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Mathematical Models for Thermally Coupled Low Speed Flows

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Abstract

In this paper we review and clarify some aspects of the asymptotic analysis of the compressible Navier Stokes equations in the low Mach number limit. In the absence of heat exchange (the isentropic regime) this limit is well understood and rigorous results are available. When heat exchange is considered different simplified models can be obtained, the most famous being the Boussinesq approximation. Here a unified formal justification of these models is presented, paying special attention to the relation between the low Mach number and the Boussinesq approximations. Precise conditions for their validity are given for classical problems in bounded domains.

Keywords: Low Mach number flows, Boussinesq approximation, Asymptotic analysis

1 Introduction

It is widely accepted that many flows of interest can be considered as incompressible. This assumption is useful as it makes the problem much simpler than if a full compressible flow is considered. For ideal fluids, in the absence of heat sources (the isentropic case), solutions of the incompressible Navier Stokes equations can be found as the limit of solutions of the compressible ones as the Mach number tends to zero under certain assumptions on the initial data. Rigorous mathematical results of this type were established in [14] (see also [20]).

When heat exchange is taken into account, the limit is quite different, since the energy equation is not uncoupled and one needs to keep the state equation to close the system. The small Mach number limit gives rise to a splitting of the pressure into a constant-in-space thermodynamic pressure p^{th} and a mechanical pressure p that has to be used in the momentum equation. This leads to a removal of the acoustic modes and the flow behaves as incompressible, in the sense that the mechanical pressure is determined by the mass conservation equation and not by the state equation. However, large variations of density due to temperature variations are allowed. This limit has been studied first in [30] in the inviscid case, and generalized to the viscous case in [28]. A rigorous derivation including combustion was presented in [21]. This zero Mach number model has also been presented in [10] and in [35]. The numerical implications of this limit have been studied in [15] and in [25], for example.

However the most widely used model in the context of thermally coupled flows is the so called Boussinesq approximation proposed in [6]. Since that moment, many authors have looked for a formal justification of the Boussinesq approximation. Expressing the thermodynamic variables as the sum of a constant static part and a fluctuating part resulting from the motion, it was shown in [32] that for a thin layer of fluid (thin in comparison to the scale of variation of the static fields) the Boussinesq approximation follows. However density variations are retained in the momentum equation even when they are of higher order based on physical arguments and not on a limiting process. The first attempt to present a rigorous derivation of the Boussinesq approximation was performed in [24] introducing an expansion in two parameters, ε_1 and ε_2, of the full compressible equations. The Boussinesq approximation is found to the lowest order in both ε_1 and ε_2. Several problems of this approach are described in [29]. On the one hand, $\varepsilon_1 \sim 10^{-4}$ and $\varepsilon_2 \sim 10^{-11}$ for typical fluids in a standard Rayleigh Bénard experiment, indicating a second order approximation for ε_1 to have the same order as a first order approximation for ε_2. On the other hand the starting point of Mihaljan's approach is the compressible equations but using an equation of state that relates temperature (ϑ) to density (ρ) only. As noted in [4] and in [3] if $\rho = \rho(\vartheta)$ convexity inequalities on thermodynamic potentials are violated. The Mihaljan's approach was improved first by Malkus (in an unpublished work mentioned in [29]) and in [29]. The new ingredient was the selection of an appropriate reference state. In [12] a derivation of the Boussinesq equations was presented taking a reference state into account and allowing temperature and pressure dependent properties. We note that all these works are concerned with natural convection as the velocity is made dimensionless using another variable as scale (viscosity or gravity, for example). An asymptotic justification of the Boussinesq approximation was developed in [36, 38, 39] and in [4, 5]. These developments dealt first with polytropic gases (in [36] and [4]) and the main conclusion was that the Mach number is a small parameter in the Boussinesq approximation. An asymptotic derivation of the Boussinesq approximation for liquids was then presented in [37]. Finally a unified approach for liquids and gases was presented in [5]. Another widely used model, the anelastic approximation, was proposed by [1] and [27] and has been used for a long time in the context of atmospheric flows (see also [9] and [11]). This approximation removes the height limitation present in Boussinesq's model.

A detailed analysis of the many aspects of low Mach number asymptotics was recently presented in [40]. Apart from the standard M^2 expansion, these aspects include singularities of this limit such as the behavior of a flow in the vicinity of the initial time or in the far field. In particular, the problem of slow motions in the atmosphere, analyzed in chapter 6, and the Rayleigh-Bénard problem, analyzed in chapter 7, are described by the Boussinesq equations, but the asymptotic analysis is different in both cases. This is due to the choice of the scales that define the flow and to the definition of a reference state about which the perturbative scheme is developed. The selection of a basic flow state and the analysis of the resulting perturbation problem is a useful procedure when a stability analysis is performed, but it is unnecessary, and in some sense confusing, when simplified models are derived by asymptotic analysis. In this work we present a unified asymptotic analysis with two particular ingredients that permit its application to different particular problems. In a first place, we do not introduce any reference state "a priori", but instead we find the reference state as a consequence of the limit. Secondly, we discuss the choice of scales after the asymptotic development, which allows us to understand different problems in the same context.

In the next section we formulate the problem and in section 3 we present a low Mach number asymptotic analysis of the Navier Stokes equations. We conclude with some remarks in section 4.

2 Problem definition

The flow of a compressible fluid in a domain Ω is described in terms of the velocity ($\boldsymbol{u}$), pressure (p), density, and temperature fields (bold characters are used to denote vectors and tensors). These fields are solutions of the compressible Navier Stokes equations that describe the dynamics of the system and that are statements of conservation of mass, momentum and energy and a state equation relating the thermodynamic variables, described in [2] and [19]. These equations of motion can be written in dimensionless form as stated by the π theorem proved in [7]. If we have r different units and we take n reference values, we have $n-r$ dimensionless numbers defining classes of similar solutions. Our approach is based on taking a different scale for each field and for each dependent property introducing the Strouhal, Mach, Reynolds, Péclet, Froude, heat release and temperature variation numbers, defined as

$$\mathrm{S} = \frac{l_0}{u_0 t_0}, \quad \mathrm{M} = \frac{u_0}{\sqrt{p_0/\rho_0}}, \quad \mathrm{R} = \frac{\rho_0 u_0 l_0}{\mu_0}$$
$$\mathrm{P} = \frac{\rho_0 c_{p_0} u_0 l_0}{k_0}, \quad \mathrm{F} = \frac{u_0}{\sqrt{g_0 l_0}}, \quad \mathrm{H} = \frac{t_0 Q_0}{\rho_0 c_{p_0} \vartheta_0}, \quad \varepsilon = \frac{\Delta\vartheta}{\vartheta_0}$$

where l_0, t_0, ρ_0, p_0, ϑ_0, u_0, μ_0, k_0, c_{p_0}, g_0, Q_0 and $\Delta\vartheta$ are the scales of length, time, density, pressure, temperature, velocity, viscosity, conductivity, constant pressure specific heat, external acceleration and external heat and temperature variation respectively (ρ_0, p_0 and ϑ_0 are assumed to be related by a state equation). We stress that we do not assume the existence of all these reference scales because if a reference scale is not available, its value can be defined eliminating a dimensionless number. For example, if an independent time scale is not available for a particular problem, we can define it from the velocity scale taking $\mathrm{S} = 1$ or, if the velocity scale is defined in terms of the fluid properties, we could take $\mathrm{M} = 1$ as it is done in [22], where compressibility effects in the Rayleigh Bénard problem are studied. The system of equations to be solved reads

