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On a Class of Hybrid Langevin Inclusions Involving Two Generalized Fractional Derivatives

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ABSTRACT

In this paper, we study the existence of solutions for a hybrid Langevin inclusion involving a combination of ϕ-Hilfer and ϕ-Caputo fractional derivatives. To this end, we construct a new operator derived from the integral solution of the given boundary value inclusion problem and subsequently apply the hypotheses of Dhage's fixed point theorem to this fractional operator. Finally, to support our theoretical findings, an illustrative example is provided.

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1 Introduction

Fractional differential inclusions (FDI's), as a generalization of fractional differential equations (FDE's), have attracted significant attention due to their relevance in optimization problems and stochastic processes [1]. They provide a framework for analyzing dynamical systems where the evolution of the system is not strictly determined by its current state, offering greater flexibility than classical formulations [2,3]. In recent years, the theory of FDE's has seen substantial development across various branches of mathematics [4–7]. This theoretical progress has been complemented by significant advances in numerical methods for solving fractional models arising in applied sciences [8–11]. Furthermore, both FDE's and their inclusions appear naturally in numerous scientific disciplines and have been applied extensively in diverse contexts [12–16]. In this direction, Almeida [17] introduced

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the ϕ-Caputo fractional derivative (FD), a novel operator that unifies various types of FD's, thereby broadening the scope of modern and complex applications.

The classical Langevin equation (LE), introduced by Paul Langevin in 1908 [18], was the first mathematical framework to describe Brownian motion. However, it fails to fully capture phenomena involving fractal disorder. In mathematical physics, such considerations are crucial for modeling fractional reaction-diffusion systems [19,20], harmonic oscillators, correlated noise sources [21], and quantum noise behavior [22,23], among others. When a macroscopic system does not exist and microscopic time-scale differential equations become inadequate, the fractional LE proves more effective than its classical counterpart [24]. In particular, the fractional LE introduced by Lutz [25], incorporates power-law memory kernels and fractional Brownian motion to describe anomalous diffusion in complex and viscoelastic media. Subsequent developments by Mainardi et al. [26] established a rigorous link between fractional Langevin dynamics and anomalous diffusion processes, emphasizing the impact of non-Markovian effects and long-time correlations in Brownian motion. Further discussions on the Langevin equation and fractional dynamics can be found in [27], while its applications span stochastic problems in physics, chemistry, and electrical engineering [28].

Over the years, significant research efforts have been devoted to investigating the existence, uniqueness, and stability of solutions for fractional differential equations involving various types of FD's, see [29,30] and the references therein. Particular attention has been given to the fractional LE and inclusions. For example:

The authors in [31] established existence results for the Langevin FDI involving a Caputo FD of the form

$$\begin{cases} {}^{C}D_{0+}^{\mathfrak{e}_1}\left({}^{C}D_{0+}^{\mathfrak{e}_2}+\lambda_1\right)u(\tau)\in\Pi(\tau,u(\tau)),\ \mathfrak{e}_1,\mathfrak{e}_2\in(0,1],\ \tau\in(0,1),\\ u(0)=\sum_{i=1}^{n}\mathfrak{m}_i\left(I_{0+}^{\mathfrak{d}_i}u\right)(\gamma_1),\ u(1)=\sum_{i=1}^{n}\mathfrak{n}_i\left(I_{0+}^{\mathfrak{e}_i}u\right)(\gamma_2),\end{cases}$$

where $0<\gamma_1,\gamma_2<1$, ${}^{C}D_{0+}^{\mathfrak{e}}$ is the Caputo FD of order $\mathfrak{e}\in\{\mathfrak{e}_1,\mathfrak{e}_2\}$, $I_{0+}^{\nu_i}$ is the Riemann-Liouville FI of order $\nu_i\in\{\mathfrak{d}_i,\mathfrak{e}_i\}$, $\lambda_i,\mathfrak{m}_i,\mathfrak{n}_i\in\mathbb{R}$, $i=\overline{1,n}$, and $\Pi:[\mathfrak{a},\overline{\rho}]\times\mathbb{R}\to\mathcal{F}(\mathbb{R})$ is a set-valued mapping, where $\mathcal{F}(\mathbb{R})$ denotes the power set of $\mathbb{R}$.

In [32], Ahmad et al. investigated a class of LE involving generalized Liouville-Caputo derivatives of distinct orders, subject to nonlocal generalized fractional integral boundary conditions

$$\begin{cases} {}^{\kappa}_{c}D_{\mathfrak{a}+}^{\mathfrak{e}_1}\left({}^{\kappa}_{c}D_{\mathfrak{a}+}^{\mathfrak{e}_2}+\lambda_1\right)u(\tau)=\pi(\tau,u(\tau)),\ \tau\in(\mathfrak{a},\overline{\rho}),\\ u(\mathfrak{a})=0,\ u(\gamma_1)=0,\ u(\overline{\rho})=\mathfrak{m}\,{}^{\kappa}I_{\mathfrak{a}+}^{\mathfrak{d}}u(\gamma_2),\\ \mathfrak{e}_1\in(1,2],\ \mathfrak{e}_2\in(0,1),\ \mathfrak{a}<\gamma_1<\gamma_2<\overline{\rho},\end{cases}$$

where ${}^{\kappa}_{c}D_{\mathfrak{a}+}^{\mathfrak{e}_1}$ and ${}^{\kappa}_{c}D_{\mathfrak{a}+}^{\mathfrak{e}_2}$ denote the generalized Liouville-Caputo FD of order $\mathfrak{e}_1$ and $\mathfrak{e}_2$, respectively, ${}^{\kappa}I_{\mathfrak{a}+}^{\mathfrak{d}}$ is the generalized fractional integral operator of order $\mathfrak{d}>0$, $\kappa>0$, $\lambda_1,\mathfrak{m}\in\mathbb{R}$ and $\pi:[\mathfrak{a},\overline{\rho}]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function.

In 2021, the authors [33] studied a class of Langevin FDI's subject to nonlocal conditions of the form

$$\begin{cases} {}^{C}D_{0+}^{\mathfrak{e}_1}\left({}^{C}D_{0+}^{\mathfrak{e}_2}u(\tau)-\lambda_1u(\tau)\right)\in\Pi(\tau,u(\tau)),\ \mathfrak{e}_1,\tau\in(0,1),\\ {}^{C}D_{0+}^{1}u(1)+{}^{C}D_{0+}^{1}u(\gamma_1)=0,\ 0<\gamma_1<1,\lambda_1\geq 0,\\ u(0)=0,{}^{C}D_{0+}^{\mathfrak{e}_2}u(0)=0,\ \mathfrak{e}_2\in(1,2).\end{cases}\tag{1}$$

In [34], the authors established the existence and uniqueness of solutions for the following nonlinear fractional Langevin integro-differential equation

$$\begin{cases} D_{0+}^{\mathfrak{e}_1}\left({}^{C}D_{0+}^{\mathfrak{e}_2}+\lambda_1\right)u(\tau)=\pi\left(\tau,u(\tau),I_{0+}^{\nu}u(\tau)\right),\ \tau\in(0,\overline{\rho}),\ \lambda_1\in\mathbb{R},\\ {}^{C}D_{0+}^{\mathfrak{e}_2}u(0)+\lambda_1u(0)={}^{C}D_{0+}^{\mathfrak{e}_2}u(\overline{\rho})=0,\\ u(0)=a\int_0^{\overline{\rho}}u(s)\,ds+b, \end{cases}$$

where $D_{0+}^{\mathfrak{e}_1}$ is the Riemann-Liouville fractional derivative of order $\mathfrak{e}_1\in(0,1)$, $a,b\in\mathbb{R}$, and $\nu\in(0,1)$.

