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ONE DISCRETE ELEMENT VS. TWO FINITE ELEMENTS AND THE ARBELOS OF ARCHIMEDES

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Abstract: In this article, a visual proof is given that for a certain simple, yet analytically challenging mechanical system a unique solution exists, which can be found by simple fixed-point iteration as easily to be performed by the reader. As turns out, this system is mostly intractable by means of the finite element method, yet being easily manageable by means of the distinct/discrete element method. Thus, this article gives evidence to the assumption that systems exist which may yield almost arbitrarily wrong results when treated with the finite element method, while giving arbitrarily accurate results when handled by means of the discrete element or similar method.

Keywords. Discrete Elements, Finite Elements, Fixed-point Iteration

1 INTRODUCTION

When a decision has to be made about which of two numerical methods is more suitable to solve a certain class of problems, a point of comparison is needed to quantify the goodness of the results of the respective methods. The more the methods in question have to be considered as some kind of black boxes and the harder they are to understand and to analyze, the simpler and clearer the referee has to be. Ideally, the decision-making method is of purely analytical nature and has nothing in common with one of the methods.

As simple this principle may sound, as hard can it be put into practice at a certain level of complexity. As can frequently be observed, when two methods A and B have to be compared to each other, they are *not* confronted with an independent method C, but the decision is made by consulting a method A', similar to or even identical with method A, plus a method A'' similar to method A', and so on.

In this article[1], for one of the simplest possible mechanical systems revealing non-linear system behavior, a *referee* is presented together with some 14 lines of code, providing the possibility to make a decision between the standard Finite Element Method on the one and the Discrete Element Method on the other side when it comes to non-linearity. This may serve as an example the reader is encouraged to accompany by others.

[1]This work is based on Chapter 4 of the author's PhD Thesis [1] (in German), submitted in 2021.

2 A SIMPLE SYSTEM OF ELASTIC RODS

In the textbook on Numerical Mathematics by Engeln-Muellges et al. [2], the following inconspicuous problem is presented, which, on closer examination, reveals a high degree of non-linearity. The system, shown in Fig. 1, consists of two elastic rods fixed at articulation points P and Q. An external force F is applied to a third, common articulation point R. Initially, the rods are supposed to be of equal length L and cross-sectional area A, while their stiffnesses E_1, E_2 shall differ by a certain factor κ,

$$E_2 = \kappa E_1 ,$$

with κ being set to the value of 2 in the original textbook.

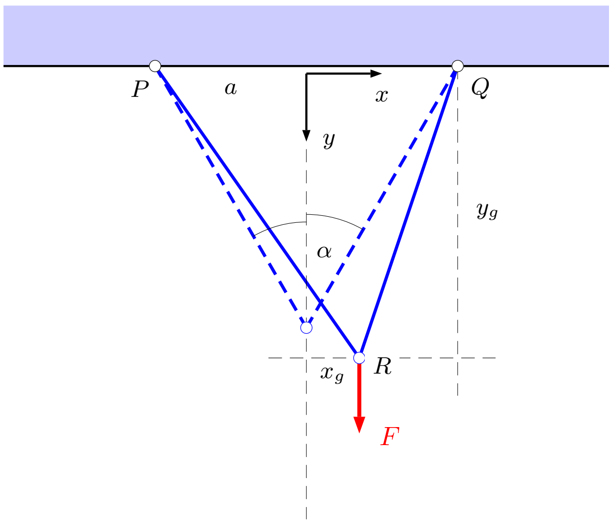


Figure 1: A system of two elastic rods, loaded with external force F.
Initial configuration and loading are assumed to be symmetric. The more the rods are assumed to differ by stiffness, the more the system reveals non-linearity.

