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Super-elements and flexibility methods exploiting hybrid equilibrium formulations.

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ABSTRACT

With the goal of formulating a hybrid equilibrium hexahedral element free from spurious kinematic modes, we develop in this paper an analogous quadrilateral element in 2D in the context of solid mechanics. The 2D element extends the original concept of a macro-element from 4 triangular primitives to 8 whilst retaining a single internal vertex. The associated spurious modes are identified and sufficient conditions for tractions to be admissible are proposed.

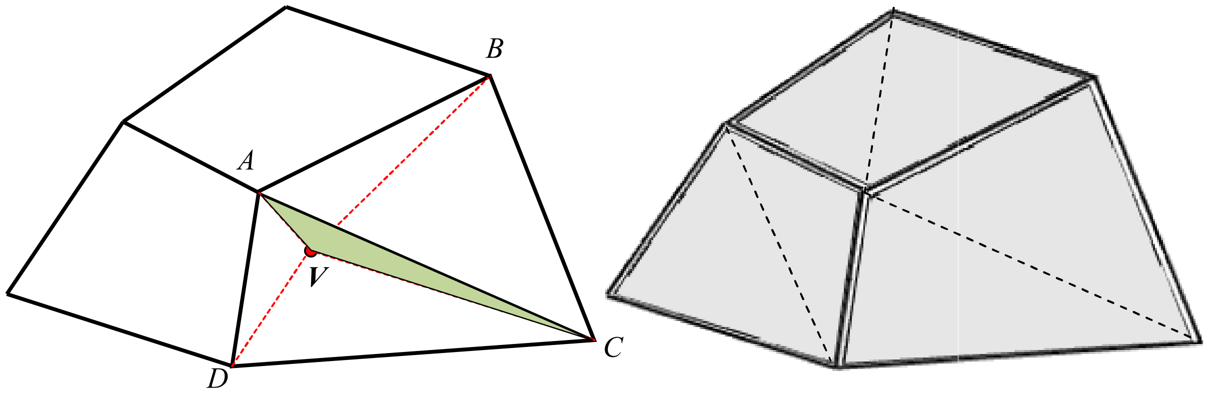
Numerical tests are reported for the single element of degree 3. The concept of cover entities are introduced as a means of controlling the spurious modes. A super-element is then defined as the macro-element with its covers.

A problem of load dispersion is presented to illustrate a mesh of super-elements, where the degree of the underlying macro-elements is 3, but the degree of the cover tractions is allowed to vary. In order to take advantage of the flexibility matrices of the super-elements, which appear directly in their formulation, a flexibility method of analysing the mesh is described.

1. INTRODUCTION

This work was inspired by the desire for a hybrid equilibrium hexahedral element without the curse of spurious kinematic modes [1]. A similar desire led to the earlier paper concerning “the exorcism of an old curse” for 2D elements composed of triangular primitives [2]. Later studies of 3D elements have revealed a more complicated curse for macro-elements in the form of hexahedra (with plane faces) composed of tetrahedra [3,4]. However certain configurations exist, e.g. vertex centred star patches of 12 or 24 primitive tetrahedra,where the spurious kinematic modes are restricted to the separate faces, and each face is a quadrilateral subdivided into 2 or 4 triangular faces of tetrahedra [3].

Figure 1(a) illustrates a hexahedron with internal vertex *V*and 2 of the 12 tetrahedra that share a diagonal edge *AC* in face *ABCD*. In Figure 1(b) each face is “covered” so that only admissible tractions are imposed to avoid exciting the spurious modes.