In [35], Khan et al. examined the existence, uniqueness, and stability of the solution for the generalized fractional LE of the form

$$\begin{cases} {}_{c}^{\kappa}D_{\mathfrak{a}+}^{\mathfrak{e}_1}\left({}_{c}^{\kappa}D_{\mathfrak{a}+}^{\mathfrak{e}_2}+\lambda_1\right)u(\tau)=\pi\left(\tau,u(\tau),u(\delta\tau),{}_{c}^{\kappa}D_{\mathfrak{a}+}^{\mathfrak{e}_2}u(\tau)\right),\ \tau\in(\mathfrak{a},\overline{\rho}),\\ u(\mathfrak{a})=0,\ u(\gamma_1)=0,\ u(\overline{\rho})=\mathfrak{m}\,{}^{\kappa}I_{\mathfrak{a}+}^{\mathfrak{d}}u(\gamma_2),\ \mathfrak{d}>0,\ \mathfrak{m}\in\mathbb{R},\\ \mathfrak{e}_1\in(1,2],\ \mathfrak{e}_2\in(0,1),\ \mathfrak{a}<\gamma_1<\gamma_2<\overline{\rho},\ \delta>0. \end{cases}$$

In [36], the authors investigated the existence, uniqueness, and stability of solutions for the LE involving ϕ-Caputo FD of different orders

$$\begin{cases} {}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\left({}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_2,\phi}+\lambda_1\right)u(\tau)=\pi(\tau,u(\tau)),\ \tau\in(\mathfrak{a},\overline{\rho}),\\ u(\mathfrak{a})=0,u'(\mathfrak{a})=0,\\ {}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_2,\phi}u(\overline{\rho})+\lambda_2u(\overline{\rho})=0, \end{cases}$$

where ${}^{C}D_{\mathfrak{a}+}^{\mathfrak{e},\phi}$ is the ϕ-Caputo FD of order $\mathfrak{e}\in\{\mathfrak{e}_1,\mathfrak{e}_2\}$ such that $\mathfrak{e}_1\in(0,1]$, $\mathfrak{e}_2\in(1,2]$, and $\lambda_1,\lambda_2\in\mathbb{R}$.

Lmou et al. [37] employed the Dhage fixed point theorem (FPT) to investigate the existence of solutions for the ϕ-Caputo fractional differential Langevin hybrid inclusion

$$\begin{cases} {}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\left({}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_2,\phi}\left(\dfrac{u(\tau)}{h(\tau,u(\tau))}\right)+\ell_1(\tau)u(\tau)\right)\in\Pi(\tau,u(\tau)),\ \tau\in(\mathfrak{a},\overline{\rho}),\\ u(\mathfrak{a})=0,\ u(\overline{\rho})=\sum\limits_{i=1}^{n}\mathfrak{m}_i\left(I_{\mathfrak{a}+}^{\mathfrak{d}_i,\phi}u\right)(\gamma_i),\\ \mathfrak{e}_1,\mathfrak{e}_2\in(0,1),\ \gamma_i\in(\mathfrak{a},\overline{\rho}), \end{cases} \tag{2}$$

where ${}^{C}D_{\mathfrak{a}+}^{\mathfrak{e},\phi}$ is the ϕ-Caputo FD of order $\mathfrak{e}\in\{\mathfrak{e}_1,\mathfrak{e}_2\}$. $I_{0+}^{\mathfrak{d}_i,\phi}$ is the ϕ-Riemann-Liouville FI of order $\mathfrak{d}_i>0$, $\mathfrak{m}_i\in\mathbb{R}$, $i=\overline{1,n}$, $\ell_1\in C(\mathbb{R}^+,\mathbb{R})$, $h\in C([\mathfrak{a},\overline{\rho}]\times\mathbb{R},\mathbb{R}\backslash\{0\})$.

Inspired by [17] and the existing literature, we analyze the existence of solutions for a nonlinear hybrid Langevin FDI with a generalized FD's under mixed boundary conditions

$$\begin{cases} {}^{H}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\mathfrak{e}_2,\phi}\left({}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\left(\dfrac{u(\tau)}{h(\tau,u(\tau))}\right)+\ell_1(\tau)u(\tau)\right)\in\Pi(\tau,u(\tau)),\ \tau\in(\mathfrak{a},\overline{\rho}),\\ \dfrac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))}=0,\ D_{\phi}^{1}\left(\dfrac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))}\right)=0,\\ {}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\left(\dfrac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}\right)+\beta\left(\dfrac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}\right)=0, \end{cases} \tag{3}$$

where ${}^{H}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\mathfrak{e}_2,\phi}$ is the ϕ-Hilfer FD of order $\mathfrak{e}_1\in(0,1)$ and type $\mathfrak{e}_2\in[0,1]$, ${}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}$ is the ϕ-Caputo FD of order $\mathfrak{e}_3\in(1,2]$, $\ell_1\in C(\mathbb{R}^+,\mathbb{R})$, $h\in C([\mathfrak{a},\overline{\rho}]\times\mathbb{R},\mathbb{R}\backslash\{0\})$, $\beta\in\mathbb{R}$, and $\Pi:[\mathfrak{a},\overline{\rho}]\times\mathbb{R}\to\digamma(\mathbb{R})$ is a set-valued mapping, where $\digamma(\mathbb{R})$ denotes the power set of $\mathbb{R}$.

We prove existence outcomes for the generalized hybrid Langevin inclusion (3), by constructing a new operator derived from the integral solution of the given boundary value inclusion problem. It does not satisfy the compactness property, which prevents the direct application of classical fixed point theorems such as the Schauder or Schaefer fixed point theorems. To overcome this difficulty, we employ Dhage's fixed point theorem, which is specifically designed to handle operators lacking compactness, and provides a suitable framework well adapted to hybrid fractional problems.

We remark that Eq. (3) represents a generalization of existing initial value problems for Langevin inclusions due to the general structure of the kernel associated with the ϕ-Hilfer and ϕ-Caputo fractional derivatives. For instance, Eq. (3) reduces to ϕ-Caputo-type inclusions problem (2) if $\mathfrak{e}_2 = 1$, and to Caputo-type inclusions problem (1) if $\mathfrak{e}_2 = 1$, $\phi(\tau) = \tau$, $h(\tau, u(\tau)) = 1$ and $\ell_1(\tau) = \lambda_1$. This hybrid formulation allows the model to capture a broader class of memory effects than those achievable by Langevin inclusions involving a single type of fractional derivative. To the best of our knowledge, such a hybrid framework has not been previously investigated and is expected to enrich the theoretical development of Langevin-type problems.

The article is organized as follows: Section 2 covers preliminary concepts. Section 3 presents our main existence results. Section 4 provides an illustrative example, and Section 5 offers concluding remarks.

2 Preliminaries

2.1 Fractional Calculus (FC)

In this portion, we present some basic concepts of FC and necessary lemmas that are required in our work.