The concrete values proposed in the textbook are $L = 2\,\mathrm{m}$, and $\alpha = 30°$. The rods are supposed to have circular cross-section of diameter $d = 10\,\mathrm{mm}$, and stiffnesses $E_1 = 100\,\mathrm{GPa}$, $E_2 = 200\,\mathrm{GPa}$. A force $F = 10\,\mathrm{kN}$ is applied at articulation point R. Below, we will keep L, α, d and F unchanged, while the values of E_1, E_2 may differ in a broad range.

As simple as this system might appear, it turns out to become more and more intractable by means of standard finite element analysis with increasing amount of asymmetry with respect to their stiffnesses.

3 HOOKE'S LAW IN VECTOR AND COMPLEX FORMULATION

When a force S is applied to an elastic rod of length L, having stiffness E and cross-sectional area A, the rod is stretched by an amount of Δl according to Hooke's Law:

$$\mathbf{S} = \frac{E\,A}{l_0}\,\Delta l \cdot \mathbf{n}\,.$$

Here $\mathbf{n}$ denotes a unit vector, $\|\mathbf{n}\| = 1$, directed along the rod's axis.

With $\mathbf{a}_0$ and $\mathbf{b}_0$ denoting the rod's endpoints if no force is applied, and $\mathbf{a}_1$, $\mathbf{b}_1$ denoting its endpoints if the rod is stretched, this relation can be expressed, using the Euclidean norm $\|\cdot\|$, as follows:

$$\begin{aligned}
\mathbf{S} &= \frac{EA}{l_0} \cdot (l_1 - l_0) \cdot \frac{\mathbf{b}_1 - \mathbf{a}_1}{\|\mathbf{b}_1 - \mathbf{a}_1\|} \\
&= \frac{EA}{\|\mathbf{b}_0 - \mathbf{a}_0\|} \cdot (\|\mathbf{b}_1 - \mathbf{a}_1\| - \|\mathbf{b}_0 - \mathbf{a}_0\|) \cdot \frac{\mathbf{b}_1 - \mathbf{a}_1}{\|\mathbf{b}_1 - \mathbf{a}_1\|} \\
&= EA \cdot \left(\frac{\|\mathbf{b}_1 - \mathbf{a}_1\| - \|\mathbf{b}_0 - \mathbf{a}_0\|}{\|\mathbf{b}_0 - \mathbf{a}_0\| \cdot \|\mathbf{b}_1 - \mathbf{a}_1\|} \right) (\mathbf{b}_1 - \mathbf{a}_1)
\end{aligned}$$

Further simplification yields

$$\mathbf{S} = EA \cdot \left(\frac{1}{\|\mathbf{b}_0 - \mathbf{a}_0\|} - \frac{1}{\|\mathbf{b}_1 - \mathbf{a}_1\|} \right) \cdot (\mathbf{b}_1 - \mathbf{a}_1) \tag{1}$$

which can be considered the *vector* formulation of Hooke's Law for the elastic rod.

For planar problems, any vector $\mathbf{v} = (v_x, v_y)^T$ can be identified by the complex number $v = v_x + v_y \cdot i$, with i denoting the imaginary unit. By doing so, Eq. (1) reads as follows:

$$s = EA \cdot \left(\frac{1}{|b_0 - a_0|} - \frac{1}{|b_1 - a_1|} \right) \cdot (b_1 - a_1) \tag{2}$$

where the Euclidean norm $\|\cdot\|$ has been replaced by complex magnitude (or absolute value) $|\cdot|$.

With this in mind, the forces appearing in our system of rods can be expressed and summarized as follows: Let the complex numbers p and q represent the articulation points of the rods, r denote their common articulation point *before* loading; also, let z represent the point of application of the external force F *after* the system has been loaded.

$$\begin{aligned}
s_1 &= E_1 A_1 \left(\frac{1}{|p - r|} - \frac{1}{|p - z|} \right) (p - z) \\
s_2 &= E_2 A_2 \left(\frac{1}{|q - r|} - \frac{1}{|q - z|} \right) (q - z)
\end{aligned}$$

Here, as before, E_1, E_2, A_1, A_2 are to be considered real numbers (or scalars), while p, q, r, s_1, s_2 are complex numbers representing or being identified with vectors.