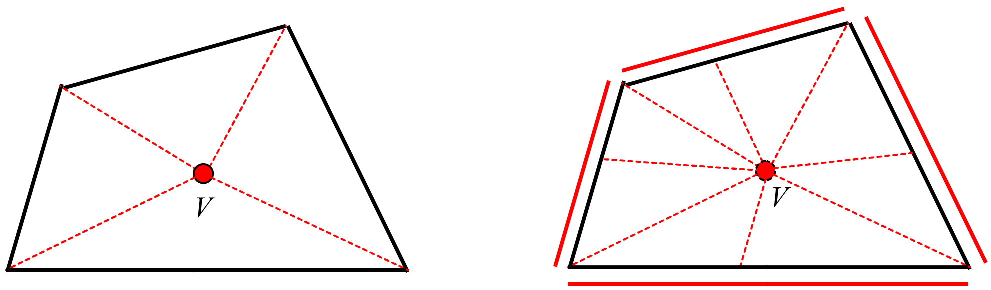


(a) tetrahedra *VACD* and *VACB* on face*ABCD*(b)covers over the flat faces.
Figure 1: Hexahedron with flat faces subdivided into 12 tetrahedra

The long-term goal is to formulate a stable (free from spurious kinematic modes) hybrid equilibrium hexahedral element. An analogous element in 2D, for plane elasticity, is the quadrilateral subdivided with one internal vertex into 8 primitive triangles. The aim in this paper is to study the formulation and feasabilityof such an element.A methodology is detailed and illustrated with a “proof of concept”.

2. FROM MACRO-ELEMENT TO SUPER-ELEMENT FOR A QUADRILATERAL

The conventional way to formulate a hybrid equilibrium quadrilateral element, as a macro-element, is to assemble four triangular elements (primitives) as illustrated in Figure 2(a) [2]. In this case there are no spurious modes involving external sides. A more refined macro-element subdivides each primitive into two as illustrated in Figure 2(b). However, this element now allows a spurious mode to exist for each side of the quadrilateral [3]. It will be referred to as a “union jack” macro, and from the kinematic stability point of view it forms a useful analogy to the hexahedral macro-element.



(a) 4 primitive elements (b) 8 primitive elements with 4 covers
Figure 2: Two forms of quadrilateral macro-elements

It is convenient to consider the undivided primitives, as illustrated in Figure 2(a), as quasi-primitives in the union jack macro-element. A quasi-primitive then has four sides but two of them are colinear.

For present illustrative purposes, the degree of the primitives is assumed to be 3. Thena primitive has a basis of 18 stress fields (none of which are hyperstatic), whereas the quasi-primitive has a basis of 28 stress fields.A basis exists in which 8 of these fields are fully continuous and the remaining 20 are partially continuous but nevertheless are statically admissible. Two of the fields are hyperstatic which implies some enrichment over the simple primitive. The quasi-primitive has 32 kinematic degrees of freedom and 3 potential spurious kinematic modes. The mode illustrated in Figure 3is the one that involves the pair of colinear sides.

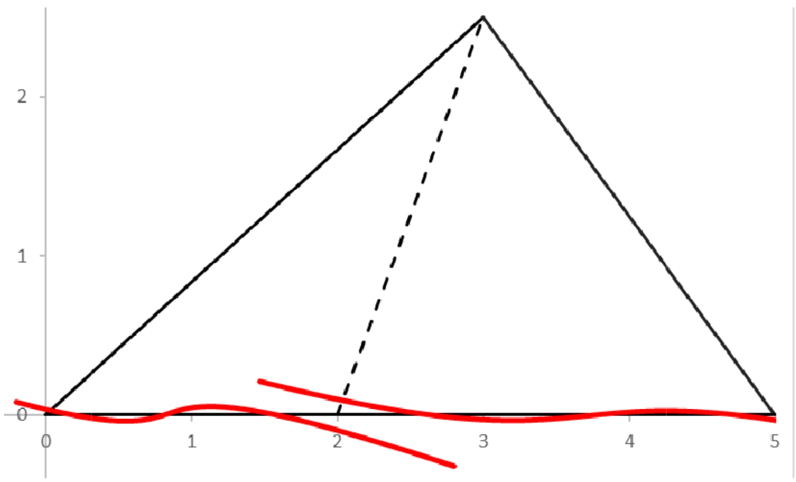


Figure 3: A quasi-primitive and the cubic spurious kinematic mode on its colinear sides.