$$\mathrm{S}\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \tag{1}$$

$$\rho\left(\mathrm{S}\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u}\right) + \frac{1}{\mathrm{M}^2}\boldsymbol{\nabla} p - \frac{1}{\mathrm{R}}\boldsymbol{\nabla} \cdot (2\mu \boldsymbol{\varepsilon}\prime(\boldsymbol{u})) = -\frac{1}{\mathrm{F}^2}\rho \hat{\boldsymbol{z}} \tag{2}$$

$$\begin{aligned}\rho c_p \left(\mathrm{S}\frac{\partial \vartheta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \vartheta\right) - \Gamma \beta \vartheta \left(\mathrm{S}\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p\right) \\ + \frac{\mathrm{M}^2}{\mathrm{R}}\Phi - \frac{1}{\mathrm{P}}\boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} \vartheta) = \mathrm{H}\mathrm{S}Q\end{aligned} \tag{3}$$

where $\hat{\boldsymbol{z}} = (0,0,1)^t$, $\boldsymbol{\varepsilon}\prime(\boldsymbol{u}) = \boldsymbol{\varepsilon} - \frac{1}{3}(\boldsymbol{\nabla} \cdot \boldsymbol{u})\boldsymbol{I}$ is the deviatoric part of the rate of deformation tensor ($\boldsymbol{\varepsilon}$ is the symmetric part of the velocity gradient, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}\boldsymbol{u}^t\right)$), β the thermal expansion coefficient (made dimensionless using the temperature scale) and Φ is the Rayleigh dissipation function defined as $\Phi = 2\mu\varepsilon\prime(\boldsymbol{u}) : \varepsilon\prime(\boldsymbol{u})$ and $\Gamma = p_0/(\rho_0 c_{p_0}\vartheta_0)$which depends on the state equation. In the case of an ideal gas $p = \rho\vartheta$ and

$$\Gamma = \frac{\gamma - 1}{\gamma}$$

The boundary conditions for the momentum equations are

$$\boldsymbol{u} = \boldsymbol{u}_D \quad \text{on} \quad \Gamma_D^{\boldsymbol{u}} \tag{4}$$

$$\left(-\frac{1}{\mathrm{M}^2} p \boldsymbol{I} + \frac{1}{\mathrm{R}} 2\mu \varepsilon\prime(\boldsymbol{u})\right) \cdot \boldsymbol{n} = \frac{1}{\mathrm{M}^2} \boldsymbol{t} \quad \text{on} \quad \Gamma_N^{\boldsymbol{u}} \tag{5}$$

where $\Gamma_D^{\boldsymbol{u}}$ is the Dirichlet part of the domain boundary where the velocity is prescribed to $\boldsymbol{u}_D$ (made dimensionless using u_0), $\Gamma_N^{\boldsymbol{u}}$ is the Neumann part of the domain boundary where the traction is prescribed to $\boldsymbol{t}$ (made dimensionless using p_0), $\partial\Omega = \overline{\Gamma_D^{\boldsymbol{u}} \cup \Gamma_N^{\boldsymbol{u}}}$ is the boundary of the domain and $\boldsymbol{n}$ its exterior normal. In the same way the boundary conditions for the energy equation are of the form

$$\vartheta = 1 + \varepsilon \vartheta_D \quad \text{on} \quad \Gamma_D^{\vartheta} \tag{6}$$

$$\frac{1}{\mathrm{P}} k \boldsymbol{n} \cdot \nabla \vartheta = \mathrm{HS} q \quad \text{on} \quad \Gamma_N^{\vartheta} \tag{7}$$

where Γ_D^{ϑ} is the Dirichlet part of the domain boundary where the temperature is prescribed to ϑ_D (made dimensionless with ϑ_0), Γ_N^{ϑ} is the Neumann part of the boundary where the heat flux is prescribed to q (made dimensionless using $Q_0 l_0$) and $\partial\Omega = \overline{\Gamma_D^{\vartheta} \cup \Gamma_N^{\vartheta}}$. Finally initial conditions are given by

$$\xi\left(\boldsymbol{x}, 0\right) = \xi_0\left(\boldsymbol{x}\right)$$

for $\xi = \boldsymbol{u}$, $\xi = p$, $\xi = \rho$ and $\xi = \vartheta$.

Having defined the equations of motion and the boundary conditions in general, let us consider some particular problems to illustrate the direct application of the unified scheme we develop in the next section. We are interested in *natural convection problems* and we consider two examples. The first one is the differentially heated cavity studied in [8] and [18] that consists of a rectangular cavity whose left (hot) wall has a fixed temperature ϑ_h and whose right (cold) wall has a fixed temperature ϑ_c. Upper and lower walls are adiabatic and initially the gas is at rest with a temperature ϑ_0 and density ρ_0. The second one is the well known Rayleigh-Bénard problem (see [16]) which consists in a layer of fluid between two infinite horizontal walls. On the lower wall a higher temperature (ϑ_h) is imposed whereas on the upper one a lower temperature is imposed (ϑ_c) and again, initially the gas is at rest with temperature ϑ_0 and density ρ_0 depending linearly on the vertical coordinate. However, we are also interested in *forced and mixed convection problems* in which a velocity field is prescribed on the boundary leading to the so-called Poiseuille-Rayleigh-Bénard (PRB) problem (see [26]). Although several boundary conditions can be applied, we assume a prescribed Poiseuille velocity profile on the inlet and prescribed temperatures on the upper (ϑ_c) and lower walls (ϑ_h). We assume initially a Poiseuille velocity distribution in the whole channel and, as in the Rayleigh Bénard problem, an initial temperature ϑ_0 and density ρ_0 depending linearly on the vertical coordinate.

Let us finally mention the class of problems that involve slow atmospheric flows described in [40] in chapter 6 and references therein. This is an entire subject that we will not consider here except in what respects to the selection of the reference state.

3 Asymptotic analysis

The limit when the Mach number tends to zero can be found using standard procedures of asymptotic analysis described for example in [13]. The first step is to expand all flow variables in power series of the small parameter considered

$$\xi\left(\boldsymbol{x},t,\mathrm{M}\right)=\xi^{(0)}\left(\boldsymbol{x},t\right)+\mathrm{M}\xi^{(1)}\left(\boldsymbol{x},t\right)+\mathrm{M}^2\xi^{(2)}\left(\boldsymbol{x},t\right)+\mathcal{O}(\mathrm{M}^3) \tag{8}$$

for $\xi=\boldsymbol{u}$, $\xi=p$, $\xi=\rho$ and $\xi=\vartheta$.