Equilibrium holds if

$$s_1 + s_2 + f = 0\,,$$

where f is a complex number representing or being identified with the external force applied at the common articulation point.

In the textbook's example, both rods have equal initial length L and cross-sectional area A. The rod to the right is considered to have twice the stiffness as the rod to the left, $E_1 = E$, $E_2 = 2E$. Under this assumption, the condition for equilibrium is:

$$EA\left(\frac{1}{|p-r|} - \frac{1}{|p-z|}\right)(p-z) + 2EA\left(\frac{1}{|q-r|} - \frac{1}{|q-z|}\right)(q-z) + f = 0$$

becoming with $L = |p-r| = |q-r|$

$$\left(\frac{1}{L} - \frac{1}{|p-z|}\right)(p-z) + 2\left(\frac{1}{L} - \frac{1}{|q-z|}\right)(q-z) + \frac{f}{EA} \quad = \quad 0\,. \tag{3}$$

This is to be considered the condition of equilibrium for the system of two rods when formulated in the complex plane.

Other than the vector formulation, which yields a *system* of two non-linear equations, this is only one equation for one unknown complex number z, which can be solved and analyzed by means of functional analysis as will be shown below.

4 SOLVING THE PROBLEM BY FIXED-POINT ITERATION

In order to tackle the problem numerically and/or analytically, it has to be transformed to fixed point form, with unknown complex z standing alone at one side of the equals sign. Multiplication yields:

$$\frac{1}{L}(p-z) - \frac{p-z}{|p-z|} + \frac{\kappa}{L}(q-z) - \kappa\,\frac{q-z}{|q-z|} + \frac{f}{EA} = 0\,.$$

Here, the special value of 2 was replaced by κ, making the problem more general than the textbook's original task. By equivalent transformations we obtain

$$\frac{1}{L}(p-z) + \frac{\kappa}{L}(q-z) = \frac{p-z}{|p-z|} + \kappa\,\frac{q-z}{|q-z|} - \frac{f}{EA}\,,$$

yielding

$$\frac{1}{L}(p+\kappa q) - \frac{1}{L}(1+\kappa)z = \frac{p-z}{|p-z|} + \kappa\,\frac{q-z}{|q-z|} - \frac{f}{EA}$$

and

$$-\frac{1}{L}(1+\kappa)z = -\frac{1}{L}(p+\kappa q) + \frac{p-z}{|p-z|} + \kappa\,\frac{q-z}{|q-z|} - \frac{f}{EA}\,.$$

Thus, we obtain Eq. (3) in fixed-point form:

$$z \quad = \quad \frac{L}{1+\kappa}\left[\frac{1}{L}(p+\kappa q) - \frac{p-z}{|p-z|} - \kappa\,\frac{q-z}{|q-z|} + \frac{f}{E\,A}\right]. \tag{4}$$

Note that the right-hand side contains only given geometrical values p, q, r (articulation points), the initial length L, plus given mechanical parameters E, A and κ. Also the external force F has to be considered to be given.

With a little luck we may succeed to solve the problem by simple fixed-point iteration, i.e. by successively updating

$$z^{k+1} \leftarrow \phi(z^k)$$

until some kind of convergence can be observed.

A simple interpreted programming language can be used to give this a try, implementing Eq. (4), using the textbook's original values:

```
E = 100000;
kappa = 2;

L = 2000;
D = 10;
A = pi/4*D^2;

p = -1000 + 0*i;
q = +1000 + 0*i;

F = 0 - 10000*i;

z = 0.73-17333*i;

for k = 1:100
  z = L/(1+kappa)*( (z-p)/abs(z-p) + ...
                    kappa*(z-q)/abs(z-q) + (p+kappa*q)/L + F/(E*A) )
end

z
```

Iteration starts at $z_0 =$ `0.73 - 1733` i, close to the textbook's stated solution. Indeed, the iterated values of z come closer and closer to some value of z^*, slowly converging towards the value

$$z^* \quad = \quad \texttt{0.7341601269 - 1733.323346 i}\,,$$

in perfect accordance with the textbook's result, which has been obtained by applying the so-called Brown method for solving non-linear systems.