Let $\mathfrak{J}_{\overline{\rho}} = [\mathfrak{a}, \overline{\rho}]$. We denote by $\mathfrak{C} = C\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{R}\right)$ the Banach space of continuous functions $u : \mathfrak{J}_{\overline{\rho}} \to \mathbb{R}$ with supremum norm

$$\|u\| = \sup\left\{|u(\tau)| : \tau \in \mathfrak{J}_{\overline{\rho}}\right\},$$

and $L^1\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{R}\right)$ be the Banach space of Lebesgue integrable functions $u : \mathfrak{J}_{\overline{\rho}} \to \mathbb{R}$ with norm

$$\|u\|_{L^1} = \int_{\mathfrak{J}_{\overline{\rho}}} |u(\tau)|\, d\tau.$$

Let $u : \mathfrak{J}_{\overline{\rho}} \to \mathbb{R}$ be an integrable function and $\phi \in C^n\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{R}\right)$ an increasing function such that $\phi'(\tau) \neq 0$, for any $\tau \in \mathfrak{J}_{\overline{\rho}}$.

Definition 1 ([6]): *The ϕ-Riemann-Liouville FI of order $\mathfrak{e}_1$ for a function u is given by*

$$I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi} u(\tau) = \frac{1}{\Gamma(\mathfrak{e}_1)} \int_{\mathfrak{a}}^{\tau} \phi'(\varsigma)\left(\phi(\tau) - \phi(\varsigma)\right)^{\mathfrak{e}_1 - 1} u(\varsigma)\, d\varsigma.$$

Definition 2 ([6]): *The ϕ-Riemann-Liouville FD of order $\mathfrak{e}_1$ for a function u is described by*

$$D_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi} u(\tau) = \left(\frac{1}{\phi'(\tau)} \frac{d}{d\tau}\right)^n I_{\mathfrak{a}+}^{(n-\mathfrak{e}_1),\phi} u(\tau),$$

where $n = [\mathfrak{e}_1] + 1$, $n \in \mathbb{N}$.

Definition 3 ([17]): *The ϕ-Caputo FD of a function $u \in C^n\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{R}\right)$ of order $\mathfrak{e}_1$ is defined by*

$$^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}u(\tau) = I_{\mathfrak{a}+}^{(n-\mathfrak{e}_1),\phi}\, u^{[n]}(\tau),$$

where $u^{[n]}(\tau) = \left(\frac{1}{\phi'(\tau)}\frac{d}{d\tau}\right)^n u(\tau)$, *and* $n = [\mathfrak{e}_1] + 1$, $n \in \mathbb{N}$.

Lemma 1 ([38]): *Let* $\mathfrak{e}_1, \mathfrak{e}_2 > 0$. *Then*

(1) $I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}(\phi(\tau) - \phi(\mathfrak{a}))^{\mathfrak{e}_2-1} = \dfrac{\Gamma(\mathfrak{e}_2)}{\Gamma(\mathfrak{e}_1+\mathfrak{e}_2)}(\phi(\tau) - \phi(\mathfrak{a}))^{\mathfrak{e}_1+\mathfrak{e}_2-1}$,

(2) $^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}(\phi(\tau) - \phi(\mathfrak{a}))^{\mathfrak{e}_2-1} = \dfrac{\Gamma(\mathfrak{e}_2)}{\Gamma(\mathfrak{e}_2-\mathfrak{e}_1)}(\phi(\tau) - \phi(\mathfrak{a}))^{\mathfrak{e}_2-\mathfrak{e}_1-1}$.

Lemma 2 ([17]): *Let* $u \in C^n\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{R}\right)$ *and* $\mathfrak{e}_1 \in (n-1, n)$, *then*

$$I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\, {}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\, u(\tau) = u(\tau) - \sum_{k=0}^{n-1}\frac{u^{[n]}(\mathfrak{a}^+)}{k!}(\phi(\tau) - \phi(\mathfrak{a}))^k.$$

Definition 4 ([38]): *The ϕ-Hilfer FD of $u \in C^n\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{R}\right)$, of order $n-1 < \mathfrak{e}_1 < n$ and type $\mathfrak{e}_2 \in [0,1]$, is defined by*

$$^{H}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\mathfrak{e}_2,\phi}u(\tau) = I^{\mathfrak{e}_2(n-\mathfrak{e}_1),\phi}\, D_{\phi}^{[n]} I^{(1-\mathfrak{e}_2)(n-\mathfrak{e}_1),\phi}u(\tau),$$

where $D_{\phi}^{[n]} = \left(\frac{1}{\phi'(\tau)}\frac{d}{d\tau}\right)^n$.

Lemma 3 ([38]): *If* $u \in C^n\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{R}\right)$, $n-1 < \mathfrak{e}_1 < n$ *and type* $\mathfrak{e}_2 \in [0,1]$, *then*

(1) $I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\, {}^{H}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\mathfrak{e}_2,\phi}u(\tau) = u(\tau) - \sum_{k=1}^{n}\dfrac{(\phi(\tau) - \phi(\mathfrak{a}))^{\delta-k}}{\Gamma(\delta-k+1)}\left(\dfrac{1}{\phi'(\tau)}\dfrac{d}{d\tau}\right)^{n-k} I^{(1-\mathfrak{e}_2)(n-\mathfrak{e}_1),\phi}u(\mathfrak{a})$, *where* $\delta = \mathfrak{e}_1 + \mathfrak{e}_2(1-\mathfrak{e}_1)$.

(2) $^{H}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\mathfrak{e}_2,\phi}\, I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\, u(\tau) = u(\tau)$.

To address (3), we require the following auxiliary result.

Lemma 4: *Let*

$$\phi_{\overline{\rho}} = (\phi(\overline{\rho}) - \phi(a)),$$

$$\Lambda = \frac{-\beta\Gamma(\delta)}{\Gamma(\mathfrak{e}_3+\delta)}\phi_{\overline{\rho}}^{\mathfrak{e}_3+\delta-1} - \phi_{\overline{\rho}}^{\delta-1} \neq 0, \tag{4}$$

and for any $\widehat{\mathfrak{z}} \in \mathfrak{C}$, *then the solution of linear-type problem*

$$\begin{cases} {}^{H}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\mathfrak{e}_2,\phi}\left({}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\left(\dfrac{u(\tau)}{h(\tau,u(\tau))}\right)+\ell_1(\tau)u(\tau)\right)=\widehat{\mathfrak{z}}(\tau),\ \tau\in(\mathfrak{a},\overline{\rho}), \\ \dfrac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))}=0,\ D_{\phi}^{1}\left(\dfrac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))}\right)=0, \\ {}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\left(\dfrac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}\right)+\beta\left(\dfrac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}\right)=0, \end{cases} \tag{5}$$

is obtained as

$$\begin{aligned} u(\tau)=h(\tau,u(\tau))\Big[&I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\widehat{\mathfrak{z}}(\tau)-I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau)\\ &+\frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\widehat{\mathfrak{z}}(\overline{\rho})-\ell_1(\overline{\rho})u(\overline{\rho})\right.\\ &\left.-\beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho})+\beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\widehat{\mathfrak{z}}(\overline{\rho})\right)\Big]. \end{aligned} \tag{6}$$