The hybrid equations for a quasi-primitive take the usual form [4] as in Equation (1),

$$\begin{bmatrix} -\boldsymbol{F} & \boldsymbol{D}^T \\ \boldsymbol{D} & \boldsymbol{0} \end{bmatrix} \begin{Bmatrix} \hat{\boldsymbol{s}} \\ \hat{\boldsymbol{v}} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{0} \\ \hat{\boldsymbol{t}} \end{Bmatrix} \tag{1}$$

from which a stiffness matrix may be formed after eliminating the stress parameters $\hat{\boldsymbol{s}}$. It is then possible to assemble the macro-element stiffness equations using the stiffness matrices of the 4 quasi-primitives. For present purposes loading is restricted to tractions applied to the 8 external sides, and so the internal displacement degrees of freedom are also eliminated to leave the 64×64 stiffness matrix $\boldsymbol{K}_{ex}$ .Due to the 4 spurious kinematic modes and 3 rigid body modes, this matrix is rank deficientby 7.

$$\boldsymbol{K}_{ex}\hat{\boldsymbol{v}}_e = \hat{\boldsymbol{t}}_e \tag{2}$$

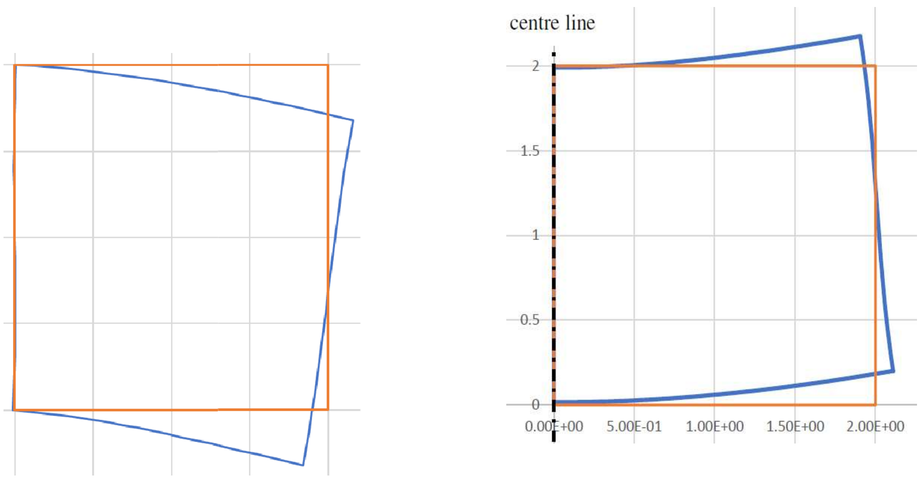
The stiffness equations in Equation (2) use suffix *e* to denote modes of displacement of the external sides $\hat{\boldsymbol{v}}_e$ and external modes of traction $\hat{\boldsymbol{t}}_e$.

Numerical tests were carried out for a single macro-element to verify the formulation and the software with problems having known exact solutions. In these tests the 3 rigid body movements were constrained and the applied tractions were admissible, i.e. they satisfied global equilibrium and did not excite the spurious modes. Two problems are presented here with exact solutions [5]:

(i) a deep cantilever with a constant parabolic shear force (having a quadratic stress field and a cubic displacement field), and

(ii) a simply supported deep beam with a uniformly distributed load on the top surface and a linearly varying parabolic shear force (cubic stress field and a quartic displacement field).

In problem(i), the theoretical displacements are cubic, and the sides of the single element conform exactly with them. In problem(ii) the stress fields agree with the Timoshenko solution [5] with cubic distributions of normal tractions over the supported face. However, the displacement fields are piecewise cubic and so they do not conform exactly with the theoretical solution. The computed displacements of both problems are presented in Figure 4, where the incompatibilities in problem (ii) are hardly noticeable.



(i) Deep cantilever with an end force(ii) Deep beam loaded on the top face
Figure 4: external side displacements of a square macro-element of side length 2.