As turns out, it is *not* simply *by chance* that fixed-point iteration is successful when applied to our problem.

5 ANALYZING THE FIXED-POINT ITERATION

The provisional nature of this section has to be emphasized. For more details, the reader must be referred to the presentation, the author's PhD thesis [1] and a future extended version of this article.

By moving the origin from the middle of P and Q to P, followed by scaling, it can be achieved that $p = 0$, $L = 1$ and $q = 1$, whereby the dimensionless term f/EA will not be affected.

This simplifies our fixed-point equation:

$$\zeta = \frac{1}{1+\kappa}\left[\frac{\zeta}{|\zeta|} + \kappa\,\frac{\zeta - 1}{|\zeta - 1|} + \kappa + \frac{f}{EA}\right] .$$

For readability, the transformed ζ is again written as z, while f is written F as is usual in most textbooks on Technical Mechanics. Now the condition of equilibrium reads as follows:

$$z = \frac{1}{1+\kappa}\left(\frac{z}{|z|} + \kappa\frac{z-1}{|z-1|} + \kappa + \frac{F}{EA}\right) = f(z) . \tag{5}$$

Note that this formulation is equivalent to the problem stated at the beginning, yet being much easier to study and investigate.

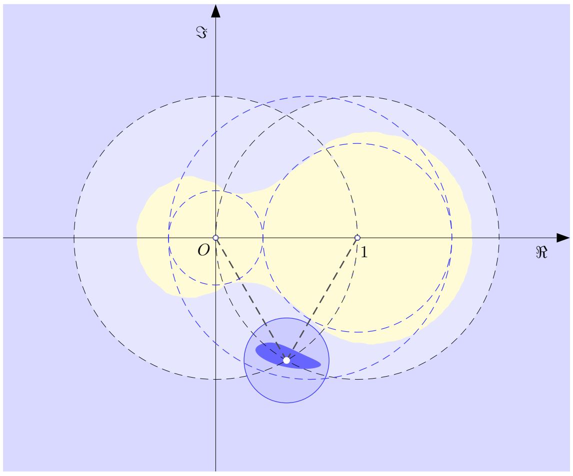
This allows to make some observations, which may be of interest for any further approach to tackle the problem by means of Analysis and Functional Analysis.

In any textbook on Functional Analysis we can find an extremely fruitful theorem by Stefan Banach; the following formulation is taken from a free encyclopedia:

Let (X, d) be a complete metric space. Then a map $T : X \to X$ is called a *contraction mapping* on X if there exists $q \in [0, 1)$ such that $d(T(x), T(y)) \leq q\, d(x, y)$ for all $x, y \in X$.

Banach Fixed-Point Theorem. Let (X, d) be a non-empty complete metric space with a contraction mapping $T : X \to X$. Then T admits a unique fixed-point x^* in X, i.e. $T(x^*) = x^*$. Furthermore, x^* can be found as follows: start with an arbitrary element $x_0 \in X$ and define a sequence $(x_n)_{n\in\mathbb{N}}$ by $x_{n+1} = T(x_n)$ for $n \geq 1$. Then $\lim_{n\to\infty} x_n = x^*$.

With this in mind, knowing that $\mathbb{C}$ is a complete metric space (and so is each closed subset of $\mathbb{C}$, in particular, a closed disk), Fig. 2 – for $\kappa = 2$ – provides a visual "proof" for the existence and uniqueness of a fixed-point inside of the dark blue disk which can be determined by fixed-point iteration.



