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**C.A. Felippa  
E. Oñate**

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# Stress, Strain and Energy Splittings for Anisotropic Elastic Solids under Volumetric Constraints

CARLOS A. FELIPPA\* AND EUGENIO OÑATE†

\* *Department of Aerospace Engineering Sciences  
and Center for Aerospace Structures  
University of Colorado  
Boulder, Colorado 80309-0429, USA*

† *International Center for Numerical Methods in Engineering (CIMNE)  
Edificio C-1, c. Gran Capitán s/n  
Universidad Politécnica de Cataluña  
08034 Barcelona, Spain*

## Summary

We define stress and strain splittings appropriate to linearly-elastic anisotropic materials with volumetric constraints. The treatment includes rigidotropic materials, which develop no strains under a stress pattern that is a null eigenvector of the compliance matrix. This model includes as special case incompressible materials, in which the eigenvector is hydrostatic stress. The main finding is that pressure and volumetric strain must be redefined as effective quantities. Using this idea, an energy decomposition that exactly separates deviatoric and volumetric energy follows.

**Keywords:** linear elasticity, solid, anisotropy, isotropy, rigidtrophy, incompressibility, isochoric, hydrochoric, volumetric constraints, stability, material, constitutive model, compliance, deviatoric, splitting.

## 1. Introduction

An isotropic solid is called incompressible if it is susceptible only of isochoric motions [1, p. 210]. If the material is linearly elastic the stress is specified by the strains only up to an arbitrary hydrostatic stress (pressure). This uncoupling forms the basis of splittings into volumetric and deviatoric strains long used in the analytical and numerical modeling of those materials.

The concept loses transparency for non-isotropic behavior because changes of shape and volume are generally coupled. A generalization aimed at linearly elastic anisotropic solids is the *rigidotropic* model introduced in [2]: the material is infinitely rigid under a specific nonzero stress pattern. The pattern is defined by the eigenvector that renders the compliance (strain-stress) matrix singular. If this eigenvector happens to be hydrostatic stress, it is shown in [2] that the material is isochoric (volume preserving) under any stress state. This specialization defines an *incompressible* anisotropic material.

With the generalization in place, useful splittings of stresses, strains and internal energy can be obtained for the numerical treatment of anisotropic solids with a singular compliance. An auxiliary tool used in this development is the concept of free-free flexibility and stiffness presented in [3,4]. These two singular matrices are the Moore-Penrose generalized inverses of each other, which can be expressed as projected ordinary inverses of modified matrices.

The splittings are worked out first for a general anisotropic material. They are then specialized to the orthotropic and isotropic cases. The treatment of the isotropic case is included to illustrate the connection to existing splitting methods.

## 2. Anisotropic Material

We consider a linearly-elastic anisotropic solid in three dimensions referred to axes  $\{x_i\}$ . Stresses  $\sigma_{ij}$  and strains  $e_{ij}$  will be arranged as 6-component column vectors constructed from the tensors through the usual conventions of structural mechanics:

$$\boldsymbol{\sigma} = [\sigma_{11} \ \sigma_{22} \ \sigma_{33} \ \sigma_{23} \ \sigma_{31} \ \sigma_{12}]^T, \quad \mathbf{e} = [e_{11} \ e_{22} \ e_{33} \ 2e_{23} \ 2e_{31} \ 2e_{12}]^T. \quad (1)$$

The mean normal stress is  $\sigma_m = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$ . A hydrostatic stress state is defined by  $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_m$ , others zero. The volumetric strain is  $e_v = e_{11} + e_{22} + e_{33}$ .

### 2.1. Constitutive Equations

The strain-stress constitutive equation is

$$\mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{13} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ & \text{symm} & & & & C_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \mathbf{C}\boldsymbol{\sigma}. \quad (2)$$

Here  $C_{ij}$  are compliance coefficients arranged into the symmetric compliance matrix  $\mathbf{C}$ . All diagonal entries  $C_{ii}$  are assumed to be nonnegative with a positive sum. The matrix  $\mathbf{C}$  is called *stable*, *semistable* or *unstable* if  $\mathbf{C}$  is positive definite, positive semidefinite, or indefinite, respectively. In the semistable case it will be assumed that  $\mathbf{C}$  has a rank deficiency of at most one to simplify the analysis.