3. THE CONCEPT OF A SUPER-ELEMENT

One way to avoid exciting the spurious kinematic modes is to ensure that only admissible tractions are applied to the external sides of the macro-element. It is thus proposed to define cover entities, one for each pair of colinear sides, and associate dual traction and displacement modes to them. A sufficient, but not necessary, condition for admissibility is that these tractions be continuous along a cover. A macro-element with its covers is termed a super-element. The underlying macro should be stable apart from spurious kinematic modes involving the colinear sides that identify with a cover.Alternative ways to impose stability of the macro would be to block the spurious modes with fictitious springs [4], or apply kinematic constraints as proposed by Parrinello [6].

Cover tractions,with parameters in vector $\hat{\boldsymbol{a}}$, are directly projectedwithout changing pointwise values onto the corresponding sides of the macro-element. This projection is represented by Equation (3) where, in the case of using degree 3 for the macro and its covers, the dimension of $\boldsymbol{A}$ is 64×32.

$$\hat{\boldsymbol{t}}_e = \boldsymbol{A}\hat{\boldsymbol{a}} \tag{3}$$

Excluding rigid body displacements from one of the covers, the remaining dual displacement modes of the covers $\hat{\boldsymbol{w}}$ can be defined by Equation (4) using the pseudoinverse $\boldsymbol{K}_{ex}^{+}$:

$$\hat{\boldsymbol{w}} = \boldsymbol{A}^T \hat{\boldsymbol{v}}_e \text{ and } \hat{\boldsymbol{v}}_e = \boldsymbol{K}_{ex}^{+}\hat{\boldsymbol{t}}_e \text{ or } \hat{\boldsymbol{w}} = \boldsymbol{G}_s\hat{\boldsymbol{a}} \text{ where } \boldsymbol{G}_s = \boldsymbol{A}^T\boldsymbol{K}_{ex}^{+}\boldsymbol{A} \tag{4}$$

$\boldsymbol{G}_s$ is a form of flexibility matrix for the super-element, which could be used to analyse a system of super-elements using a flexibility method, or alternatively its inverse could be used in the more usual stiffness method. The dual nature of the relations implies the work equivalence as in Equation (5).

$$\hat{\boldsymbol{a}}^T \hat{\boldsymbol{w}} = \hat{\boldsymbol{a}}^T \boldsymbol{A}^T \hat{\boldsymbol{v}}_e = \hat{\boldsymbol{t}}_e^T \hat{\boldsymbol{v}}_e \tag{5}$$

It should be noted that (i) in practise we can identify the bases of $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{w}}$ with Legendre polynomials, and (ii)reference to the pseudoinverse matrix is for conciseness,and does not imply the best computational procedure.The mappings of tractions and displacements from and onto the covers in Equations (3) and (4) are illustrated in Figure 5.

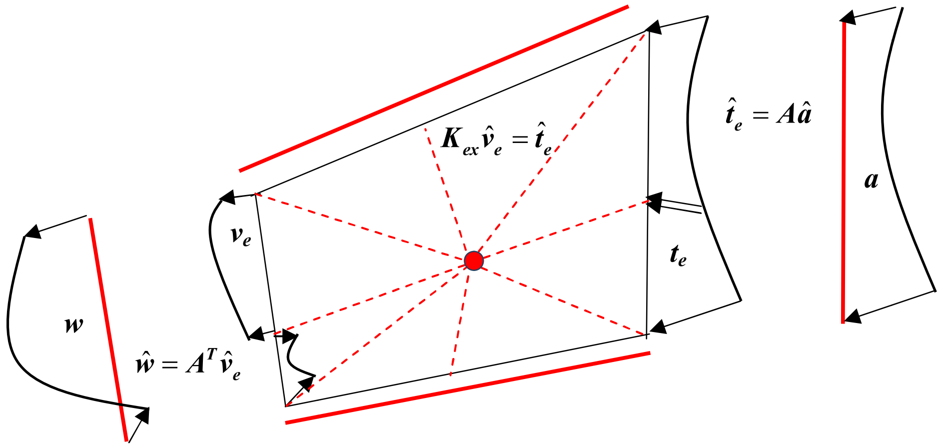


Figure 5: diagrammatic view of dual maps from and to covers of a super-element.