Proof: Taking the ϕ-FI $I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}$ to the first equation of (5), and by using Lemma 3, we get

$${}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\left(\frac{u(\tau)}{h(\tau,u(\tau))}\right)=I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\widehat{\mathfrak{z}}(\tau)+c_1\left(\phi(\tau)-\phi(a)\right)^{\delta-1}-\ell_1(\tau)u(\tau),\ \tau\in\mathfrak{J}_{\overline{\rho}}\text{ and }c_1\in\mathbb{R}, \tag{7}$$

where $\delta=\mathfrak{e}_1+\mathfrak{e}_2(1-\mathfrak{e}_1)$. Now, by applying the ϕ-FI $I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}$ on the both sides of (7), and from Lemma 1 we get

$$\begin{aligned} \frac{u(\tau)}{h(\tau,u(\tau))}=&I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\widehat{\mathfrak{z}}(\tau)-I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau)+\frac{c_1\Gamma(\delta)}{\Gamma(\mathfrak{e}_3+\delta)}\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}\\ &+c_3(\phi(\tau)-\phi(\mathfrak{a}))+c_2,\ c_2,c_3\in\mathbb{R}. \end{aligned} \tag{8}$$

By applying the conditions $\dfrac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))}=0$ and $D_{\phi}^{1}\left(\dfrac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))}\right)=0$ in (8), we find $c_2=c_3=0$. Therefore, we have

$$\frac{u(\tau)}{h(\tau,u(\tau))}=I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\widehat{\mathfrak{z}}(\tau)-I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau)+\frac{c_1\Gamma(\delta)}{\Gamma(\mathfrak{e}_3+\delta)}\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}. \tag{9}$$

Using the second condition ${}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\left(\dfrac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}\right)+\beta\dfrac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}=0$, Eqs. (7) and (9), we get

$$c_1=\frac{1}{\Lambda}\left[I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\widehat{\mathfrak{z}}(\overline{\rho})-\ell_1(\overline{\rho})u(\overline{\rho})-\beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho})+\beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\widehat{\mathfrak{z}}(\overline{\rho})\right].$$

By substituting the value of c_1 into (9), we arrive at the solution of (5) which is given by (6). □

2.2 Multi-Function Theory

We begin by introducing several fundamental concepts related to set valued maps (multi-functions) as outlined in [39]. Let $(\mathbb{E},\|.\|)$ be a Banach space, and consider a multi-function $\varrho:\mathbb{E}\to\digamma(\mathbb{E})$. The following properties are defined:

- Closed (Convex) values: The multi-function ϱ is said to be closed-valued (respectively, convex-valued) if for every $u \in \mathbb{E}$, the set $\varrho(u)$ is closed (respectively, convex) in $\mathbb{E}$.
- Boundedness: The multi-function ϱ is bounded if, for any bounded subset $\mathcal{B} \subset \mathbb{E}$, $\varrho(\mathcal{B}) = \cup_{u\in\mathcal{B}}\varrho(u)$ is a bounded subset of $\mathbb{E}$, i.e.,

$$\sup_{u\in\mathcal{B}} \{\sup\{|\xi| : \xi \in \varrho(u)\}\} < \infty,$$

- Measurability: The multi-function ϱ is measurable if, for each $\xi \in \mathbb{R}$, the function

$$\tau \to d(\xi, \varrho(\tau)) = \inf\{|\xi - \lambda| : \lambda \in \varrho(\tau)\}.$$

is measurable.

For additional concepts such as complete continuity and upper semi-continuity (u.s.c.), we refer the reader to [39]. Furthermore, the selections set of Π is defined as:

$$\mathcal{R}_{\Pi,\xi} = \left\{\varkappa \in L^1\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{R}\right) \mid \varkappa(\tau) \in \Pi(\tau,\xi) \ \ \forall \text{ (a.e.) } \tau \in \mathfrak{J}_{\overline{\rho}}\right\}.$$

In the next, we define

$$\digamma_{k\mu}(\mathfrak{C}) = \{\ominus \in \digamma(\mathbb{E}) : \ominus \neq \emptyset \text{ and has a property } \mu\}.$$

In particular, we use the notations:

- $\digamma_{cl}$ for the class of all closed subsets of $\mathbb{E}$,
- $\digamma_{c}$ for convex subsets,
- $\digamma_{b}$ for bounded subsets, and
- $\digamma_{cp}$ for compact subsets.

Definition 5 ([40]): *The multi-function $\Pi : \mathfrak{J}_{\overline{\rho}} \times \mathbb{R} \to \digamma(\mathbb{R})$ is Carathéodory if $\tau \to \Pi(\tau, u)$ is measurable for each $u \in \mathbb{R}$, and for almost all $\tau \in \mathfrak{J}_{\overline{\rho}}$ the mapping $u \to \Pi(\tau, u)$ is u.s.c.*

Moreover, Π is said to be L^1-Carathéodory if, for every $l > 0$, there is $m^* \in L^1\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{R}^+\right)$ such that $\|m^*\| \le l$, the norm of $\Pi(\tau, u)$, defined as

$$\|\Pi(\tau, u)\| = \sup\{|\varkappa| : \varkappa \in \Pi(\tau, u)\} \le m^*(\tau),$$

satisfies

$$\|\Pi(\tau, u)\| \le m^*(\tau),$$

for almost every $\tau \in \mathfrak{J}_{\overline{\rho}}$.

The following lemmas are essential for establishing the main results developed in this study.

Lemma 5 ([40]): *Let $Gb(\varrho) = \{(u, \varphi) \in \mathbb{E} \times \mathbb{X},\ u \in \varrho(u)\}$ be a graph of ϱ. If $\varrho : \mathbb{E} \to \digamma_{cl}(\mathbb{X})$ is u.s.c., then $Gb(\varrho)$ is closed in $\mathbb{E} \times \mathbb{X}$. As well as, if ϱ is completely continuous and has a closed graph, it implies that ϱ is u.s.c.*

Lemma 6 ([41]): *Let $\mathbb{E}$ be a separable Banach space, $\Pi : \mathfrak{J}_{\overline{\rho}} \times \mathbb{E} \to \digamma_{cp,c}(\mathbb{E})$ be the L^1-Carathéodory, $\Upsilon : L^1\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{E}\right) \to C\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{E}\right)$ be linear and continuous. Then*

$$\Upsilon \circ \mathcal{R}_{\Pi} : C\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{E}\right) \to \digamma_{cp,c}\left(C\left(\mathfrak{J}_{\overline{\rho}}, \mathbb{E}\right)\right),\ u \to (\Upsilon \circ \mathcal{R}_{\Pi})(u) = \Upsilon\left(\mathcal{R}_{\Pi,u}\right),$$

is a map with closed graph in $C\left(\mathfrak{J}_{\overline{\rho}},\mathbb{E}\right)\times C\left(\mathfrak{J}_{\overline{\rho}},\mathbb{E}\right)$.

Theorem 1 (Dhage FPT [42]): *Let* $\mathfrak{C}$ *be a Banach algebra, and consider two operators* $\widehat{\mathbb{Z}}_1:\mathfrak{C}\to\mathfrak{C}$, *and* $\widehat{\mathbb{Z}}_2:\mathfrak{C}\to\mathcal{F}_{cp,c}(\mathfrak{C})$. *Assume the following conditions hold:*

- $\widehat{\mathbb{Z}}_1$ *is Lipschitz with constant* $\widetilde{w}_1^*$,
- $\widehat{\mathbb{Z}}_2$ *is compact and u.s.c.*,
- $2\widetilde{w}_1^*\Psi<1$ *is satisfied, where* $\Psi=\left\|\widehat{\mathbb{Z}}_2(\mathfrak{C})\right\|$.