The eigenvalues of  $\mathbf{C}$  are  $\gamma_i$  for  $i = 1, 2, \dots, 6$ , with  $\mathbf{v}_i$  the corresponding eigenvector normalized to length  $\sqrt{3}$ . (This nonstandard normalization simplifies linkage to the incompressible case.) Accordingly the spectral decomposition is

$$\mathbf{C} = \frac{1}{3} \sum_{i=1}^6 \gamma_i \mathbf{v}_i \mathbf{v}_i^T, \quad \mathbf{v}_i^T \mathbf{v}_j = 3\delta_{ij}, \quad (3)$$

where  $\delta_{ij}$  is the Kronecker delta. The eigenvalues are arranged so that  $\gamma_1 = \gamma_{min}$  and  $\gamma_6 = \gamma_{max}$  are the algebraically smallest and largest, respectively. For stable models  $\gamma_i > 0$  for  $i = 1, \dots, 6$ . For semistable models  $\gamma_1 = 0$ . Unstable models are not accepted.

If  $\mathbf{C}$  is stable the ordinary inverse is called  $\mathbf{E} = \mathbf{C}^{-1}$ . This matrix collects the moduli  $E_{ij}$  and relates stresses to strains:  $\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$ . It is called the elasticity or rigidity matrix in the continuum mechanics literature. If  $\mathbf{C}$  is semistable (singular) a generalized inverse is defined in Section 2.6.

The Rayleigh quotients of the  $\mathbf{v}_i$  are noted for later use:

$$\gamma_i = \frac{\mathbf{v}_i^T \mathbf{C} \mathbf{v}_i}{\mathbf{v}_i^T \mathbf{v}_i} = \frac{1}{3} \mathbf{v}_i^T \mathbf{C} \mathbf{v}_i. \quad (4)$$

### 2.2. Volumetric Constraints

The volumetric constraints defined in the Introduction are mathematically expressed in terms of  $\mathbf{C}$  as

$$\text{Rigidropic: } \gamma_1 = 0, \ \gamma_i > 0, \ i = 2, \dots, 6. \quad (5)$$

$$\text{Incompressible: } \text{rigidropic and } C_{1j} + C_{2j} + C_{3j} = 0, \ j = 1, 2, 3. \quad (6)$$

In both cases

$$\det(\mathbf{C}) = 0, \quad \mathbf{C} \mathbf{v}_1 = \mathbf{0}. \quad (7)$$

If  $\mathbf{v}_1$  is the hydrostatic stress mode  $[1 \ 1 \ 1 \ 0 \ 0 \ 0]^T$  the zero-column-sum condition (6) is verified and the material is incompressible. In the sequel no particular attention is placed on incompressibility because it is a subset of the more general rigidotropic model.

The following example matrix, referred to principal material axes, has the entry pattern appropriate to an hexagonal (trigonal-pyramidal, rhombohedral) material governed by the  $C_8$  symmetry group [1, p. 89]

$$\mathbf{C}_{hex} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & 0 \\ & C_{11} & C_{13} & -C_{14} & -C_{15} & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & -2C_{15} \\ & & & & C_{55} & 2C_{14} \\ \text{symm} & & & & & 2(C_{11}-C_{12}) \end{bmatrix} = \begin{bmatrix} 12 & -6 & -12 & -7 & 10 & 0 \\ -6 & 12 & -12 & 7 & -10 & 0 \\ -12 & -12 & 48 & 0 & 0 & 0 \\ -7 & 7 & 0 & 20 & 0 & -20 \\ 10 & -10 & 0 & 0 & 20 & -14 \\ 0 & 0 & 0 & -20 & -14 & 36 \end{bmatrix} \quad (8)$$

The set numerical entries are exact integers. The eigenvalues (listed to 6 places if noninteger) are  $\gamma_1 = 0$ ,  $\gamma_2 = 1.70838$ ,  $\gamma_3 = 2.30953$ ,  $\gamma_4 = 36.2916$ ,  $\gamma_5 = 53.6905$  and  $\gamma_6 = 54$ . The compliance (8) is semistable and rigidotropic. The null eigenvector is  $\mathbf{v}_1 \equiv \mathbf{w} = (1/\sqrt{3}) [2 \ 2 \ 1 \ 0 \ 0 \ 0]^T$ ,  $\mathbf{w}^T \mathbf{w} = 3$ .