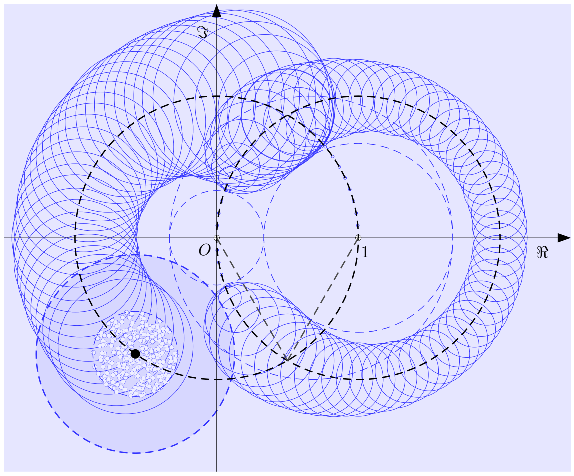


Figure 2: *Visual "proof" for existence of a fixed point of the function f described in Eq. (5)*
Left: Outside of the yellow region function f is contractive. The blue disk centered at the lower articulation point lies entirely inside of the area of contractivity. Function f maps the disk to the dark blue region within, which resembles an airfoil. Thus, on the blue disk, function f matches the condition of Banach's fixed-point theorem.
Right: Function f can be proven (analytically) to be contractive outside of the circles (dashed, black) of radius 1 centered at 0 and 1. The area of non-contractivity can be constructed by Monte-Carlo simulation. Disks (blue) with their midpoints on the dashed black circles are widened and shrunk in a bisection process up to a maximum of size no contraction could be detected within. The remaining region is to be considered non-contractive. This is the region shown in yellow in the figure to the left.

Remark: Function f maps the entire complex plane to a region bounded by three "kissing circles", a figure whose upper or lower half is known as the Arbelos of Archimedes. This can be seen from evaluating function f 1. for values far from the origin, 2. for values close to 0, and 3. for values close to 1.

1. If z is located far from the origin we have $z/|z| \approx (z-1)/|z-1|$, thus:

$$\begin{aligned} f(z) &\approx \frac{1}{1+\kappa}\left((1+\kappa)\frac{z}{|z|}+\kappa+\frac{F}{EA}\right) \\ &\approx \frac{z}{|z|}+\frac{\kappa}{1+\kappa}\,. \end{aligned}$$

(In general, F/EA is small, and divided by $1+\kappa$ it becomes even smaller.) Thus, the outer contour of $f(\mathbb{C})$ is the unit circle, shifted to the right by a distance of $\kappa/(1+\kappa)$.

2. For z close to zero we have:

$$\begin{aligned} f(z) &\approx \frac{1}{1+\kappa}\left(\frac{z}{|z|}+\kappa\frac{-1}{|-1|}+\kappa+\frac{F}{EA}\right) \\ &\approx \frac{1}{1+\kappa}\left(\frac{z}{|z|}-\kappa+\kappa\right) \\ &\approx \frac{1}{1+\kappa}\frac{z}{|z|}\,, \end{aligned}$$

describing a circle of radius $1/(1+\kappa)$ with its center at the origin.

3. Correspondingly, for $z \approx 1$ we obtain :

$$\begin{aligned} f(z) &\approx \frac{1}{1+\kappa}\left(\frac{1}{|1|} + \kappa\frac{z-1}{|z-1|} + \kappa + \frac{F}{EA}\right) \\ &\approx 1 + \frac{\kappa}{1+\kappa}\frac{z-1}{|z-1|}\,, \end{aligned}$$

describing a circle of radius $\dfrac{\kappa}{1+\kappa}$ centered at $(1,0)$.

6 FINITE AND DISCRETE ELEMENTS

Now that we are provided with a point of comparison, different methods can be investigated with respect to the given non-linear system.

There are, principally, two approaches to tackle the problem, see Fig. 3. The first one, being the structural engineer's approach, is to look at the system as consisting of two constructive elements, i.e. rods, that are fixed, joined and loaded at certain articulation points, finding themselves in equilibrium after loading not asking what happened before this state was reached.