Then either,

(i) *The operator inclusion* $u\in\left(\widehat{\mathbb{Z}}_1u\right)\left(\widehat{\mathbb{Z}}_2u\right)$ *admits a solution,*

(ii) *The set* $\Theta=\left\{u\in\mathfrak{C}:\xi u(\tau)\in\left(\widehat{\mathbb{Z}}_1u\right)\left(\widehat{\mathbb{Z}}_2u\right),\ \xi>1\right\}$ *is unbounded.*

3 Existence Results for Set-Valued Problem

Definition 6: *A function* $u\in C^1\left(\mathfrak{J}_{\overline{\rho}},\mathbb{R}\right)$ *is a solution of (3), if there is a function* $\varkappa\in L^1\left(\mathfrak{J}_{\overline{\rho}},\mathbb{R}\right)$ *with* $\varkappa(\tau)\in\Pi(\tau,u)$ *for every* $\tau\in\mathfrak{J}_{\overline{\rho}}$ *fulfilling the mixed BC's*

$$\frac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))}=0,\left(\frac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))}\right)'=0,$$

$${}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_2,\phi}\left(\frac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}\right)+\beta\left(\frac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}\right)=0,$$

and

$$\begin{aligned}u(\tau)&=h(\tau,u(\tau))\left[I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\tau)-I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau)\right.\\&\quad+\frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa(\overline{\rho})-\ell_1(\overline{\rho})u(\overline{\rho})\right.\\&\quad\left.\left.-\beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho})+\beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\overline{\rho})\right)\right].\end{aligned}$$

For the sake of convenience, we put

$$\begin{aligned}\theta_1&=\left[\frac{\phi_{\overline{\rho}}^{\mathfrak{e}_1+\mathfrak{e}_3}}{\Gamma(\mathfrak{e}_1+\mathfrak{e}_3+1)}+\frac{\Gamma(\delta)\phi_{\overline{\rho}}^{\mathfrak{e}_1+\mathfrak{e}_3+\delta-1}}{|\Lambda|\Gamma(\mathfrak{e}_3+\delta)\Gamma(\mathfrak{e}_1+1)}+\frac{|\beta|\Gamma(\delta)\phi_{\overline{\rho}}^{\mathfrak{e}_1+2\mathfrak{e}_3+\delta-1}}{\Gamma(\mathfrak{e}_1+\mathfrak{e}_3+1)|\Lambda|\Gamma(\mathfrak{e}_3+\delta)}\right],\\\theta_2&=\left[\frac{\ell_1^*\phi_{\overline{\rho}}^{\mathfrak{e}_3}}{\Gamma(\mathfrak{e}_3+1)}+\frac{\Gamma(\delta)|\beta|\ell_1^*\phi_{\overline{\rho}}^{2\mathfrak{e}_3+\delta-1}}{|\Lambda|\Gamma(\mathfrak{e}_3+\delta)\Gamma(\mathfrak{e}_3+1)}+\ell_1^*\frac{\Gamma(\delta)\phi_{\overline{\rho}}^{\mathfrak{e}_3+\delta-1}}{|\Lambda|\Gamma(\mathfrak{e}_3+\delta)}\right].\end{aligned}\tag{10}$$

Theorem 2: *Let* $h\in C\left(\mathfrak{J}_{\overline{\rho}}\times\mathbb{R},\mathbb{R}^*\right)$ *and* $\Pi:\mathfrak{J}_{\overline{\rho}}\times\mathbb{R}\to\mathcal{F}_{cp,c}(\mathbb{R})$ *be* L^1*-Carathéodory and continuous. Also, suppose that:*

(As01) There is a bounded function $\widetilde{w}_1:\mathfrak{J}_{\overline{\rho}}\times\mathbb{R}\to\mathbb{R}^+$ *such that*

$|h(\tau,u_1(\tau))-h(\tau,u_2(\tau))|\leq\widetilde{w}_1(\tau)|u_1-u_2|,\ \forall(\tau,u_1,u_2)\in\mathfrak{J}_{\overline{\rho}}\times\mathbb{R}^2.$

(As02) There exists $\widetilde{w}_2\in C\left(\mathfrak{J}_{\overline{\rho}},\mathbb{R}^+\right)$ *and a nondecreasing function* $\widetilde{w}_3\in C(\mathbb{R}^+,\mathbb{R}^+)$ *such that*

$\|\Pi(\tau,u)\|_{\mathcal{F}}=\sup\{|\xi|:\xi\in\Pi(\tau,u)\}\leq\widetilde{w}_1(\tau)\widetilde{w}_2(\|u\|),\ \forall(\tau,u)\in\mathfrak{J}_{\overline{\rho}}\times\mathbb{R}.$

(As03) There is $\Delta > 0$ satisfy

$$\Delta > \left(\widetilde{w}_1^*\Delta + h^*\right)\left(\theta_1 \|\widetilde{w}_1\| \widetilde{w}_2(\Delta) + \theta_2\Delta\right). \tag{11}$$

If

$$\widetilde{w}_1^*\theta_1 \|\widetilde{w}_1\| \widetilde{w}_2(\Delta) + \theta_2\Delta < \frac{1}{2}, \tag{12}$$

then, Eq. (3) possesses a solution.

Proof: To reformulate problem (3) as a FP problem, we first define the multi-function operator $\widehat{\mathbb{Z}} : \mathfrak{C} \to \digamma(\mathfrak{C})$ by

$$\widehat{\mathbb{Z}}(u) = \left\{\begin{array}{l} \widetilde{g} \in \mathfrak{C} : \\ \widetilde{g}(\tau) = \left\{\begin{array}{l} h(\tau, u(\tau))\left[I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau)\right. \\ +\dfrac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa(\overline{\rho}) - \ell_1(\overline{\rho})u(\overline{\rho})\right. \\ \left.\left. -\beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\overline{\rho})\right)\right], \end{array}\right. \end{array}\right\}, \tag{13}$$

for $\varkappa \in \mathcal{R}_{\Pi,u}$. Clearly, solutions of (3) correspond to FP of $\widehat{\mathbb{Z}}$. Consider two operators $\widehat{\mathbb{Z}}_1 : \mathfrak{C}\to\mathfrak{C}$ and $\widehat{\mathbb{Z}}_2 : \mathfrak{C} \to \digamma(\mathfrak{C})$ as

$$\left(\widehat{\mathbb{Z}}_1 u\right)(\tau) = h(\tau, u(\tau)),$$

and

$$\left(\widehat{\mathbb{Z}}_2 u\right)(\tau) = \left\{\begin{array}{l} \widetilde{k} \in \mathfrak{C} : \\ \widetilde{k}(\tau) = \left\{\begin{array}{l} I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau) \\ +\dfrac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa(\overline{\rho}) - \ell_1(\overline{\rho})u(\overline{\rho})\right. \\ \left. -\beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\overline{\rho})\right), \ \varkappa \in \mathcal{R}_{\Pi,u}. \end{array}\right. \end{array}\right\}.$$

Obviously $\widehat{\mathbb{Z}}(u) = \widehat{\mathbb{Z}}_1(u)\widehat{\mathbb{Z}}_2(u)$. In the subsequent analysis, we will demonstrate that these operators satisfy the necessary conditions of (1). The proof will proceed according to the following steps:

Step 1. $\widehat{\mathbb{Z}}_1$ is Lipschitz.