### 2.3. QRT Materials and the Reference Model

Models that satisfy (7) exactly or to high numerical accuracy are rare if entries come from experimental data. It is far more common to find “quasi-rigidotropic” (QRT) behavior in the sense that  $\mathbf{C}$  has a tiny positive compliance in a stress eigenmode. Mathematically:

$$\gamma_1 > 0, \quad \gamma_1 < \epsilon_r \gamma_6, \quad \epsilon_r \ll 1. \quad (9)$$

Here  $\epsilon_r$  is a preset tolerance, for example  $10^{-3}$ , which triggers treatment of the material model as QRT. Note that the reciprocal  $1/\epsilon_r$  is a spectral condition number. A tiny but negative  $\gamma_1$  may also occur due to experimental noise; stabilization of such unstable models is discussed in Section 2.6.

If (9) is verified, one proceeds to compute two auxiliary quantities: the *reference rigidotropic model*  $\bar{\mathbf{C}}$  and an *effective bulk modulus*  $K$ . These are used to split the constitutive equations and internal energy in Sections 2.8ff.

The reference rigidotropic model  $\bar{\mathbf{C}}$  is that nearest  $\mathbf{C}$  satisfying (5). But how to define “nearness”? The answer is not unique. Two methods for constructing  $\bar{\mathbf{C}}$  are described below: projection and scaling. The key relations are summarized in Figure 1.

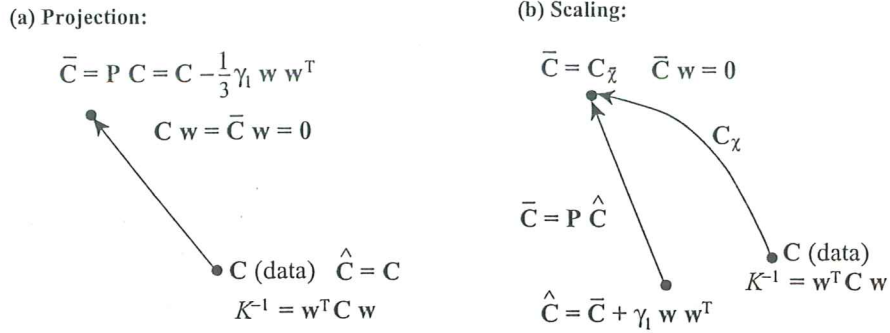
#### 2.4. $\bar{\mathbf{C}}$ by Projection

To apply the projection method, compute the eigensystem of  $\mathbf{C}$ . Pick the smallest eigenvalue  $\gamma_1$  and check whether the model classifies as QRT (For a tiny but negative  $\gamma_1$  see Section 2.6.) Call the associated eigenvector  $\mathbf{w} \equiv \mathbf{v}_1$ ,  $\mathbf{w}^T \mathbf{w} = 3$ . Redo the spectral decomposition (3) subtracting off  $\gamma_1$ :

$$\bar{\mathbf{C}} = \mathbf{C} - \frac{1}{3} \gamma_1 \mathbf{w} \mathbf{w}^T = \sum_{i=2}^6 \frac{1}{3} \gamma_i \mathbf{v}_i \mathbf{v}_i^T. \quad (10)$$

This is called a projection method since it is equivalent to pre- or post-multiplying  $\mathbf{C}$  by a  $\mathbf{w}$ -defined orthogonal projector  $\mathbf{P}$ :

$$\bar{\mathbf{C}} = \mathbf{P} \mathbf{C} = \mathbf{C} \mathbf{P} \mathbf{P} = \mathbf{I} - \frac{1}{3} \mathbf{w} \mathbf{w}^T, \quad \mathbf{P}^2 = \mathbf{P}, \quad (11)$$



where  $\mathbf{I}$  is the  $6 \times 6$  identity matrix. The effective bulk modulus  $K$  is computed as

$$K^{-1} = 3\gamma_1 = w^T C w. \quad (12)$$

The constitutive meaning of  $K$  as ratio of effective pressure to effective volumetric strain is discussed later after those quantities are introduced.