4. PROOF OF CONCEPT

A problem of load dispersion through a block is considered as a “proof of concept”. A square domain of side length 2 and unit thickness is illustrated in Figure 6 modelled by a uniform mesh of 4 super elements, and Figure 7 displays the underlying macro-elements with a total of 32 triangular primitives of degree 3.

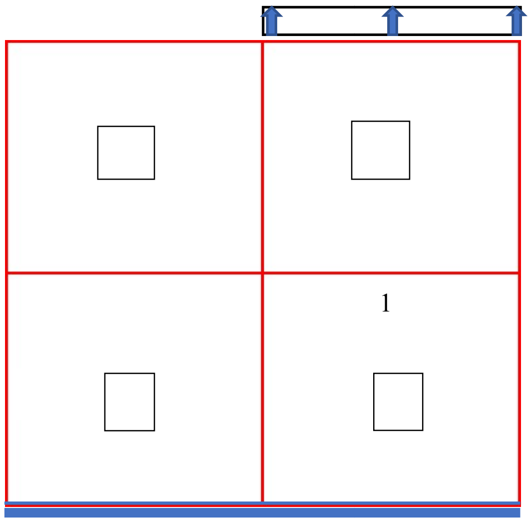


Figure 6: super-element model

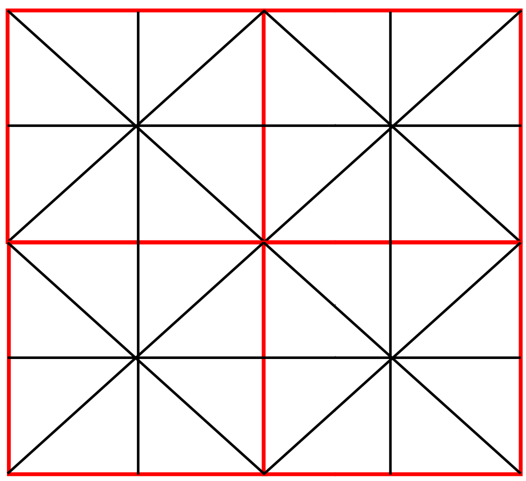


Figure 7: underlying meshes in super-elements

The domain is loaded by a constant unit mode of traction applied to the top cover of element 3, and the bottom covers of elements 2 and 4 are fixed. Fixity for this model implies that all modes of cover displacement up to the degree of a cover are zero. Young’s modulus $E = 100$, Poisson’s ratio $v = 0.3$.

As equilibrium models, the energy of a force driven problem is expected to be greater than the exact theoretical value, and it is observed in Table 1 that the energy decreases as the degree of the covers increases. It should be noted that the basic modes of traction are the 3 linear modes: constant normal and tangential modes together with a linear normal mode having a resultant moment. Then the complete linear modes includes the self-balancing linear tangential mode. The higher degree modes add further self-balancing traction distributions along a cover.

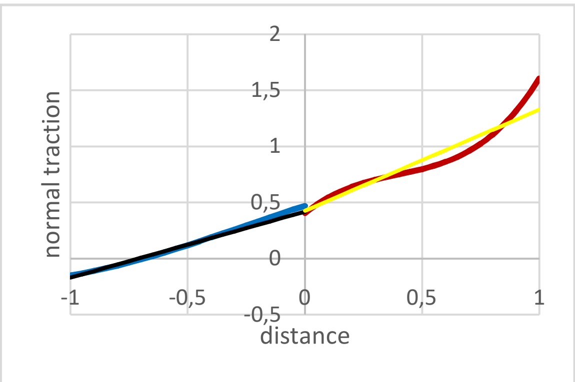
Model 0 to 5	Strain energy × 1000
Model 0: covers with basic modes of traction	8.981039
Model 1: covers withfull linear modes of traction	8.900991
Model 2: covers with quadratic modes of traction	8.891849
Model 3: covers with cubic modes of traction	8.889394
Model 4: macro-elements of degree 3	8.887848
Model 5: reference solution	8.883919

Table 1: strain energies of the solutions for different models

The use of the basic modes of traction, and their dual modes of displacement, may be particularly advantageous when the main goal of analysis is to establish load paths through a structure. This aspect requires further study to establish an appropriate balance between the fineness of the mesh and the quality of the load path.