This asymptotic setting cannot be used in any situation and in particular we have to mention the problem of the behavior near the initial time. In this case it is necessary to introduce a fast time scale $\tau=t/\mathrm{M}$ and assume an expansion of the form

$$\xi\left(\boldsymbol{x},t,\mathrm{M}\right)=\xi^{(0)}\left(\boldsymbol{x},t,\tau\right)+\mathrm{M}\xi^{(1)}\left(\boldsymbol{x},t,\tau\right)+\mathrm{M}^2\xi^{(2)}\left(\boldsymbol{x},t,\tau\right)+\mathcal{O}(\mathrm{M}^3) \tag{9}$$

This is done in [38] and [25]. We also have to mention the problem of the behavior of the flow in the far field when unbounded domains are considered. In this case it is necessary to introduce a long space variable $\boldsymbol{\eta}=\mathrm{M}\boldsymbol{x}$ that ¨looks" to the infinity and to assume an expansion of the form

$$\xi\left(\boldsymbol{x},t,\mathrm{M}\right)=\xi^{(0)}\left(\boldsymbol{x},\boldsymbol{\eta},t\right)+\mathrm{M}\xi^{(1)}\left(\boldsymbol{x},\boldsymbol{\eta},t\right)+\mathrm{M}^2\xi^{(2)}\left(\boldsymbol{x},\boldsymbol{\eta},t\right)+\mathcal{O}(\mathrm{M}^3)$$

This is done in [15] and [23]. The objective of these variables is to separate scales and to perform a multiple scale analysis of the problem. The multiple scale analysis of the compressible Navier Stokes equations is of crucial importance to analyze acoustic phenomena that involve a fast time scale and the propagation to the infinite. We do not consider the acoustic problem here but we will include a long space variable $\eta=\mathrm{M}z$ in the $\hat{\boldsymbol{z}}$ direction to include the case of slow atmospheric motion considered in [40]. Therefore, we propose the expansion

$$\xi\left(\boldsymbol{x},t,\mathrm{M}\right)=\xi^{(0)}\left(\boldsymbol{x},\eta,t\right)+\mathrm{M}\xi^{(1)}\left(\boldsymbol{x},\eta,t\right)+\mathrm{M}^2\xi^{(2)}\left(\boldsymbol{x},\eta,t\right)+\mathcal{O}(\mathrm{M}^3) \tag{10}$$

that in a bounded domain reduces to 8 simply omitting the dependence with η, and we have

$$\left.\frac{\partial\xi}{\partial z}\right|_{\mathrm{M}}=\frac{\partial\xi^{(0)}}{\partial z}+\left(\frac{\partial\xi^{(0)}}{\partial\eta}+\frac{\partial\xi^{(1)}}{\partial z}\right)\mathrm{M}+\left(\frac{\partial\xi^{(1)}}{\partial\eta}+\frac{\partial\xi^{(2)}}{\partial z}\right)\mathrm{M}^2+\mathcal{O}(\mathrm{M}^3)$$

Any physical property χ (where χ can be μ, k, c_p, or β) can be expanded as

$$\chi(\vartheta, p) = \chi^{(0)} + \mathrm{M}\chi^{(1)} + \mathrm{M}^2\chi^{(2)} + \mathcal{O}(\mathrm{M}^3)$$

where $\chi^{(0)} = \chi\left(\vartheta^{(0)}, p^{(0)}\right)$ and $\chi^{(i)}$ depends on $\xi^{(i)}$ and on the derivatives of χ with respect to the temperature and pressure.

The second step is to substitute expansion 10 into equations 1 to 3 and to require that all terms in the expanded equations that are multiplied by the same power of M vanish to obtain a hierarchy of equations. When the Mach number tends to zero and the rest of the numbers remain $\mathcal{O}(1)$, keeping the first set of equations of the hierarchy, we obtain the low Mach number approximation discussed in subsection 3.1. The same procedure is followed when other numbers (apart from the Mach number) tend to zero at the same time but different results are obtained depending on the relation between them. The Boussinesq and the anelastic approximations, discussed in subsections 3.2 and 3.3 respectively, are obtained when also F, H and eventually ε tend to zero. These numbers represent the strength of the external driving, either by volumetric forces (F and H) or by the boundary conditions (ε). In order to obtain all these limits in a unified way, let us consider the expansion of the compressible Navier Stokes operator (i.e. the left hand side of 1 to 3)

- Continuity

$$\mathrm{S}\frac{\partial \rho^{(0)}}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho^{(0)}\boldsymbol{u}^{(0)}\right) = \mathcal{O}(\mathrm{M}^0) \tag{11}$$

- Momentum

$$\mathrm{M}^{-2}\boldsymbol{\nabla} p^{(0)} = \mathcal{O}(\mathrm{M}^{-2}) \tag{12}$$

$$\mathrm{M}^{-1}\left(\frac{\partial p^{(0)}}{\partial \eta}\hat{\boldsymbol{z}} + \boldsymbol{\nabla} p^{(1)}\right) = \mathcal{O}(\mathrm{M}^{-1}) \tag{13}$$

$$\begin{aligned}\rho^{(0)}\left(\mathrm{S}\frac{\partial \boldsymbol{u}^{(0)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{(0)}\right) + \frac{\partial p^{(1)}}{\partial \eta}\hat{\boldsymbol{z}} + \boldsymbol{\nabla} p^{(2)} \\ -\frac{1}{\mathrm{R}}\boldsymbol{\nabla}\cdot\left(2\mu\boldsymbol{\varepsilon}\prime(\boldsymbol{u}^{(0)})\right) = \mathcal{O}(\mathrm{M}^0)\end{aligned} \tag{14}$$

- Energy

$$\begin{aligned}\rho^{(0)}c_p^{(0)}\left(\mathrm{S}\frac{\partial \vartheta^{(0)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\boldsymbol{\nabla}\vartheta^{(0)}\right) \\ -\Gamma\beta^{(0)}\vartheta^{(0)}\left(\mathrm{S}\frac{\partial p^{(0)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\boldsymbol{\nabla} p^{(0)}\right) \\ -\frac{1}{\mathrm{P}}\boldsymbol{\nabla}\cdot\left(k^{(0)}\boldsymbol{\nabla}\vartheta^{(0)}\right) = \mathcal{O}(\mathrm{M}^0)\end{aligned} \tag{15}$$

Each term in the operator expansion must be equal to a term of the same order in the external forces as shown in the following subsections. The state equation, in the case of an ideal gas, is expanded as

$$\begin{aligned} \mathcal{O}(1): &\qquad p^{(0)} = \rho^{(0)}\vartheta^{(0)} \\ \mathcal{O}(\mathrm{M}): &\qquad p^{(1)} = \rho^{(0)}\vartheta^{(1)} + \vartheta^{(0)}\rho^{(1)} \end{aligned}$$

3.1 The low Mach number approximation

This case is defined by $\mathrm{M} \to 0$,$\mathrm{F} = \mathcal{O}(1)$, $\mathrm{H} = \mathcal{O}(1)$ and $\varepsilon = \mathcal{O}(1)$. Then the external forces are $\mathcal{O}(1)$ and from 12 we have $p^{(0)} = p^{(0)}(\eta, t)$ whereas from 13 we have $p^{(1)} = p^{(1)}(\eta, t)$. The multiple scale analysis in an unbounded domain is presented by [23]. In a bounded domain we have $p^{(0)} = p^{(0)}(t)$ and $p^{(1)} = p^{(1)}(t)$ which becomes irrelevant and can be taken constant (it is not necessary to solve the resulting system, as shown below). Therefore, the M^2 expansion presented in [30], [28] and [21] is recovered. The pressure splits into two contributions: $p^{(0)}$, a reference thermodynamic pressure and $p^{(2)}$, a mechanical pressure. The first one, constant over the whole domain, changes its value only by global heating or mass adding as shown below. The mechanical pressure component $p^{(2)}$ is determined from a velocity constraint playing the same role as in incompressible flows. In the zero Mach number limit a system of equations for $\rho^{(0)}$, $\vartheta^{(0)}$, $p^{(2)}$ and $\boldsymbol{u}^{(0)}$ has to be solved.