To build an example QRT matrix, the nonzero entries of the example matrix (8) are perturbed on the order of 1% by a random number generator. The results of the process is

$$C = \begin{bmatrix} 11.78029 & -5.90183 & -11.79849 & -6.96337 & 10.14722 & 0 \\ -5.90183 & 11.93375 & -11.82934 & 7.10696 & -9.92362 & 0 \\ -11.79849 & -11.82934 & 47.38653 & 0 & 0 & 0 \\ -6.96337 & 7.10696 & 0 & 20.01094 & 0 & -19.85809 \\ 10.14722 & -9.92362 & 0 & 0 & 20.04357 & -13.97715 \\ 0 & 0 & 0 & -19.85809 & -13.97715 & 36.16219 \end{bmatrix}. \quad (13)$$

The eigenvalues are now  $\gamma_1 = 0.04409$ ,  $\gamma_2 = 1.53218$ ,  $\gamma_3 = 2.50946$ ,  $\gamma_4 = 36.2653$ ,  $\gamma_5 = 53.2816$  and  $\gamma_6 = 53.6847$ . The eigenvector corresponding to  $\gamma_1 = 0.04409$  is

$$w = v_1 = [1.20725 \quad 1.09464 \quad 0.57438 \quad -0.02075 \quad -0.10602 \quad -0.05244]^T. \quad (14)$$

The projected reference model is

$$\bar{C} = C - \frac{1}{3} \gamma_1 w w^T = \begin{bmatrix} 11.75887 & -5.92125 & -11.80868 & -6.96300 & 10.14910 & 0.00093 \\ -5.92125 & 11.91614 & -11.83858 & 7.10729 & -9.92191 & 0.00084 \\ -11.80868 & -11.83858 & 47.38169 & 0.00018 & 0.00089 & 0.00044 \\ -6.96300 & 7.10729 & 0.00018 & 20.01094 & -0.00003 & -19.85810 \\ 10.14910 & -9.92191 & 0.00089 & -0.00003 & 20.04340 & -13.97723 \\ 0.00093 & 0.00084 & 0.00044 & -19.85810 & -13.97723 & 36.16215 \end{bmatrix} \quad (15)$$

The eigenvalues of (15) are the same as those of  $C$  except for  $\gamma_1 = 0$ . The eigenvectors do not change. The effective bulk modulus is  $K = 1/(3\gamma_1) = 1/(w^T C w) = 7.56067$ .

## 2.5. $\bar{C}$ by Scaling

Projection preserves eigenvectors (and all eigenvalues but  $\gamma_1$ ) but generally changes all entries of  $C$  as can be observed comparing (15) to (13). Occasionally it is desirable to keep the same diagonal entries or to preserve zero off-diagonal entries resulting from material symmetries known *a priori*.

In the scaling procedure the off-diagonal entries of the upper  $3 \times 3$  minor are multiplied by  $(1 + \chi)$ , where  $\chi$  is a real parameter:

$$C_\chi = \begin{bmatrix} C_{11} & (1 + \chi)C_{12} & (1 + \chi)C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & (1 + \chi)C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & & & & C_{55} & C_{56} \\ \text{symm} & & & & & C_{66} \end{bmatrix}. \quad (16)$$

Expand  $\det(C_\chi)$  as a polynomial in  $\chi$ . Solve  $\det C_\chi = 0$  for the smallest real root  $\bar{\chi}$ . Apply  $\bar{\chi}$  to  $C_\chi$  to get  $\bar{C} = C_{\bar{\chi}}$ . Although scaling keeps diagonal entries invariant and preserves zero off-diagonal ones, it will generally change all eigenvalues and eigenvectors. Hence it is necessary to redo the eigensystem analysis to get the spectral decomposition

$$\bar{C} = \frac{1}{3} \sum_{i=1}^6 \bar{\gamma}_i \bar{v}_i \bar{v}_i^T, \quad \bar{v}_i^T \bar{v}_i = 3. \quad (17)$$

