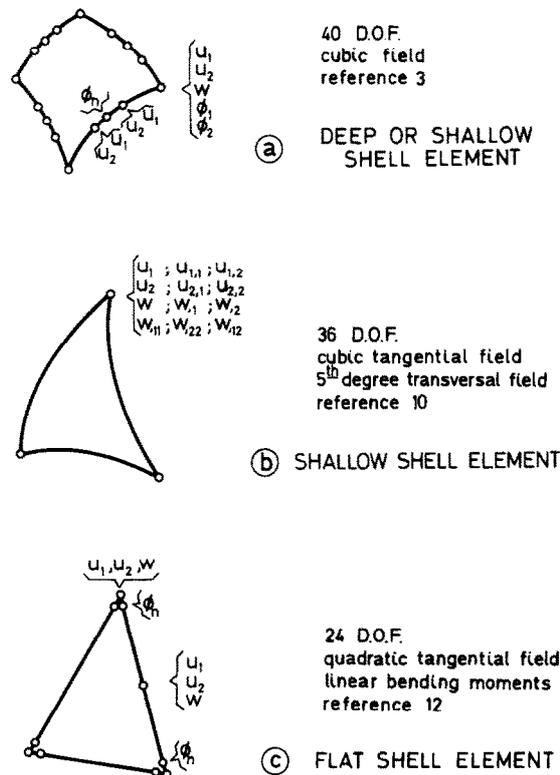


Fig. 3. Some shell finite elements.

4.2. Cartesian components

On the other hand, if the strain tensors are taken as function of the cartesian displacement components, the terms neglected in the shallow shell theory are  $\rho_{\alpha\beta}^N$ . All the terms of (19) are multiplied by  $Z_{,\alpha}$  or  $Z_{,\beta}$  and therefore tend to zero when the element number increases. In this case the strain energy converges to the same solution as that arrived at by using the deep shell theory.

It is important to emphasize here that although the assumptions of the two shallow shell theories are the same, the terms neglected are not. This is because in the latter, the simplifications are made on the cartesian displacement components instead of the curvilinear displacement components. This difference explains the different behaviours of those two shallow shell theories at the limit.

5. The flat shell elements

One of the most important aspects of the finite element analysis of shells are the flat shell elements. They are the simplest to generate and are easy to define geometrically. They can be obtained from eqs. (21) and (22) by neglecting the last two terms of (21).

The equations of the strain tensor and curvature change tensor become the well known stretching and plate bending tensors

$$\gamma_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta}^e + u_{\beta,\alpha}^e), \tag{26}$$

$$\rho_{\alpha\beta} = -u_{3,\alpha\beta}. \tag{27}$$

It is interesting to see that here again all the neglected terms tend to zero when the element number is increased. This deduction explains why the flat shell elements converge to the correct energy value when they are used in the analysis of deep shell structures [7].

The flat shell approximation cannot be obtained from the curvilinear component shell theory. This is because flat elements are by definition functions of the cartesian displacement components.