For $u_1, u_2 \in \mathfrak{C}$ and by (As01), we have

$$\begin{aligned} \left|\left(\widehat{\mathbb{Z}}_1 u_1\right)(\tau) - \left(\widehat{\mathbb{Z}}_1 u_2\right)(\tau)\right| &= |h(\tau, u_1(\tau)) - h(\tau, u_2(\tau))| \\ &\leq \widetilde{w}_1(\tau)|u_1 - u_2| \\ &\leq \widetilde{w}_1^*|u_1 - u_2|, \end{aligned}$$

where $\widetilde{w}_1^* = \sup_{\tau\in\mathfrak{I}_{\overline{\rho}}} |\widetilde{w}_1(\tau)|$ .Thus, $\widehat{\mathbb{Z}}_1$ is a $\widetilde{w}_1^*$-Lipschitz operator.

Step 2. For each $u \in \mathfrak{C}$, the set $\widehat{\mathbb{Z}}_2(u)$ is convex.

Let $\widetilde{k}_1, \widetilde{k}_2 \in \widehat{\mathbb{Z}}_2(u)$. Then there exist $\varkappa_1, \varkappa_2 \in \mathcal{R}_{\Pi,u}$ such that for each $\tau \in \mathfrak{J}_{\overline{\rho}}$, we have

$$\widetilde{k}_j(\tau) = I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa_j(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau) + \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\Big(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa_j(\overline{\rho}) - \ell_1(\overline{\rho})u(\overline{\rho}) - \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa_j(\overline{\rho})\Big),\ j=1,2.$$

Let $\vartheta \in [0,1]$. Then for any $\tau \in \mathfrak{J}_{\overline{\rho}}$

$$\begin{aligned}&\left[\vartheta\widetilde{k}_1 + (1-\vartheta)\widetilde{k}_2\right](\tau)\\ &= I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\left(\vartheta\varkappa_1(\tau) + (1-\vartheta)\varkappa_2(\tau)\right) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau)\\ &+ \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\Big(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\left(\vartheta\varkappa_1(\overline{\rho}) + (1-\vartheta)\varkappa_2(\overline{\rho})\right) - \ell_1(\overline{\rho})u(\overline{\rho})\\ &- \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\left(\vartheta\varkappa_1(\overline{\rho}) + (1-\vartheta)\varkappa_2(\overline{\rho})\right)\Big).\end{aligned}$$

Since Π has convex values, $\mathcal{R}_{\Pi,u}$ is convex and $\left(\vartheta\varkappa_1(\overline{\rho}) + (1-\vartheta)\varkappa_2(\overline{\rho})\right) \in \mathcal{R}_{\Pi,u}$. Thus, $\vartheta\widetilde{k}_1 + (1-\vartheta)\widetilde{k}_2 \in \widehat{\mathbb{Z}}_2(u)$.

Step 3. $\widehat{\mathbb{Z}}_2$ maps bounded sets in $\mathfrak{C}$ to bounded sets.

Let $r > 0$ and $B_r = \{u \in \mathfrak{C} : \|u\| \le r\}$ be a bounded set in $\mathfrak{C}$. Then for each $\widetilde{k} \in \widehat{\mathbb{Z}}_2(u)$ and $u \in B_r$, there exists $\varkappa \in \mathcal{R}_{\Pi,u}$ such that

$$\widetilde{k}(\tau) = I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau) + \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\Big(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa(\overline{\rho}) - \ell_1(\overline{\rho})u(\overline{\rho}) - \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\overline{\rho})\Big).$$

Under the assumption (As02) and for any $\tau \in \mathfrak{J}_{\overline{\rho}}$, we attain

$$\begin{aligned}&\left|\widetilde{k}(\tau)\right|\\ &\le I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\left|\ell_1(\tau)\right|\left|u(\tau)\right| + I^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\left|\varkappa(\tau)\right|\\ &+ \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{|\Lambda|\Gamma(\mathfrak{e}_3+\delta)}\Big(I^{\mathfrak{e}_1,\phi}\left|\varkappa(\overline{\rho})\right| + \left|\ell_1(\overline{\rho})\right|\left|u(\overline{\rho})\right|\\ &+ |\beta| I^{\mathfrak{e}_3,\phi}\left|\ell_1(\overline{\rho})\right|\left|u(\overline{\rho})\right| + |\beta| I^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\left|\varkappa(\overline{\rho})\right|\Big)\\ &\le \|\widetilde{w}_1\|\,\widetilde{w}_2(r)\left[\frac{\phi_{\overline{\rho}}^{\mathfrak{e}_1+\mathfrak{e}_3}}{\Gamma(\mathfrak{e}_1+\mathfrak{e}_3+1)} + \frac{\Gamma(\delta)\,\phi_{\overline{\rho}}^{\mathfrak{e}_1+\mathfrak{e}_3+\delta-1}}{|\Lambda|\Gamma(\mathfrak{e}_3+\delta)\Gamma(\mathfrak{e}_1+1)} + \frac{|\beta|\Gamma(\delta)\,\phi_{\overline{\rho}}^{\mathfrak{e}_1+2\mathfrak{e}_3+\delta-1}}{\Gamma(\mathfrak{e}_1+\mathfrak{e}_3+1)|\Lambda|\Gamma(\mathfrak{e}_3+\delta)}\right]\\ &+ r\left[\frac{\ell_1^*\,\phi_{\overline{\rho}}^{\mathfrak{e}_3}}{\Gamma(\mathfrak{e}_3+1)} + \frac{\Gamma(\delta)|\beta|\,\ell_1^*\phi_{\overline{\rho}}^{2\mathfrak{e}_3+\delta-1}}{|\Lambda|\Gamma(\mathfrak{e}_3+\delta)\Gamma(\mathfrak{e}_3+1)} + \ell_1^*\frac{\Gamma(\delta)\,\phi_{\overline{\rho}}^{\mathfrak{e}_3+\delta-1}}{|\Lambda|\Gamma(\mathfrak{e}_3+\delta)}\right],\end{aligned}$$

where $\ell_1^* = \sup_{\tau\in\mathfrak{J}_{\overline{\rho}}} |\ell_1(\tau)|$. Thus

$$\left\|\widetilde{k}\right\| \leq \theta_1 \|\widetilde{w}_1\| \widetilde{w}_2(r) + \theta_2 r.$$

Step 4. $\widehat{\mathbb{Z}}_2$ maps bounded sets in $\mathfrak{C}$ to equicontinuous sets.