Verify the rigidropy and stability conditions:  $\bar{\gamma}_1 = 0$ ,  $\bar{\gamma}_i > 0$  for  $i = 2, \dots, 6$ . If  $\bar{C}$  is unstable the projection method should be used as fall-back. If  $\bar{C}$  is semistable, pick  $w \equiv v_1$ . The effective bulk modulus is computed via the Rayleigh quotient of  $w$  on the original  $C$ :

$$K^{-1} = 3 \frac{w^T C w}{w^T w} = w^T C w. \quad (18)$$

Why not scale all off-diagonal entries? The answer is that scaling is only recommended when rigidropy is associated with lateral contraction (Poisson) effects controlled by  $C_{12}$ ,  $C_{13}$  and  $C_{23}$ , so it is applied where it is most effective. For instance the null eigenvector  $w$  of the exact rigidropic example compliance (8) has zeros in the last 3 components, while in the QRT perturbed matrix (13) those are very small.

To scale (13), entries  $C_{12}$ ,  $C_{13}$  and  $C_{23}$  are multiplied by  $(1 + \chi)$ . The determinant equation  $\det C_\chi = 0$  has three roots:  $\chi_1 = -3.00511$ ,  $\chi_2 = -0.467716$  and  $\chi_3 = 0.002822$ . Pick  $\bar{\chi} = \chi_3$  as that closest to zero, and substitute  $\bar{\chi}$  to get

$$\bar{C} = \begin{bmatrix} 11.78029 & -5.91849 & -11.83179 & -6.96337 & 10.14722 & 0 \\ -5.91849 & 11.93375 & -11.86272 & 7.10696 & -9.92362 & 0 \\ -11.83179 & -11.86272 & 47.38653 & 0 & 0 & 0 \\ -6.96337 & 7.10696 & 0 & 20.01094 & 0 & -19.85809 \\ 10.14722 & -9.92362 & 0 & 0 & 20.04357 & -13.97715 \\ 0 & 0 & 0 & -19.85809 & -13.97715 & 36.16219 \end{bmatrix} \quad (19)$$

This has eigenvalues  $\bar{\gamma}_1 = 0$ ,  $\bar{\gamma}_2 = 1.54084$ ,  $\bar{\gamma}_3 = 2.50934$ ,  $\bar{\gamma}_4 = 36.273$ ,  $\bar{\gamma}_5 = 53.3091$  and  $\bar{\gamma}_6 = 53.685$ . The null eigenvector  $w \equiv \bar{v}_1$  is

$$w = \bar{v}_1 = [1.20524 \quad 1.09659 \quad 0.57545 \quad -0.02113 \quad -0.10313 \quad -0.05146]^T. \quad (20)$$

The effective bulk modulus is  $K = 1/(w^T C w) = 7.55588$ . Comparing with the results of the projection method the answer are very similar. The main visible difference between the reference models (15) and (19) is that zero entries are exactly preserved in the latter.

## 2.6. Recommendations on Method Selection

Figure 1 graphically depicts both methods. Projection has the important advantage of being observer invariant since it works with the spectrum of  $C$ . If  $C$  is stable,  $\bar{C}$  is guaranteed to be semistable since  $\gamma_2$  through  $\gamma_6$  are unchanged. Scaling would produce different results as axes  $\{x_i\}$  are rotated; moreover it may be necessary to extend the  $(1 + \chi)$  factors to all off-diagonal components. For this reason scaling should be restricted to models satisfying three conditions:

- The projection method would erase zeros entries that should be preserve to express symmetries
- Rigidtrophy is due to lateral contraction (Poisson) effects governed by  $C_{12}$ ,  $C_{13}$  and  $C_{23}$
- Preservation of zero entries is important

Although the presumption (9) excludes QRT unstable models, in practice discovering a tiny but negative  $\gamma_1$  is as likely as having a positive value. Experimental noise coupled to preselection of symmetry groups may cause that to happen. The good news is that projection and scaling are unaffected by the sign of  $\gamma_1$ : if negative the computation of  $\bar{C}$  can be viewed as a model stabilization step.