The reference pressure $p^{(0)}$, also called thermodynamic pressure, depends on the boundary conditions of the problem. If $\Gamma_N^{\boldsymbol{u}} \neq \emptyset$ the thermodynamic pressure is determined by the boundary condition. This can be seen introducing the asymptotic expansion 8 in the boundary condition 5, from where

$$\mathcal{O}\left(\mathrm{M}^{-2}\right): \qquad p^{(0)} = \boldsymbol{t}^{(0)} \cdot \boldsymbol{n} \tag{16}$$

$$\mathcal{O}\left(\mathrm{M}^{-1}\right): \qquad p^{(1)} = \boldsymbol{t}^{(1)} \cdot \boldsymbol{n} \tag{17}$$

$$\mathcal{O}\left(\mathrm{M}^{0}\right): \qquad \left(-p^{(2)}\boldsymbol{I} + \frac{1}{\mathrm{R}} 2\mu\varepsilon\prime(\boldsymbol{u}^{(0)})\right) \cdot \boldsymbol{n} = \boldsymbol{t}^{(2)} \tag{18}$$

This justifies what was noted in [30]: if the domain is "open" to the atmosphere, the reference pressure is determined by the external pressure. In a "closed" domain ($\Gamma_N^{\boldsymbol{u}} = \emptyset$) the thermodynamic pressure is determined by a global balance. Using the zero order mass and energy conservation equations and the state equation an equation relating the velocity divergence and the thermodynamic pressure can be found. In the case of an ideal gas, this constraint is

$$p^{(0)} \boldsymbol{\nabla} \cdot \boldsymbol{u}^{(0)} = -\frac{1}{\gamma} \mathrm{S} \frac{dp^{(0)}}{dt} + \frac{1}{\mathrm{P}} \boldsymbol{\nabla} \cdot \left(k^{(0)} \boldsymbol{\nabla} \vartheta^{(0)}\right) + \mathrm{HS}Q \tag{19}$$

This equation, integrated over the domain gives an ordinary differential equation for the reference pressure. In the case of an ideal gas this equation is explicit and given by

$$p^{(0)} \int_{\partial\Omega} \boldsymbol{u}^{(0)} \cdot \boldsymbol{n} = -\frac{V_\Omega}{\gamma} \mathrm{S} \frac{dp^{(0)}}{dt} + \frac{1}{\mathrm{P}} \int_{\partial\Omega} \boldsymbol{q}^{(0)} \cdot \boldsymbol{n} + \mathrm{HS} \int_\Omega Q \tag{20}$$

where $V_\Omega = \mathrm{meas}\,(\Omega)$ is the volume of the domain and $\boldsymbol{q}^{(0)}$ is the zero order term of the heat flux on the boundary (either prescribed as boundary condition or computed from the temperature). In general this equation will be an implicit equation for the reference pressure. In the case of an ideal gas, a physical interpretation is possible. The constant-in-space thermodynamic pressure changes in time due to the mass addition (subtraction) or to the domain expansion (contraction) (left hand side term) or to heat addition (subtraction) either by the boundary (second right hand side term) or by volumetric sources (last right hand side term). Note that in a closed flow without mass addition or domain expansion the first term in 20 vanishes. The asymptotic expansion 8 applied to the initial pressure defines initial conditions $p_0^{(0)}$, $p_0^{(1)}$ and $p_0^{(2)}$ for $p^{(0)}$, $p^{(1)}$ and $p^{(2)}$. If the zero order initial pressure is not uniform, there is an initial transient that should be studied by a multiple scale asymptotic analysis introduce a fast time scale $\tau = t/\mathrm{M}$ as mentioned before.

If we consider a *non-conducting fluid in absence of heat sources*, the case of open flows or the case of closed flows without addition of mass or domain expansion, we have a constant thermodynamic pressure and 19 gives

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}^{(0)} = 0$$

The system to be solved, called the *non-homogeneous Navier Stokes equations* by [19], is given by

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{u}^{(0)} &= 0 \\ \mathrm{S}\frac{\partial \rho^{(0)}}{\partial t} + \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \rho^{(0)} &= 0 \\ \rho^{(0)} \left(\mathrm{S}\frac{\partial \boldsymbol{u}^{(0)}}{\partial t} + \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(0)} \right) + \boldsymbol{\nabla} p^{(2)} - \frac{1}{\mathrm{R}} \boldsymbol{\nabla} \cdot \left(2\mu^{(0)} \boldsymbol{\varepsilon}(\boldsymbol{u}^{(0)}) \right) &= -\frac{1}{\mathrm{F}^2} \rho^{(0)} \hat{\boldsymbol{z}} \end{aligned}$$

Further, if the density distribution is initially constant, it remains constant for all times and we have the *homogeneous incompressible Navier Stokes equations*

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{u}^{(0)} &= 0 \\ \rho^{(0)} \left(\mathrm{S}\frac{\partial \boldsymbol{u}^{(0)}}{\partial t} + \boldsymbol{u}^{(0)} \cdot \boldsymbol{\nabla} \boldsymbol{u}^{(0)} \right) + \boldsymbol{\nabla} p^{(2)} - \frac{1}{\mathrm{R}} \boldsymbol{\nabla} \cdot \left(2\mu^{(0)} \boldsymbol{\varepsilon}(\boldsymbol{u}^{(0)}) \right) &= -\frac{1}{\mathrm{F}^2} \rho^{(0)} \hat{\boldsymbol{z}} \end{aligned}$$

However, in the general case of a conducting fluid, the system to be solved is made up of the *zero Mach number equations*, given by

$$
\begin{aligned}
\mathrm{S}\frac{\partial \rho^{(0)}}{\partial t} + \boldsymbol{\nabla}\cdot\left(\rho^{(0)}\boldsymbol{u}^{(0)}\right) &= 0 \\
\rho^{(0)}\left(\mathrm{S}\frac{\partial \boldsymbol{u}^{(0)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{(0)}\right) + \boldsymbol{\nabla}p^{(2)} - \frac{1}{\mathrm{R}}\boldsymbol{\nabla}\cdot\left(2\mu^{(0)}\boldsymbol{\varepsilon}\prime(\boldsymbol{u}^{(0)})\right) &= -\frac{1}{\mathrm{F}^2}\rho^{(0)}\hat{\boldsymbol{z}} \\
\rho^{(0)}c_p^{(0)}\left(\mathrm{S}\frac{\partial \vartheta^{(0)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\boldsymbol{\nabla}\vartheta^{(0)}\right) - \Gamma\beta\vartheta^{(0)}\mathrm{S}\frac{dp^{(0)}}{dt} & \\
-\frac{1}{\mathrm{P}}\boldsymbol{\nabla}\cdot\left(k^{(0)}\boldsymbol{\nabla}\vartheta^{(0)}\right) &= \mathrm{HS}Q
\end{aligned}
$$

which has to be completed with the state equation $p^{(0)} = \rho^{(0)}\vartheta^{(0)}$where the thermodynamic pressure $p^{(0)}$ is either given by 20 or determined by the boundary condition 16.

This system of equations does not present acoustic phenomena that are present in a compressible flow as was shown in [30], [28] and [25]. Acoustic phenomena are pressure and density waves of small amplitude and fast propagation velocity (the sound speed c) that satisfy the compressible Navier Stokes equations. It is easy to see that a wave equation for the pressure can be deduced from the full compressible equations taking the temporal derivative of 1 and 3, the divergence of 2 and using the state equation. When the Mach number is small the hyperbolic wave equation for the pressure becomes an elliptic equation for the first order pressure $p^{(2)}$, thus showing the implicit ("incompressible" or "mechanical") character of this pressure component. It is not an evolving variable but can be understood as an implicit Lagrange multiplier determined by the mass conservation.