It is observed that for model4 with macro-elements of degree 3, the energy is slightly reduced compared with that of model 3. This reduction may be explained by referring back to the statement that it is assumed for admissibility purposes that cover tractions are fully continuous, but this is a sufficient and not a necessary condition. In the solution using macro-elements this assumption is not made and other discontinuous codiffusive tractions may exist, thus raising the possibility of further enriching the stress fields.
It should also be noted that the geometric regularity of the macro-mesh of model (4) means that this mesh contains 10 potential spurious kinematic modes which make the global stiffness matrix singular and rank deficient by 10. This deficiency has to be accounted for in the solution algorithm.

Solutions for the reactive tractions on the bottom faces of elements 2 and 4 are shownin Figures 8(a) and (b)in order to compare models 0 and 3.



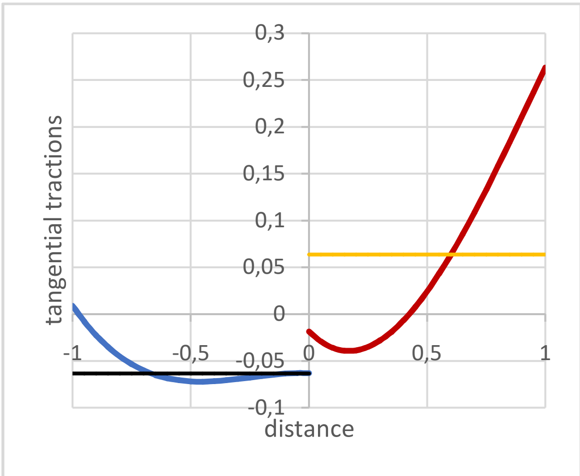


Figure 8(a): normal tractionsFigure 8(b): tangential tractions
In both Figures the tractions are from covers on face 3 of elements 2 and 4. The blue and brown lines are with cover degree 3, and the black and yellow lines are with basic covers.

5. A SUMMARY OF A FLEXIBILITY METHOD OF ANALYSIS.

The availability of a flexibility matrix provides the option of mesh analysis using a flexibility method [7] which can lead to a significant reduction in the dimension of the equations for a system.

The analysis starts with generating statically admissible fields of traction as described by Equation (6),

$$\hat{\boldsymbol{a}} = \hat{\boldsymbol{a}}^{0} + \boldsymbol{Bx} \tag{6}$$

where $\hat{\boldsymbol{a}}^{0}$ is a particular vector of element tractions that equilibrate with the prescribed load,and hyperstatic parameters in vector $\boldsymbol{x}$are transformed by matrix$\boldsymbol{B}$ to element actions in vector $\hat{\boldsymbol{a}}$. Then compatibility of displacements is expressed in the form of Equation (7).

$$\boldsymbol{B}^{T}\boldsymbol{G}_{s}\left(\boldsymbol{Bx}+\hat{\boldsymbol{a}}^{0}\right)=\boldsymbol{0} \text{ or } \boldsymbol{Gx}+\hat{\boldsymbol{e}}^{0}=\boldsymbol{0} \text{ where } \boldsymbol{G}=\sum_{e}\boldsymbol{B}_{h}^{e^{T}}\boldsymbol{G}_{s}^{e}\boldsymbol{B}_{h}^{e} \text{ and } \hat{\boldsymbol{e}}^{0}=\boldsymbol{B}^{T}\boldsymbol{G}_{s}\hat{\boldsymbol{a}}^{0}\ . \tag{7}$$

Matrix $\boldsymbol{G}_{s}$ is in block diagonal form with element flexibility submatrix $\boldsymbol{G}_{s}^{e}$ on the diagonal. The system flexibility matrix $\boldsymbol{G}$ is summed for all elements *e* where matrix $\boldsymbol{B}_{h}^{e}$ transforms the hyperstatic parameter*h* in vector $\boldsymbol{x}$ to the cover tractions for element *e*.