## 2.7. The Stress-Strain Matrices

From now on a compliance that exactly satisfies (5) will be called  $\bar{C}$ . This has the spectral decomposition (17). This matrix generally comes from adjusting a QRT compliance  $C$  by one of the methods discussed in the three previous sections. The null eigenvector of  $\bar{C}$  is always called  $w$  for brevity:

$$\bar{C}w = 0, \quad w = [w_1 \quad w_2 \quad w_3 \quad w_4 \quad w_5 \quad w_6]^T, \quad w^T w = 3. \quad (21)$$

The projected elasticity matrix is defined as the Moore-Penrose generalized inverse of  $\bar{C}$ :

$$\bar{E} = P(\bar{C} + \frac{1}{3}ww^T)^{-1} = \frac{1}{3} \sum_{i=2}^6 \frac{1}{\bar{\gamma}_i} \bar{v}_i \bar{v}_i^T, \quad P = I - \frac{1}{3}ww^T. \quad (22)$$

The  $w$  used above is always the null eigenvector of the singular compliance. These expressions follow from the theory of the free-free flexibility developed in [3,4]. Matrix  $\bar{E}$  is symmetric, singular and positive semidefinite with null eigenvector  $w$ :  $\bar{E}w = 0$ . It verifies  $\bar{E}\bar{C} = \bar{C}\bar{E} = P$ . Relation (22) persists if  $\bar{C}$  and  $\bar{E}$  are switched and  $1/\bar{\gamma}_i$  replaced by  $\bar{\gamma}_i$ .

As an example let us compute the  $\bar{E}$  of (8), for which  $w^T = [2 \quad 2 \quad 1 \quad 0 \quad 0 \quad 0]/\sqrt{3}$ :

$$\bar{E}_{hex} = (I - \frac{1}{3}ww^T)(C_{hex} + ww^T)^{-1} = \frac{1}{30132} \begin{bmatrix} 4891 & -4829 & -124 & 3402 & -4860 & 0 \\ -4829 & 4891 & -124 & -3402 & 4860 & 0 \\ -124 & -124 & 496 & 0 & 0 & 0 \\ 3402 & -3402 & 0 & 8748 & 0 & 4860 \\ -4860 & 4860 & 0 & 0 & 8748 & 3402 \\ 0 & 0 & 0 & 4860 & 3402 & 4860 \end{bmatrix}. \quad (23)$$

As a visual check, the pattern of zero entries is exactly that of  $C_{hex}$ . Furthermore  $\bar{E}_{44} = \frac{1}{2}(\bar{E}_{11} - \bar{E}_{12})$ ,  $\bar{E}_{46} = -\bar{E}_{15}$ , etc., as appropriate for the hexagonal symmetry group [1, p. 89]. The ordinary inverse of  $C_{hex}$  does not exist.

For future use define

$$\hat{C} = \bar{C} + \frac{1}{9}K^{-1}ww^T, \quad \hat{E} = \bar{E} + Kww^T. \quad (24)$$



These have the useful properties

$$\hat{\mathbf{E}}\hat{\mathbf{C}} = \hat{\mathbf{C}}\hat{\mathbf{E}} = \mathbf{I}, \quad \hat{\mathbf{C}}\mathbf{w} = \frac{1}{3}K^{-1}\mathbf{w}, \quad \hat{\mathbf{E}}\mathbf{w} = 3K\mathbf{w}, \quad \bar{\mathbf{C}} = \mathbf{P}\hat{\mathbf{C}} = \hat{\mathbf{C}}\mathbf{P}, \quad \bar{\mathbf{E}} = \mathbf{P}\hat{\mathbf{E}} = \hat{\mathbf{E}}\mathbf{P}. \quad (25)$$

Of course  $\hat{\mathbf{E}}$  blows up as  $K \rightarrow \infty$  but this matrix is never assembled as such; it is a convenient “splitting tool” in the spirit of the delta function, which works behind the scenes and disappears once done.

If the projection method is used to get  $\bar{\mathbf{C}}$ , matrices  $\hat{\mathbf{C}}$  and  $\hat{\mathbf{E}}$  reduce to  $\mathbf{C}$  and  $\mathbf{E}$ , respectively.