3.2 The Boussinesq approximation

This case is defined by $\mathrm{M} \to 0$, $\mathrm{F}^2 = \mathcal{O}(\mathrm{M})$, $\mathrm{H} = \mathcal{O}(\mathrm{M})$ and $\varepsilon = \mathcal{O}(\mathrm{M})$. In order to study this limit it is useful to introduce the Boussinesq number, defined as

$$\mathrm{B} = \frac{\rho_0 g l_0}{p_0} = \frac{\mathrm{M}^2}{\mathrm{F}^2}$$

This number was defined first in [36] but its importance in vertically stratified flows was already noted in [1]. Note that $\mathrm{M} \to 0$ with $\mathrm{F}^2 = \mathcal{O}(\mathrm{M})$ implies $\mathrm{B} \to 0$ with $\mathrm{B} = \mathcal{O}(\mathrm{M})$. A small Boussinesq number is a restriction on the height of the flow analyzed, as the quantity $p_0/\rho_0 g_0$ is the height scale of the thermodynamic field.

Now the external forces are $\mathcal{O}(\mathrm{M}^{-1})$ and from 12 we have $p^{(0)} = p^{(0)}(\eta, t)$ whereas from 13 we have

$$\frac{\partial p^{(0)}}{\partial \eta}\hat{\boldsymbol{z}} + \boldsymbol{\nabla} p^{(1)} = -\mathrm{MF}^{-2}\rho^{(0)}\hat{\boldsymbol{z}} \tag{21}$$

from where $p^{(1)} = p^{(1)}(\eta, z, t)$. In the problem of slow atmospheric motion (unbounded domain) the variable η is the standard altitude as defined in [39] and the standard atmosphere satisfies 21 with $p^{(1)}$ constant and $p^{(0)} = p^{(0)}(\eta)$. The analysis presented in [38, 39, 40] and in [5] considers $p^{(1)}$ constant and then, motivated by 21, the zero order fields independent of z and t. We are not aware of any formal derivation of this result that should be therefore considered as a hypothesis, certainly reasonable anyway. Note that the dependence on η is weaker than the dependence on z giving a different description of the standard atmosphere.

In a bounded domain, from 12 we have $p^{(0)} = p^{(0)}(t)$ whereas from 13 $p^{(1)} = p^{(1)}(z, t)$ and $\rho^{(0)} = \rho^{(0)}(z, t)$. Using the state equation we obtain $\vartheta^{(0)} = \vartheta^{(0)}(z, t)$. Then, the zero order energy equation reads

$$\rho^{(0)} c_p^{(0)} \left(\mathrm{S}\frac{\partial \vartheta^{(0)}}{\partial t} + \boldsymbol{u}^{(0)} \cdot \hat{\boldsymbol{z}} \frac{\partial \vartheta^{(0)}}{\partial z} \right) - \Gamma \beta^{(0)} \vartheta^{(0)} \mathrm{S} \frac{dp^{(0)}}{dt} = \frac{1}{\mathrm{P}} \frac{\partial}{\partial z} \left(k^{(0)} \frac{\partial \vartheta^{(0)}}{\partial z} \right) \tag{22}$$

Introducing the asymptotic expansion 8 in the boundary condition 6 and 7 we obtain

$$\mathcal{O}\left(\mathrm{M}^{0}\right): \qquad \vartheta^{(0)} = 1 \tag{23}$$

$$\mathcal{O}\left(\mathrm{M}^{1}\right): \qquad \vartheta^{(1)} = \vartheta_D^{(0)} \tag{24}$$

and

$$\mathcal{O}\left(\mathrm{M}^{0}\right): \qquad \frac{1}{\mathrm{P}} k \boldsymbol{n} \cdot \nabla \vartheta^{(0)} = 0 \tag{25}$$

$$\mathcal{O}\left(\mathrm{M}^{1}\right): \qquad \frac{1}{\mathrm{P}} k \boldsymbol{n} \cdot \nabla \vartheta^{(1)} = \mathrm{S} q^{(0)} \tag{26}$$

The solution $\vartheta^{(0)} = 1$ and $p^{(0)} = 1$ satisfies 22, 20 and boundary conditions 23 and 25 except for closed flows with addition of mass (note that the last two terms in 20 vanish). As in the previous case, the asymptotic expansion 8 applied to the initial pressure defines the initial condition for $p^{(0)}$ which is assumed to be uniform as discussed before. In the same way if the initial temperature for $\vartheta_0^{(0)}$ depends on x or y there is an initial transient and a multiple scale asymptotic analysis should be performed, and if $\vartheta^{(0)}(\boldsymbol{x}, 0) = \vartheta_0^{(0)}(z)$ the evolution equations 22 and 20 must be solved. However, if the initial zero order temperature is constant (compatible with the boundary conditions) then $\vartheta^{(0)} = 1$ and $p^{(0)} = 1$ and from the state equation $\rho^{(0)} = 1$. In this case, from 21 we have $p^{(1)} = -\mathrm{MF}^{-2}\rho^{(0)}z$ and the *Boussinesq equations* read

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}^{(0)} = 0$$

$$\rho^{(0)}\left(\mathrm{S}\frac{\partial\boldsymbol{u}^{(0)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\boldsymbol{\nabla}\boldsymbol{u}^{(0)}\right) + \boldsymbol{\nabla}p^{(2)} - \frac{1}{\mathrm{R}}\boldsymbol{\nabla}\cdot\left(2\mu\boldsymbol{\varepsilon\prime}(\boldsymbol{u}^{(0)})\right) = -\mathrm{MF}^{-2}\rho^{(1)}\hat{\boldsymbol{z}}$$

$$\rho^{(0)}c_p^{(0)}\left(\frac{\partial\vartheta^{(1)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\boldsymbol{\nabla}\vartheta^{(1)}\right) + \Gamma\beta^{(0)}\vartheta^{(0)}w^{(0)}\frac{dp^{(1)}}{dz} - \frac{1}{\mathrm{P}}\boldsymbol{\nabla}\cdot\left(k^{(0)}\boldsymbol{\nabla}\vartheta^{(1)}\right) = \mathrm{S}Q$$

This system has to be completed with a state equation, which for an ideal gas is

$$\rho^{(1)} = \frac{p^{(1)}}{\vartheta^{(0)}} - \frac{\rho^{(0)}}{\vartheta^{(0)}}\vartheta^{(1)}$$

In the case of an unbounded domain the same system is obtained replacing $\frac{dp^{(1)}}{dz}$ by $\frac{dp^{(0)}}{d\eta}$, as shown in [38, 39]. In this case also $p^{(0)}(\eta)$, $\rho^{(0)}(\eta)$, and $\vartheta^{(0)}(\eta)$ describe the reference static atmosphere. However, in a bounded domain the reference state is simply a constant and the hydrostatic pressure $p^{(1)}$ naturally appears. We contrast this approach with the one followed in [39, 40] and in [5] where a motionless basic state is introduced a priori and the asymptotic analysis is performed for the perturbations.