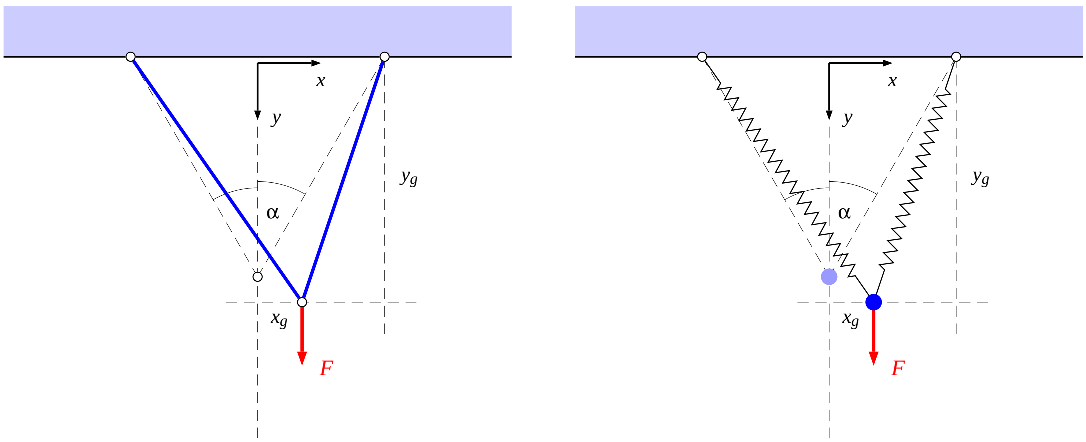


Figure 3: Two different points of view at a system of elastic rods
Left: From a structural engineer's point of view, the system consists of two constructive – or finite – elements. The engineer is interested in the state of equilibrium.
Right: From the physicist's point of view, a mass point is attracted by elastic forces, if an external force is applied. The mass point is called a discrete element. The physicist is interested in the mass point's trajectory on its way to equilibrium.

This point of view completely differs from the physicist's, who is interested in the system's behavior after the load has been applied, until equilibrium is reached. Thus, the physicist will not concentrate on the constructive elements, i.e the rods, but the element the force is applied to, i.e. the articulation point which is assumed to be a mass point, fixed to the wall by two elastic rods (or springs).

Mathematically, the difference between both approaches is that the first one, i.e. the engineer's one, considers the problem to be a boundary value problem, while the second one, i.e. the physicist's one, is treating the problem as an initial value problem.

7 FINITE ELEMENTS

A system may consist of elastic rods, joined and articulated at points $P_1, P_2, \ldots, P_m$, while the whole system may be fixed at points $P_{m+1}, \ldots, P_n$. Then, if external forces $\mathbf{F}_i$ are applied at P_i $(i = 1, \ldots, m)$, the displacements $\mathbf{x}_j$ of articulation points can be determined by

$$\mathbf{F}_j = \sum_{i \sim j} c_{i,j} \frac{(\mathbf{p}_j - \mathbf{p}_i)^{\mathrm{T}}(\mathbf{x}_j - \mathbf{x}_i)}{(\mathbf{p}_j - \mathbf{p}_i)^{\mathrm{T}}(\mathbf{p}_j - \mathbf{p}_i)}(\mathbf{p}_j - \mathbf{p}_i), \qquad j = 1, \ldots, m\,,$$

where $c_{i,j}$ refer to the elastic properties of the respective rods. Here, the notation $i \sim j$ means that articulation points P_i and P_j are connected by an elastic rod. Articulation points $P_{m+1}, \ldots, P_n$ are fixed, thus $\mathbf{x}_j = 0$ for $j = m+1, \ldots, n$.

The assembly of the stiffness matrix and further proceeding is described in detail by Nowottny[3], pp. 75–79; the script below can be considered the shortest finite element program to be found in literature.