Let $u \in B_r$ and $\widetilde{k} \in \widehat{\mathbb{Z}}_2(u)$. Then there exists a function $\varkappa \in \mathcal{R}_{\Pi,u}$ such that

$$\begin{aligned}\widetilde{k}(\tau) &= I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau)\\ &+ \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa(\overline{\rho}) - \ell_1(\overline{\rho})u(\overline{\rho})\right.\\ &\left.- \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\overline{\rho})\right), \ \tau\in\mathfrak{J}_{\overline{\rho}}.\end{aligned}$$

Let $\tau_1, \tau_2 \in \mathfrak{J}_{\overline{\rho}}$, $\tau_1 < \tau_2$. Then

$$\begin{aligned}&\left|\widetilde{k}(\tau_2) - \widetilde{k}(\tau_1)\right|\\ &\leq \frac{\ell_1^* r}{\Gamma(\mathfrak{e}_3+1)}\left((\phi(\tau_2)-\phi(\mathfrak{a}))^{\mathfrak{e}_3} - (\phi(\tau_1)-\phi(\mathfrak{a}))^{\mathfrak{e}_3}\right)\\ &+ \frac{\|\widetilde{w}_1\|\widetilde{w}_2(r)}{\Gamma(\mathfrak{e}_1+\mathfrak{e}_3+1)}\left((\phi(\tau_2)-\phi(\mathfrak{a}))^{\mathfrak{e}_1+\mathfrak{e}_3} - (\phi(\tau_1)-\phi(\mathfrak{a}))^{\mathfrak{e}_1+\mathfrak{e}_3}\right)\\ &+ \frac{\Gamma(\delta)}{|\Lambda|\Gamma(\mathfrak{e}_3+\delta)}\left((\phi(\tau_2)-\phi(\mathfrak{a}))^{\mathfrak{e}_3+\delta-1} - (\phi(\tau_1)-\phi(\mathfrak{a}))^{\mathfrak{e}_3+\delta-1}\right)\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa(\overline{\rho}) - \ell_1(\overline{\rho})u(\overline{\rho})\right.\\ &\left.+|\beta| I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho}) + |\beta| I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\overline{\rho})\right).\end{aligned}$$

As $\tau_1 \to \tau_2$, we obtain

$$\left|\widetilde{k}(\tau_2) - \widetilde{k}(\tau_1)\right| \to 0.$$

Hence $\widehat{\mathbb{Z}}_2(B_r)$ is equicontinuous. From the above-mentioned steps $3-4$ along with Arzela-Ascoli theorem, we deduce that $\widehat{\mathbb{Z}}_2$ is completely continuous.

Step 5. We prove that the graph of $\widehat{\mathbb{Z}}_2$ is closed.

Let $u_n \to u_*$, $\widetilde{k}_n \in \widehat{\mathbb{Z}}_2(u_n)$ and $\widetilde{k}_n$ tends to $\widetilde{k}_*$. We show that $\widetilde{k}_* \in \widehat{\mathbb{Z}}_2(u_*)$. Since $\widetilde{k}_n \in \widehat{\mathbb{Z}}_2(u_n)$, there exists $\varkappa_n \in \mathcal{R}_{\Pi,u_n}$ such that

$$\begin{aligned}\widetilde{k}_n(\tau) &= I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa_n(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u_n(\tau)\\ &+ \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa_n(\overline{\rho}) - \ell_1(\overline{\rho})u_n(\overline{\rho})\right.\\ &\left.- \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u_n(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa_n(\overline{\rho})\right), \ \tau\in\mathfrak{J}_{\overline{\rho}}.\end{aligned}$$

Therefore, we must prove that there exists $\varkappa_* \in \mathcal{R}_{\Pi,u_*}$ such that, for each $\tau \in \mathfrak{J}_{\overline{\rho}}$,

$$\widetilde{k}_*(\tau) = I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa_*(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u_*(\tau) + \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa_*(\overline{\rho}) - \ell_1(\overline{\rho})u_*(\overline{\rho}) - \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u_*(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa_*(\overline{\rho})\right), \ \tau \in \mathfrak{J}_{\overline{\rho}}.$$

Define the continuous linear operator $\Upsilon : L^1\left(\mathfrak{J}_{\overline{\rho}},\mathbb{R}\right) \to C\left(\mathfrak{J}_{\overline{\rho}},\mathbb{R}\right)$ as follows

$$\varkappa \to \Upsilon(\varkappa)(\tau) = I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau) + \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa(\overline{\rho}) - \ell_1(\overline{\rho})u(\overline{\rho}) - \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\overline{\rho})\right), \ \tau \in \mathfrak{J}_{\overline{\rho}}.$$

Notice that

$$\left\|\widetilde{k}_n - \widetilde{k}_*\right\| = \Big\| I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\left(\varkappa_n(\tau)-\varkappa_*(\tau)\right) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)\left(u_n(\tau)-u_*(\tau)\right) + \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\left(\varkappa_n(\overline{\rho})-\varkappa_*(\overline{\rho})\right) - \ell_1(\overline{\rho})\left(u_n(\overline{\rho})-u_*(\overline{\rho})\right) - \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})\left(u_n(\overline{\rho})-u_*(\overline{\rho})\right) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\left(\varkappa_n(\overline{\rho})-\varkappa_*(\overline{\rho})\right)\right) \to 0\Big\|,$$

when $n \to \in fty$. So in view of Lemma 6 that $\Upsilon \circ \mathcal{R}_{\Pi,u}$ is a closed graph operator. Moreover, we have $\widetilde{k}_n \in \Upsilon\left(\mathcal{R}_{\Pi,u_n}\right)$.

Since $u_n \to u_*$, Lemma 6 gives

$$\widetilde{k}_*(\tau) = I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa_*(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u_*(\tau) + \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa_*(\overline{\rho}) - \ell_1(\overline{\rho})u(\overline{\rho}) - \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u_*(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa_*(\overline{\rho})\right), \ \tau \in \mathfrak{J}_{\overline{\rho}},$$

for some $\varkappa_* \in \mathcal{R}_{\Pi,u_*}$.

Hence, we deduce that the multivalued operator $\widehat{\mathbb{Z}}_2$ is both u.s.c. and compact.

Step 6. We show that $2\widetilde{w}_1^*\Psi < 1$.

For $\Delta > 0$ and according to Step 2, we have

$$\Psi = \widehat{\mathbb{Z}}_2(\mathfrak{C}) = \sup\left\{\left|\widehat{\mathbb{Z}}_2(\mathfrak{C})\right|, \ u \in \mathfrak{C}\right\} \leq \theta_1\|\widetilde{w}_1\|\,\widetilde{w}_2(\Delta) + \theta_2\Delta.$$

In view of (12), we get

$$\widetilde{w}_1^*\theta_1\|\widetilde{w}_1\|\,\widetilde{w}_2(\Delta) + \theta_2\Delta < \frac{1}{2}.$$

From this, it follows that $2\widetilde{w}_1^*\Psi < 1$.

Hence, the operators $\widehat{\mathbb{Z}}_1$ and $\widehat{\mathbb{Z}}_2$ fulfill the assumptions stated in (1). Consequently, it follows that either (i) or (ii) will be true. We now proceed to show that (ii) is not possible.

Let $u \in \Theta$ be such that $\|u\| = \Delta$. Then $\xi u(\tau) \in \left(\widehat{\mathbb{Z}}_1 u\right)(\tau)\left(\widehat{\mathbb{Z}}_2 u\right)(\tau)$ for all $\xi > 1$. Choose $\varkappa \in \mathcal{R}_{\Pi,u}$. Then $\forall \xi > 1$, we have

$$\begin{aligned} u(\tau) = \frac{h(\tau,u(\tau))}{\xi}\Big[& I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\tau) - I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\tau)u(\tau) \\ & + \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{\Lambda\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}\varkappa(\overline{\rho}) - \ell_1(\overline{\rho})u(\overline{\rho})\right. \\ & \left. - \beta I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\ell_1(\overline{\rho})u(\overline{\rho}) + \beta I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}\varkappa(\overline{\rho})\right)\Big], \ \forall\tau \in \mathfrak{J}_{\overline{\rho}}. \end{aligned}$$