3.3 The anelastic and quasistatic approximations

This case is defined by $\mathrm{M} \to 0$ and $\mathrm{F} = \mathcal{O}(\mathrm{M})$. From 12 and 13 we have

$$\mathrm{M}^{-2}\boldsymbol{\nabla}p^{(0)} = -\mathrm{F}^{-2}\rho^{(0)}\hat{\boldsymbol{z}} \tag{27}$$

$$\mathrm{M}^{-1}\frac{\partial p^{(0)}}{\partial\eta}\hat{\boldsymbol{z}} + \mathrm{M}^{-1}\boldsymbol{\nabla}p^{(1)} = -\mathrm{F}^{-1}\rho^{(1)}\hat{\boldsymbol{z}} \tag{28}$$

from where $p^{(0)} = p^{(0)}(\eta, z, t)$, $\rho^{(0)} = \rho^{(0)}(\eta, z, t)$, $p^{(1)} = p^{(1)}(\eta, z, t)$ and $\rho^{(1)} = \rho^{(1)}(\eta, z, t)$. Asuming $\mathrm{H} = \mathcal{O}(\mathrm{M}^2)$, the zero order energy equation is

$$\rho^{(0)}c_p^{(0)}\left(\mathrm{S}\frac{\partial\vartheta^{(0)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\hat{\boldsymbol{z}}\frac{\partial\vartheta^{(0)}}{\partial z}\right) - \Gamma\beta^{(0)}\vartheta^{(0)}\left(\mathrm{S}\frac{\partial p^{(0)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\hat{\boldsymbol{z}}\frac{\partial p^{(0)}}{\partial z}\right) - \frac{1}{\mathrm{P}}\boldsymbol{\nabla}\cdot\left(k^{(0)}\boldsymbol{\nabla}\vartheta^{(0)}\right) = 0 \tag{29}$$

If $\mathrm{H} = \mathcal{O}(1)$ the heat source appears on the right hand side of 29 but this term does not affect the asymptotic analysis, it only changes the reference order zero fields. Again, boundary conditions are split as 16-18, 23-24 and 25-26.

This problem is quite complex and no closed solution has been found, although some comments can be made. This problem is analyzed in [38, 40] *assuming* $p^{(0)} = p^{(0)}(z)$, $\rho^{(0)} = \rho^{(0)}(z)$, $\vartheta^{(0)} = \vartheta^{(0)}(z)$, $p^{(1)} = 0$, $\rho^{(1)} = 0$ and $\vartheta^{(1)} = 0$. Note that to assume $p^{(1)} = 0$, $\rho^{(1)} = 0$ and $\vartheta^{(1)} = 0$ is equivalent to assume an expansion in powers of M^2 instead of M. This analysis is valid both in bounded and unbounded domains as the long variable η is not used. In the problem of slow atmospheric motions this implies a stronger dependence of the standard reference fields on the standard altitude, now defined using z in [40].

Under these assumptions 29 becomes

$$\boldsymbol{u}^{(0)} \cdot \hat{\boldsymbol{z}} \left[\rho^{(0)} c_p^{(0)} \frac{d\vartheta^{(0)}}{dz} - \Gamma \beta^{(0)} \vartheta^{(0)} \frac{dp^{(0)}}{dz} \right] = \frac{1}{\mathrm{P}} \boldsymbol{\nabla} \cdot \left(k^{(0)} \boldsymbol{\nabla} \vartheta^{(0)} \right)$$

and two different cases can be considered

- If

$$\rho^{(0)} c_p^{(0)} \frac{d\vartheta^{(0)}}{dz} - \Gamma \beta^{(0)} \vartheta^{(0)} \frac{dp^{(0)}}{dz} \neq 0 \tag{30}$$

then

$$\boldsymbol{u}^{(0)} \cdot \hat{\boldsymbol{z}} = \frac{\frac{1}{\mathrm{P}} \boldsymbol{\nabla} \cdot \left(k^{(0)} \boldsymbol{\nabla} \vartheta^{(0)} \right)}{\rho^{(0)} \frac{d\vartheta^{(0)}}{dz} - \Gamma \beta^{(0)} \vartheta^{(0)} \frac{dp^{(0)}}{dz}}$$

This case is called *quasi-static approximation* in [4, 5] and [38, 39, 40]. The vertical velocity is constrained by an hydrostatic equilibrium in the vertical direction and only plane motions can occur.

- If

$$\rho^{(0)} c_p^{(0)} \frac{d\vartheta^{(0)}}{dz} - \Gamma \beta^{(0)} \vartheta^{(0)} \frac{dp^{(0)}}{dz} = 0 \tag{31}$$

the *anelastic approximation* follows. This condition together with the zero order momentum and energy equations define the reference (zero order) state

$$\rho^{(0)} c_p^{(0)} \frac{d\vartheta^{(0)}}{dz} - \Gamma \beta^{(0)} \vartheta^{(0)} \frac{dp^{(0)}}{dz} = 0 \tag{32}$$

$$\frac{1}{\mathrm{P}} \frac{d}{dz} \left(k^{(0)} \frac{d\vartheta^{(0)}}{dz} \right) = 0 \tag{33}$$

$$\frac{dp^{(0)}}{dz} = \mathrm{B} \rho^{(0)} \tag{34}$$

where also the zero order state equation needs to be considered. For an ideal gas $p^{(0)} = \rho^{(0)}\vartheta^{(0)}$. The final set of *anelastic equations* to be solved, using this reference state, is given by

$$\nabla \cdot \left(\rho^{(0)}\boldsymbol{u}^{(0)}\right) = 0$$

$$\mathrm{S}\rho^{(0)}\frac{\partial \boldsymbol{u}^{(0)}}{\partial t} + \rho^{(0)}\boldsymbol{u}^{(0)} \cdot \nabla \boldsymbol{u}^{(0)} + \nabla p^{(2)} - \frac{1}{\mathrm{R}}\nabla \cdot \left(2\mu\boldsymbol{\varepsilon}\prime(\boldsymbol{u}^{(0)})\right) = -\mathrm{B}\rho^{(2)}\hat{\boldsymbol{z}}$$

$$\rho^{(0)}c_p^{(0)}\left(\mathrm{S}\frac{\partial\vartheta^{(2)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\nabla\vartheta^{(2)}\right) + \left(\rho^{(2)}c_p^{(0)} + \rho^{(0)}c_p^{(2)}\right)w^{(0)}\frac{d\vartheta^{(0)}}{dz}$$
$$-\Gamma\left(\beta^{(2)}\vartheta^{(0)} + \beta^{(0)}\vartheta^{(2)}\right)\frac{dp^{(0)}}{dz} - \Gamma\beta^{(0)}\vartheta^{(0)}\left(\mathrm{S}\frac{\partial p^{(2)}}{\partial t} + \boldsymbol{u}^{(0)}\cdot\nabla p^{(2)}\right)$$
$$-\frac{1}{\mathrm{P}}\frac{d}{dz}\left(k^{(0)}\frac{d\vartheta^{(2)}}{dz} + k^{(2)}\frac{d\vartheta^{(0)}}{dz}\right) = \frac{1}{\mathrm{R}}\Phi^{(0)} + \mathrm{S}Q$$

which also needs to be closed by the state equation, that in the case of an ideal gas is

$$p^{(2)} = \rho^{(0)}\vartheta^{(2)} + \vartheta^{(0)}\rho^{(2)}$$

These equations were presented in [28], where it is mentioned that they were written in this form in [17] and that they are a generalization of those obtained in [9] and [11]. As noted in [28], in the case of an ideal gas with constant c_p equations 32-33-34 can be solved and the reference state can be written as

$$\vartheta^{(0)} = (1 - \Gamma\mathrm{B}z)$$
$$\rho^{(0)} = (1 - \Gamma\mathrm{B}z)^{\frac{1}{\gamma-1}}$$
$$p^{(0)} = (1 - \Gamma\mathrm{B}z)^{\frac{\gamma}{\gamma-1}}$$

If now the limit of $\mathrm{B} \to 0$ as $\mathrm{B} = \mathcal{O}(\mathrm{M})$ is considered, the Boussinesq approximation presented in the previous subsection is recovered (as $\mathrm{B} = \mathcal{O}(\mathrm{M})$ implies $\mathrm{F}^2 = \mathcal{O}(\mathrm{M})$). The Boussinesq approximation is obtained in this way in [29] defining a reference state in equations 32, 33 and 34.