In this example, hyperstatic fields are defined as two types: (i) involving basic modes of traction that are transmitted at interfaces between adjacent covers and follow a path around a common vertex, and (ii) involving higher degree self-equilibrating modes of traction at an interface. Type (i) is illustrated in Figure 9 by the blue and yellow paths, where the yellow path includes the ground element. In general these paths can easily be defined using the topological and geometrical properties of a mesh.

A particular stress field is associated with a path that connects a loaded face of an element to a ground element. This is defined for example by the green route through elements 3 and 4 in Figure 9.

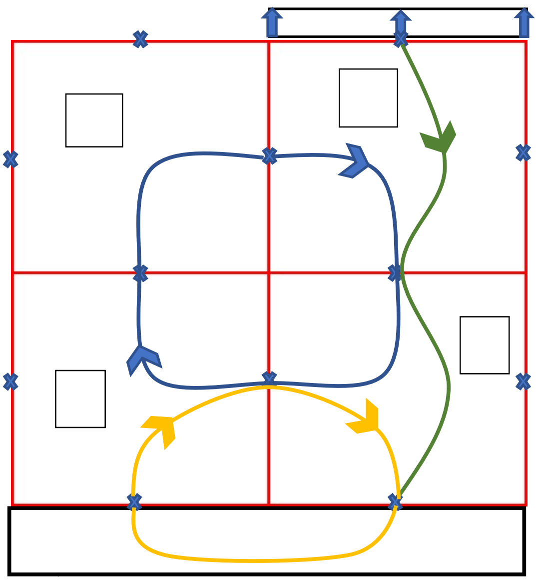


Figure 9: Topological definition of force paths for basic traction modes

In this example dimension $\boldsymbol{x}$ increases from 3×2 = 6 for model 0, to 3×2 + 5×6 = 36 for model 3. On the other hand a stiffness method would have 3×10 = 30 and 8×10 = 80 kinematic variables for the corresponding models of super-elements.

6. CONCLUSIONS

- The underlying macro-element configuration of a quadrilateral super-element has been successfully tested to provide the correct stress fields for problems where the exact solutions are knownto be of degree ≤ 3.
- When super-elements are assembled in a mesh, they can provide good quality solutions for internal stress fields whilst the degree of interaction with adjacent elements via their covers can be reduced to a minimal level to quantify stress-resultants. This should be a useful feature for

generating an understanding of load paths in early stages of design.

- A flexibility method of analysing a mesh of super-elements has been demonstrated to be feasible for 2D models. Further studies are required to realise possible benefits in 3D meshes.
- The study reported in this paper has provided a proof of concept regarding the formulation of super-elements in 2D, and future research will address a similar formulation for hexahedral elements.

7. REFERENCES

[1] E.A.W. Maunder and J.P. Moitinho de Almeida,"Towards a hexahedral hybrid equilibrium macro-element", ADMOS 2019, 27-29 May, 2019, El Campello.

[2] Maunder EAW, Moitinho de Almeida JP, Ramsay ACA, A general formulation of equilibrium macro-elements with control of spurious kinematic modes: the exorcism of an old curse, *International Journal for Numerical Methods in Engineering*, 39, 1996:3175-3194.

[3] Maunder EAW, Moitinho de Almeida JP, Almeida Pereira OJB, The stability of stars of simplicial hybrid equilibrium finite elements for solid mechanics, *International Journal for Numerical Methods in Engineering*, 107, 2016:633-668.

[4] J.P.Moitinho de Almeida and E.A.W.Maunder, *Equilibrium Finite Element Formulations*, Wiley,2017.

[5] Timoshenko S, Goodier JN, *Theory of Elasticity*, McGraw-Hill, 2nd edition, 1951

[6] F.Parrinello, Restraining approach for the spurious kinematic modes in hybrid equilibrium element, *Computational Mechanics*, 52, 2013: 885-901.

[7] Maunder EAW, Savage GJ, A graph-theoretic model for finite elements with side variables, *Civil Eng. Sys*. 11, 1994:111-141.