Remark: The terminology "finite element program" used for this code fragment is somewhat archaic; the author is also aware of the existence of non-linear finite element approaches. Nonetheless, the above view on finite element analysis is the one still used in many textbooks written for practitioners and engineers. Elastic bars or rods *as finite elements* are still a common starting point in lectures on the basics of structural analysis.

With the code given by Nowottny [3], the given problem can be tackled, obtaining the solution

$$u_x = \texttt{0.735105}\,, \qquad u_y = \texttt{-1733.32404}\,,$$

which is in good, yet not perfect, accordance with the textbook's solution of $u_x = \texttt{0.73416}$, $u_y = \texttt{-1733.32335}$.

8 DISCRETE ELEMENTS

The Discrete Element Method, applied to aour problem, is described by Newton's equation of motion:

$$m\,\ddot{\mathbf{x}} = \sum \mathbf{F} - \delta\,\dot{\mathbf{x}}\,,$$

```
E = 100000;

L = 2000;
d = 10;
A = pi/4*d^2;

alpha = 30;

x_1 = -L*sin(30/180*pi);
y_1 =  0;
x_2 =  L*sin(30/180*pi);
y_2 =  0;
x_F =  0;
y_F = -L*cos(30/180*pi);

c_1 =   E*A/L;
c_2 = 2*E*A/L;

P = [x_F x_1 x_2;
     y_F y_1 y_2];

F = [     0;
     -10000];

A = [ 1   1;
      2   3;
     c_1 c_2];

network(P,F,A)
```

```
function X = network(P,F,A)
% Input:  P ... Articulation points
%         F ... External forces
%         A ... Adjacencies
% Output: X ... Displacements
%
% Taken from Nowottny [3], pp. 77-78

[dummy,n] = size(P);
[dummy,m] = size(F);
[dummy,z] = size(A);

F = F(:);
M = zeros(2*n,2*n);
for k = 1:z
  i = A(1,k);
  j = A(2,k);
  c = A(3,k);
  Diff = P(:,j)-P(:,i);
  denom = Diff'*Diff;
  H = c/denom*Diff*Diff';
  H = [H -H; -H H];
  index = [2*i-1 2*i 2*j-1 2*j];
  M(index,index) = M(index,index)+H;
end
M = M(1:2*m,1:2*m);

X = M\F;
X = reshape(X,2,m);
```

where m is the mass of the imaginary mass point (or discrete element), $\sum \mathbf{F}$ is the sum of all forces applied to the mass point, and δ is a value to incorporate viscous damping into the system. Without damping, the system's motion would not come to an end.

Nonetheless, it is crucial to use a numerical scheme which *does* preserve energy in a sense that no energy is lost by numerical errors. Thus, it is recommended to use a numerical scheme that well suits Hamiltonian problems, as does, for instance, the well-known velocity Stoermer-Verlet method:

$$\begin{aligned} x^{n+1} &= x^n + h\left(1 - \frac{h}{2m}\delta\right) v^n + \frac{h^2}{2m} F(x^n)\,, \\ v^{n+1} &= \frac{1 - \frac{h}{2m}\delta}{1 + \frac{h}{2m}\delta}\, v^n + \frac{1}{1 + \frac{h}{2m}\delta}\, \frac{h}{2m}\left[F(x^n) + F(x^{n+1})\right]\,. \end{aligned}$$

Again, performing the calculation in the complex plane simplifies the code by replacing two equations to one.

A straight-forward implementation leads to a program that tracks the oscillatory motion (Fig. 4) of the articulation point R after force F is applied, which, due to damping, finally comes to an end at position

$$\texttt{0.73415694448615 - 1.27257287220801 i},$$

or – shifting it downwards by an amount of $L\,\cos 30° = 2000 \cdot \sqrt{3}$ – at the position

$$\texttt{0.73415694448615 - 1733.32338044108520 i},$$

in excellent accordance with the textbook's solution of $u_x = \texttt{0.73416}$, $u_y = \texttt{-1733.32335}$.