3.4 The choice of reference scales

In this subsection we apply the developed framework to the problems defined at the end of section 2. Let us start considering the case of natural convection problems. We consider $\mathrm{S} = 1$, that is to say that we take l_0/u_0 as a reference time and we have the scales l_0, ρ_0, ϑ_0, μ_0, k_0, c_{p_0}, g_0 and $\Delta\vartheta$. Therefore these problems are described in terms of five parameters: M, F, R, P and ε. In the natural convection context, the Rayleigh-Bénard problem and the differentially heated cavity, it is common to consider the Rayleigh number Ra and the Prandtl number Pr, defined as

$$\mathrm{Ra} = \frac{g l_0^3}{\nu_0 \alpha_0} \frac{\Delta\vartheta}{\vartheta_0}, \quad \mathrm{Pr} = \frac{\nu_0}{\alpha_0}$$

where ν_0 is the kinematic viscosity ($\nu_0 = \mu_0/\rho_0$), which satisfy

$$\mathrm{Ra} = \frac{\varepsilon}{\mathrm{F}^2}\mathrm{R}^2\mathrm{Pr}, \quad \mathrm{R} = \mathrm{P}\mathrm{Pr}$$

and to describe the problems in terms of M, F, Ra, Pr, and ε. A definition of the velocity scale eliminates one of these numbers. The problem of the differentially heated cavity has been analyzed in [8] and the Rayleigh Bénard problem has been analyzed in [24], in both cases defining *the velocity scale of the problem as the diffusive speed* given by

$$u_0 = \frac{k_0}{\rho_0 c_{p_0} l_0} = \frac{\alpha_0}{l_0}$$

where α_0 is the thermal diffusivity scale ($\alpha_0 = k_0/\rho_0 c_{p_0} l_0$), which corresponds to make $\mathrm{P} = 1$. The differentially heated cavity problem has also been analyzed in [33] assuming that *the velocity scale problem is the viscous speed* given by $u_0 = \nu_0/l_0$, what corresponds to make $\mathrm{R} = 1$. However, in our view, the most appropriate scaling for the velocity is the one used in [29] in the context of the Rayleigh Bénard problem and in [12], given by

$$u_0 = (\beta_0 \Delta\vartheta g l_0)^{1/2}$$

that we may call "buoyancy speed" and that is obtained from

$$\mathrm{F}^2 = \varepsilon$$

With this definition of the velocity scale $\mathrm{M}^2 = \mathrm{B}\varepsilon$. Therefore, the low Mach number approximation is valid for thin layers ($\mathrm{B} \to 0$) *or* small temperature differences ($\varepsilon \to 0$). The Boussinesq approximation also requires $\mathrm{F}^2 = \mathcal{O}(\mathrm{M})$ and $\varepsilon = \mathcal{O}(\mathrm{M})$ from where $\varepsilon = \mathcal{O}(\mathrm{F}^2)$, that is automatically satisfied with this definition of the velocity scale, which also implies $\mathrm{B} = \mathcal{O}(\varepsilon)$. Therefore, the Boussinesq approximation is valid for thin layers ($\mathrm{B} \to 0$) *and* small temperature differences ($\varepsilon \to 0$). In the case of the Rayleigh-Bénard problem and the differentially heated cavity the Boussinesq number is defined in terms of the height. Assuming it is small, the low Mach number approximation will be valid and the Boussinesq approximation will be valid only for small temperature differences. A comparison between both results in the differentially heated cavity can be found in [8].

Another important aspect of the proposed approach is the possibility of analyzing forced or mixed convection problems. In the case of the Poiseuille-Rayleigh-Bénard problem we have a velocity u_0 defined by the boundary condition. The low Mach number approximation is valid when the velocity u_0 is small compared to the sound speed. The Boussinesq approximation also requires $\mathrm{F}^2 = \mathcal{O}(\mathrm{M})$, what is equivalent to $\mathrm{B} = \mathcal{O}(\mathrm{M})$, and implies a restriction on the height scale, and $\varepsilon = \mathcal{O}(\mathrm{F}^2)$, what implies a velocity u_0 of the order of the buoyancy speed or, equivalently, small temperature differences (i.e. $\varepsilon = \mathcal{O}(\mathrm{M})$). However, if the velocity prescribed on the boundary is much smaller than the buoyancy speed ($u_0 \ll (\beta_0 \Delta\vartheta g l_0)^{1/2}$ or $\mathrm{F}^2 \ll \varepsilon$), the problem will be similar to the Rayleigh-Bénard problem and the fluid motion will be driven by temperature differences. Therefore, when $\mathrm{F}^2 \ll \varepsilon$ we redefine the velocity scale by $\mathrm{F}^2 = \varepsilon$ and the case of natural convection is recovered. If $\mathrm{F}^2 \gg \varepsilon$ we keep the velocity scale given by the boundary conditions and the validity of the low Mach number approximation will depend directly on the Mach number. The Boussinesq approximation will be valid if the Froude number is also small, as the asymptotic analysis shows. Note that, in this case, the temperature difference is small because $\mathrm{F}^2 \gg \varepsilon$. It is the comparison between F^2 and ε what defines the transition from natural convection to mixed and finally forced convection.

4 Summary and conclusions

The zero Mach number limit of the compressible flow equations yields different sets of equations depending on the type of flow analyzed. If an isentropic flow is considered, the incompressible Navier Stokes equations are recovered. When heat exchange is taken into account, different sets of equations are found. This limit is obtained using an expansion of the unknowns in series of the Mach number, which according to [31] is valid (i.e. yields a convergent solution) in the near field (see also [34]). When the Froude number is also small several situations can be found. On the one hand, the anelastic and the quasistatic approximations are found when $\mathrm{M}, \mathrm{F} \to 0$ and $\mathrm{M} \simeq \mathrm{F}$, if a reference state depending on z is assumed. On the other hand, the Boussinesq approximation is found when $\mathrm{M}, \mathrm{F} \to 0$ and $\mathrm{M} \simeq \mathrm{F}^2$, $\mathrm{H} \simeq \mathrm{F}^2$ and $\varepsilon \simeq \mathrm{F}^2$ assuming appropriate initial and boundary conditions. This approximation is also valid in an unbounded domain if the reference state depends weakly on z through $\zeta = \mathrm{B}z$ as shown in [4, 5] and [36, 38, 39].

These limits have been obtained under the unified asymptotic setting proposed here which presents two particular ingredients. On the one hand, the reference state is obtained as a consequence of the limit and not “a priori” as it is usually made. On the other hand, the choice of reference scales is discussed after the asymptotic development. The physical meaning of the similarity rules introduced in the asymptotic analysis has been made precise in the case of bounded domains for both natural and mixed convection problems.

The three approximations considered when heat exchange is taken into account (the zero Mach number approximation, the anelastic approximation and the Boussinesq approximation) describe the basic mechanism of thermal coupling which is due to the dependence of the density on the temperature. When a fluid element is heated, it expands and moves up. None of the three approximations describe acoustic phenomena, what is certainly desirable from a numerical point of view. The main difference between them is how they take into account the compressibility of the medium. While in the Boussinesq approximation the flow is incompressible, in the zero Mach number model the density distribution is predicted and the velocity field is affected by expansions or contractions due to heating and mass addition or domain expansion. A clear example is that of a fluid in a domain closed by a piston on the top. If we heat the fluid, it will expand and the piston will move up. Whereas the low Mach number model correctly predict this behavior, the Boussinesq approximation will predict a non moving piston because the flow does not expand. Between the low Mach number and the Boussinesq approximation, the anelastic approximation (mainly used in atmospheric sciences) takes into account the density of a stratified medium in the mass balance.