Thus, we can write

$$\begin{aligned} u(\tau) & \\ \leq & \frac{|h(\tau,u(\tau)) - h(\tau,0)| + |h(\tau,0)|}{\xi}\Big[I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}|\ell_1(\tau)||u(\tau)| + I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}|\varkappa(\tau)| \\ & + \frac{\Gamma(\delta)\left(\phi(\tau)-\phi(\mathfrak{a})\right)^{\mathfrak{e}_3+\delta-1}}{|\Lambda|\Gamma(\mathfrak{e}_3+\delta)}\left(I_{\mathfrak{a}+}^{\mathfrak{e}_1,\phi}|\varkappa(\overline{\rho})| + |\ell_1(\overline{\rho})||u(\overline{\rho})|\right. \\ & \left. + |\beta| I_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}|\ell_1(\overline{\rho})||u(\overline{\rho})| + |\beta| I_{\mathfrak{a}+}^{\mathfrak{e}_1+\mathfrak{e}_3,\phi}|\varkappa(\overline{\rho})|\right)\Big] \\ \leq & \left(\widetilde{w}_1^*\Delta + h^*\right)\left(\theta_1\|\widetilde{w}_1\|\widetilde{w}_2(\Delta) + \theta_2\Delta\right), \end{aligned}$$

where $h^* = \sup_{\tau\in\mathfrak{J}_{\overline{\rho}}}|h(\tau,0)|$. Therefore, we obtain

$$\Delta \leq \left(\widetilde{w}_1^*\Delta + h^*\right)\left(\theta_1\|\widetilde{w}_1\|\widetilde{w}_2(\Delta) + \theta_2\Delta\right).$$

In view of condition (11), we observe that (ii) cannot hold. Thus, $u(\tau) \in \left(\widehat{\mathbb{Z}}_1 u\right)\left(\widehat{\mathbb{Z}}_2 u\right)$. Therefore, it follows from Theorem 1 that the problem (3) admits at least one solution. □

4 Example

To demonstrate the applicability of our theoretical results, we now examine a particular case of hybrid Langevin FDI's.

Consider the general hybrid Langevin inclusion problem

$$\begin{cases} {}^{H}D_{\mathfrak{a}+}^{\mathfrak{e}_1,\mathfrak{e}_2,\phi}\left({}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_3,\phi}\left(\dfrac{u(\tau)}{h(\tau,u(\tau))}\right) + \ell_1(\tau)u(\tau)\right) \in \Pi(\tau,u(\tau)), \tau \in (\mathfrak{a},\overline{\rho}), \\ \dfrac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))} = 0, \ D_{\phi}^{1}\left(\dfrac{u(\mathfrak{a})}{h(\mathfrak{a},u(\mathfrak{a}))}\right) = 0, \\ {}^{C}D_{\mathfrak{a}+}^{\mathfrak{e}_2,\phi}\left(\dfrac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}\right) + \beta\left(\dfrac{u(\overline{\rho})}{h(\overline{\rho},u(\overline{\rho}))}\right) = 0. \end{cases} \tag{14}$$

By taking $\phi(\tau)=\tau$, $\mathfrak{e}_1=\frac{1}{2}$, $\mathfrak{e}_2=\frac{1}{4}$, $\mathfrak{e}_3=\frac{3}{2}$, $\ell_1(\tau)=\tau^2$, $\beta=\frac{1}{6}$, $\mathfrak{a}$=0 and $\overline{\rho}=1$ in (14). Then, the problem (14) reduces to

$$\begin{cases} {}^{H}D_{0+}^{\frac{1}{2},\frac{1}{4},\tau}\left({}^{C}D_{0+}^{\frac{3}{2},\tau}\left(\frac{u(\tau)}{h(\tau,u(\tau))}\right)+\tau^2 u(\tau)\right)\in\Pi(\tau,u(\tau)),\ \tau\in(0,1),\\ \frac{u(0)}{h(0,u(0))}=0,\ D_{\tau}^{1}\left(\frac{u(0)}{h(0,u(0))}\right)=0,\\ {}^{C}D_{0+}^{\frac{3}{2},\tau}\left(\frac{u(1)}{h(1,u(1))}\right)+\frac{1}{6}\left(\frac{u(1)}{h(1,u(1))}\right)=0, \end{cases} \tag{15}$$

Using these parameter values, we compute $\Lambda=-1.2257\neq 0$. We then define the function h and multi-function $\Pi:[0,1]\times\mathbb{R}\to\mathcal{F}(\mathbb{R})$ as follows:

$$h(\tau,u)=\frac{\cos(u)}{\exp(t^2)+35},\ \forall(t,\varphi)\in[0,1]\times\mathbb{R},$$

and

$$\Pi(\tau,u)=\left[\frac{1}{(\tau^3+8\exp(\tau^2))}\frac{u^2}{6(u^2+2)},\frac{1}{\sqrt{\tau+4}}\frac{|u|}{|u|+1}\right].$$

For $u_1,u_2\in\mathbb{R}$, we have

$$\begin{aligned}|h(\tau,u_1)-h(\tau,u_2)| &= \left|\frac{1}{\exp(t^2)+35}(\cos(u_1)-\cos(u_1))\right|\\ &\leq \frac{1}{\exp(t^2)+35}|u_1-u_2|\leq\frac{1}{36}|u_1-u_2|.\end{aligned}$$

Therefore the assumption (As01) holds with $\widetilde{w}_1(\tau)=\frac{1}{\exp(t^2)+35}$, which gives $\widetilde{w}_1^*=\frac{1}{36}$.

The multi-function Π clearly fulfills assumption (As02). Moreover, we have

$$\|\Pi(\tau,u)\|_{\mathcal{F}}=\sup\{|\xi|:\xi\in\Pi(\tau,u)\}\leq\frac{1}{\sqrt{\tau+4}}=\widetilde{w}_1(\tau)\widetilde{w}_2(\|u\|),$$

where $\|\widetilde{w}_1\|=\frac{1}{2}$ and $\widetilde{w}_2(\|u\|)=1$. By choosing $\Delta=\frac{1}{8}$, we have

$$\Delta>\left(\widetilde{w}_1^*\Delta+h^*\right)(\theta_1\|\widetilde{w}_1\|\widetilde{w}_2(\Delta)+\theta_2\Delta)\approx 0.036087,$$

which means that (11) holds. The condition '

$$\widetilde{w}_1^*\theta_1\|\widetilde{w}_1\|\widetilde{w}_2(r)+\theta_2\Delta\approx 0.27457<\frac{1}{2},$$

is satisfied. Since all assumptions of Theorem 2 hold, the hybrid Langevin inclusion problem (15) possesses at least one solution on the interval [0, 1].

5 Concluding Remarks

Generalized fractional operators extend classical fractional calculus by employing more general kernel functions. These operators, combined with fixed point theory, provide a robust framework for analyzing the qualitative behavior of solutions to certain fractional dynamical systems, especially those modeling complex and chaotic phenomena. In this study, we investigated existence results for a hybrid Langevin FDI involving a power law-type generalized kernel. We transformed the inclusion problem into an equivalent fixed-point formulation using an operator constructed from the integral solution and then applied Dhage's theorem for set-valued mapping to obtain the main existence outcomes. The theoretical result was supported by a demonstrative example. This approach can be applied to mathematical modeling in science, engineering, and real-world phenomena [43,44].

In future work, we plan to extend the problem presented in this article to a general structure by using the Mittag-Leffler power law [45] and fractal-fractional operators [46]. We also aim to investigate other qualitative properties such as Ulam–Hyers stability and controllability for certain classes of stochastic fractional differential equations involving fractional derivatives with variable orders.

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