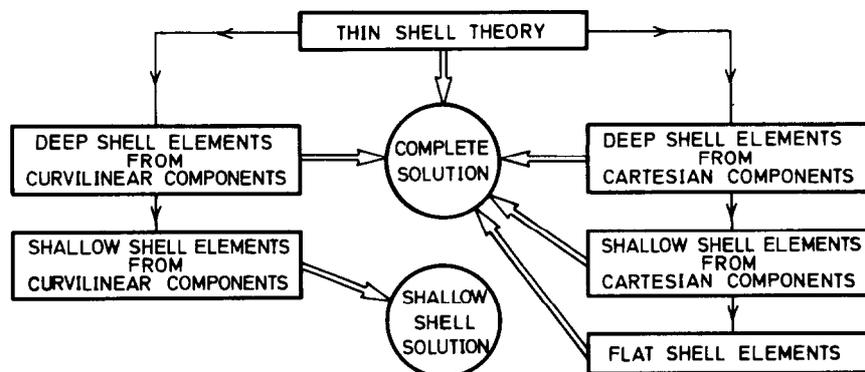
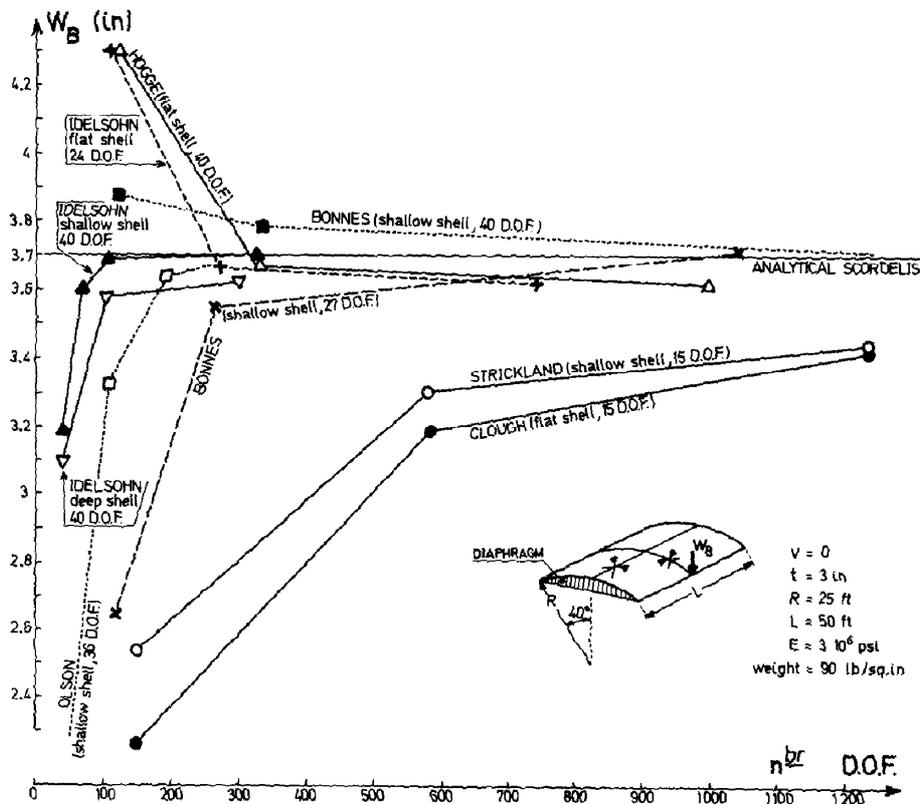


Fig. 4. Flow diagram of the different shell theories.



Fig. 6. Cylindrical shell, deflection  $W_B$ .

The flat shell elements tested here were a simple flat element developed by Clough with linear tangential displacements and cubic transversal field, a flat shell element with quadratic tangential displacement and cubic transversal field [13] and a mixed flat element developed by the author with quadratic displacement field for the stretching part and linear moments  $M_x$ ,  $M_y$  and  $M_{xy}$  for the bending part [12].

It is important to note that a difference of 3.5% is obtained between shallow shell elements and deep shell elements although this is a structure that is nearly a shallow shell. For a real deep shell structure, this difference can be much greater.

Many other authors have obtained the deep shell solution with thin shell or degenerate thick shell elements [15–18] but more sophisticated deep shell elements, like SHEBA [14, 19], converge with few elements to the shallow shell results. It should be interesting to confirm this behaviour with other deep shell elements (e.g. [5]) and the new flat shell elements described in [20].

## 7. Conclusions

Shallow thin shell theory is an interesting simplification of the general thin shell theory but significant errors can be found when shallow shell elements in curvilinear components are used for the analysis of deep shell structures. On the other hand, flat shell elements are apparently

more interesting. They are simple to generate, their geometry is easy to define and the most important conclusion of this paper is that flat shell elements have the property of converging to the correct solution of deep shell theory.

An alternative solution is to build shallow shell elements using the cartesian displacement components (eqs. (21) and (22)). To the author's knowledge, no thin shell elements based on this theory have yet been developed.

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