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References

[1] G. K. Batchelor. The conditions for dynamical similarity of motions of a frictionless perfect gas atmosphere. *Q. J. Roy. Meteor. Soc.*, 79:224–235, 1953.

[2] G. K. Batchelor. *An Introduction to Fluid Dynamics.* Cambridge University Press, Cambridge, UK, 1967.

[3] S. E. Bechtel, M. G. Forest, F. J. Rooney, and Q. Wang. Thermal expansion models of viscous fluids based on limits of free energy. *Phys. Fluids*, 15(9):2681–2693, 2003.

[4] P. A. Bois. Propagation linéare et non linéaire d'ondes atmosphériques. *J. de Mécanique*, 15(5):781–811, 1976.

[5] P. A. Bois. Asymptotic aspects of the Boussinesq approximation for gases and liquids. *Geophys. Astrophys. Fluid Dyn.*, 58:45–55, 1991.

[6] J. Boussinesq. *Théorie analytique de la chaleur*, volume 2. Gauthier-Villars, Paris (France), 1903. Reproduction Bibliothèque Nationale de France, 1995.

[7] E. Buckingham. On physically similar systems; illustrations of the use of dimensional equations. *Phys. Rev.*, 4(4):345–376, 1914.

[8] D. R. Chenoweth and S. Paolucci. Natural convection in an enclosed vertical air layer with large horizontal temperature differences. *J. Fluid Mech.*, 169:173–210, 1986.

[9] J. A. Dutton and G. H. Fichtl. Approximate equations of motion for gases and liquids. *J. Atmos. Sci.*, 26:241–253, 1969.

[10] A. T. Fedorchenko. A model of unsteady subsonic flow with acoustics excluded. *J. Fluid Mech.*, 334:135–155, 1997.

[11] D. O. Gough. The anelastic approximation for thermal convection. *J. Atmos. Sci.*, 26:448–456, May 1969.

[12] D. D. Gray and A. Giorgini. The validity of the Boussinesq approximation for liquids and gases. *Int. J. Heat Mass Transf.*, 19(5):545–551, 1976.

[13] J. Kevorkian and J. D. Cole. *Perturbation Methods in Applied Mathematics*, volume 34 of *Applied mathematical sciences*. Springer-Verlag, New York, 1981.

[14] S. Klainerman and A. Majda. Compressible and incompressible fluids. *Commun. Pure Appl. Math.*, XXXV:629–651, 1982.

[15] R. Klein. Semi-implicit extension of a Godunov-type scheme based on low Mach number asymptotics I: One-dimensional flow. *J. Comput. Phys.*, 121(2):213–237, Oct 1995.

[16] E. L. Koschmieder. *Bénard cells and Taylor vortices*. Cambridge monographs on mechanics and applied mathematics. Cambridge University Press, Cambridge, UK, 1993.

[17] V. I. Kovshov. Three approximations for thermal convection. *Sov. Astron.*, 22:288–296, June 1978.

[18] P. Le Quéré and M. Behnia. From onset to unsteadiness to chaos in a differentially heated cavity. *J. Fluid Mech.*, 359:81–107, 1998.

[19] P. L. Lions. *Mathematical Topics in Fluid Dynamics, Volume 1. Incompressible Models*. Oxford University Press, 1996.

[20] P. L. Lions. *Mathematical Topics in Fluid Dynamics, Volume 2. Compressible Models*. Oxford University Press, 1998.

[21] A. Majda and J. Sethian. The derivation and numerical solution of the equations for zero Mach number combustion. *Combust. Sci. Technol.*, 42:185–205, 1985.

[22] A. Manela and I. Frankel. On the Rayleigh Bénard problem: dominant compressibility effects. *J. Fluid Mech.*, 565:461–475, 2006.

[23] A. Meister. Asymptotic single and multiple scale expansions in the low Mach number limit. *SIAM J. Appl. Math.*, 60(1):256–271, 1999.

[24] J. M. Mihaljan. A rigorous exposition of the Boussinesq approximations applicable to a thin layer of fluid. *Astrophys. J.*, 136(3):1126–1133, 1962.

[25] B. Müller. Low Mach number asymptotics of the Navier-Stokes equations. *J. Eng. Math.*, 34:97–109, 1998.

[26] X. Nicolas. Bibliographical review on the Poiseuille-Rayleigh-Bénard flows: the mixed convection flows in horizontal rectangular ducts heated from below. *Int. J. Therm. Sci.*, 41(10):961–1016, Oct 2002.

[27] Y. Ogura and N. A. Phillips. Scale analysis of deep and shallow convection in atmosphere. *J. Atmos. Sci.*, 19:173–179, 1962.

[28] S. Paolucci. On the filtering of sound from the Navier-Stokes equations. Technical Report 82-8257, Sandia National Laboratories, 1982.

[29] R. Perez Cordon and M. G. Velarde. On the (non-linear) foundations of Boussinesq approximation applicable to a thin-layer of fluid. *J. Phys.*, 36(7):591–601, 1975.

[30] R. G. Rehm and H. R. Baum. The equations of motion for thermally driven bouyant flows. *J. Res. Nat. Bur. Stand.*, 83(3):297–308, 1978.

[31] S. A. Slimon, M. C. Soteriou, and D. W. Davis. Development of computational aeroacoustics equations for subsonic flows using a mach number expansion approach. *J. Comput. Phys.*, 159(2):377–406, Apr 2000.

[32] E. A. Spiegel and G. Veronis. On the Boussinesq approximation for a compressible fluid. *Astrophys. J.*, 131:442–447, 1960.

[33] S. A. Suslov and S. Paolucci. Nonlinear stability of mixed convection flow under non-Boussinesq conditions. Part 1. Analysis and bifurcations. *J. Fluid Mech.*, 398:61–86, Nov 1999.

[34] M. Van Dyke. *Perturbation methods in fluid mechanics*, volume 8 of *Applied mathematics and mechanics*. Academic Press, New York, 1964.

[35] G. P. Zank and W. H. Matthaeus. The Equations of Nearly Incompressible Fluids. I. Hydrodynamics, Turbulence, and Waves. *Phys. Fluids A*, 3:69–82, January 1991.

[36] R. Kh. Zeytounian. A rigorous derivation of the equations of compresible viscous fluid motion with gravity at low Mach number. *Archiv. Mech.*, 26:499–509, 1974.

[37] R. Kh. Zeytounian. Sur une formulation asymptotique du problème de Rayleigh-Bénard via l'aproximation de Boussinesq pour les liquides dilatables. *C. R. Acad. Sci. Paris,*, 297(11):271–274, 1983.

[38] R. Kh. Zeytounian. *Asymptotic modelling of atmospheric flows.* Springer Verlag, Heidelberg, 1990.

[39] R. Kh. Zeytounian. Joseph Boussinesq and his approximation: a contemporary view. *C. R. Mec.*, 331(8):575–586, Aug 2003.

[40] R. Kh. Zeytounian. *Topics in Hyposonic Flow Theory*, volume 672/2006 of *Lecture Notes in Physics.* Springer Verlag, Berlin Heidelberg, 2006.

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