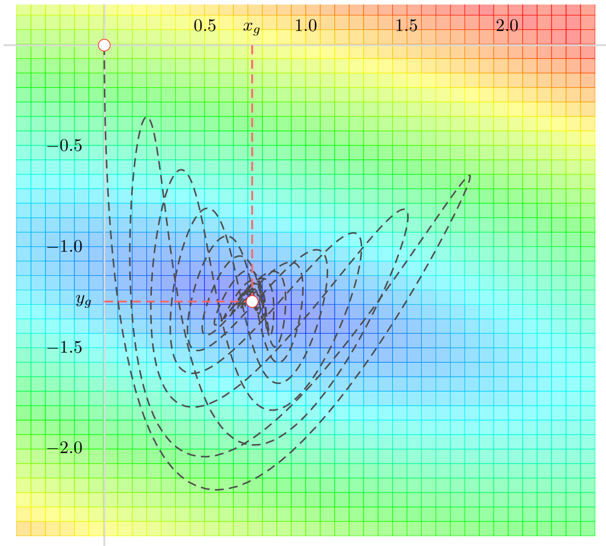


Figure 4: Trajectory of articulation point R.
Here, the origin has been moved to the articulation point the external force is initially applied to. After a somewhat unpredictable oscillatory motion the point the external force is applied at reaches equilibrium in the center of the blue area where the system's potential energy becomes a minimum.

A more detailed analysis shows that the point of equilibrium is located at the deepest point inside of an oblong and very narrow valley. This makes it hard to be found by hand-made methods such as the Newton or Golden Search methods. This may also explain why the fixed-point iteration scheme takes about 100 steps to reach the solution.

9 COMPARISON OF SIMULATION RESULTS

The reader is invited to rerun the above MATLAB script to confirm the following observations: 1. For high values of E, i.e. for (very) small displacements, all three methods provide the same solution, as given by the original textbook. 2. For decreasing values of E, i.e. for larger displacements, the fixed-point and the discrete element solution are in excellent accordance with each other, see Table 1. For decreasing values of E the finite element solution more and more misses the correct solution.

Table 1: Simulation results for fixed-point iteration, DEM and FEM
Identical results were obtained by the fixed-point iteration and the discrete element method at the given level of accuracy. For FEM, the x-values do not change, but shift, revealing the method's linear approach.

E	Fixed-point Iteration & DEM		FEM	
[MPa]	x_g	y_g	x_g	y_g
1000 000	0.073501	-1732.178125	0.073511	-1732.178132
100 000	0.734160	-1733.323346	0.735105	-1733.324047
10 000	7.257087	-1744.714027	7.351051	-1744.783203
1 000	64.691468	-1853.282073	73.510519	-1859.374762
100	271.322121	-2742.273306	735.105193	-3005.290352

Remark: In Table 1, the FEM values for x, or their digits, do not change but shift, according to the relation $x_g(0.1 \cdot E) = 10 \cdot x_g(E)$. Similar is the case for the y-values, which can be seen by adding a value of $L\cos 30° = 2000 \cdot \sqrt{3}$ to all of them. This reveals the intrinsic linear nature of the finite element code used here. It is recommended to the reader to compare the above results with the solution found by his or her preferred finite element program, which may handle non-linearity in a better way. Note, however, that the "finite element" code used here is far from being primitive, since vertex displacements *are* taken into consideration.

10 CONCLUSION

Three completely different methods have been presented which allow to tackle one and the same problem numerically. The problem itself has to be considered the prototype of a whole class of problems being of both theoretical and practical interest. All three methods are independent from each other, the first being of purely analytical nature, the second being the well-established, static/stationary Finite Element Method. With the so-called Discrete Element Method a third method has been introduced, which can be described a dynamical method. The simulation results show that the analytical way can be taken only for extremely simple configurations, while the Finite Element Method yields reliable results in the – important – case of small displacements. The dynamical method may become of broader interest in the near future, either been used as a reference or as the method of choice for non-linear systems.

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