## 2.8. Effective Pressure and Volumetric Strain

Define effective pressure  $p$  and effective volumetric strain  $\theta$  as

$$\begin{aligned} p &= \frac{1}{3}\mathbf{w}^T\boldsymbol{\sigma} = \frac{1}{3}(w_1\sigma_{11} + w_2\sigma_{22} + w_3\sigma_{33} + w_4\sigma_{23} + w_5\sigma_{31} + w_6\sigma_{12}), \\ \theta &= \mathbf{w}^T\mathbf{e} = w_1e_{11} + w_2e_{22} + w_3e_{33} + 2w_4e_{23} + 2w_5e_{31} + 2w_6e_{12}. \end{aligned} \quad (26)$$

The ratio of  $p$  to  $\theta$  is called the effective bulk modulus:

$$p = K\theta, \quad \theta = K^{-1}p. \quad (27)$$

The proof that  $K^{-1} = \mathbf{w}^T\mathbf{C}\mathbf{w}$  is given in Section 2.10.

For the incompressible specialization  $\mathbf{w} = [1 \ 1 \ 1 \ 0 \ 0 \ 0]^T$ . In this case  $p$  and  $\theta$  reduce to the mean normal stress and usual volumetric strain, respectively:

$$p = \frac{1}{3}\mathbf{w}^T\boldsymbol{\sigma} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33}) = \sigma_m, \quad \theta = \mathbf{w}^T\mathbf{e} = e_{11} + e_{22} + e_{33} = e_v. \quad (28)$$

Incompressibility may be also characterized by  $\theta = \mathbf{div} \mathbf{u} = \partial u_i / \partial x_i = 0$  for any admissible motion, where  $\mathbf{u} = [u_1 \ u_2 \ u_3]^T$  is the displacement field.

## 2.9. Stress-Strain Splitting

The definitions (26) are used to introduce effective deviatoric stresses  $s_{ij}$  and deviatoric strains  $e_{ij}$  by

$$\mathbf{s} = \boldsymbol{\sigma} - \mathbf{w}p, \quad \mathbf{g} = \mathbf{e} - \frac{1}{3}\mathbf{w}\theta, \quad (29)$$

in which the vector arrangements mimic those of (1):

$$\mathbf{s} = [s_{11} \ s_{22} \ s_{33} \ s_{23} \ s_{31} \ s_{12}]^T, \quad \mathbf{g} = [g_{11} \ g_{22} \ g_{33} \ 2g_{23} \ 2g_{31} \ 2g_{12}]^T, \quad (30)$$

The key decoupling relations that follow from this definition are

$$\mathbf{w}^T\mathbf{s} = 0, \quad \mathbf{w}^T\mathbf{g} = 0. \quad (31)$$

From these easily follow the projective properties

$$\mathbf{s} = \mathbf{P}\boldsymbol{\sigma}, \quad \mathbf{g} = \mathbf{P}\mathbf{e}, \quad \mathbf{s}^T\mathbf{e} = \mathbf{s}^T\mathbf{g}, \quad \mathbf{g}^T\mathbf{s} = \mathbf{g}^T\boldsymbol{\sigma}, \quad (32)$$

where  $\mathbf{P} = \mathbf{I} - \frac{1}{3}\mathbf{w}\mathbf{w}^T$ .

## 2.10. QRT Constitutive Equations

To derive the split constitutive equations of a QRT material,  $\mathbf{e} = \mathbf{C}\boldsymbol{\sigma}$  is replaced by  $\mathbf{e} = \hat{\mathbf{C}}\boldsymbol{\sigma}$ . The splitting (29) and the bulk equation (27) are introduced, and the decoupling relations (31)-(32) used:

$$\mathbf{e} = \hat{\mathbf{C}}\boldsymbol{\sigma} = (\bar{\mathbf{C}} + \frac{1}{3}K^{-1}\mathbf{w}\mathbf{w}^T)(\mathbf{s} + \mathbf{w}p) = \bar{\mathbf{C}}\mathbf{s} + \frac{1}{3}\mathbf{w}K^{-1}p = \mathbf{g} + \frac{1}{3}\mathbf{w}\theta. \quad (33)$$

Identifying gives  $\mathbf{g} = \bar{\mathbf{C}}\mathbf{s}$ . To work out the inverse relation,  $\boldsymbol{\sigma} = \mathbf{E}\mathbf{e}$  is replaced by  $\boldsymbol{\sigma} = \hat{\mathbf{E}}\mathbf{e}$  and use made of (25):

$$\boldsymbol{\sigma} = \hat{\mathbf{E}}\mathbf{e} = (\bar{\mathbf{E}} + K\mathbf{w}\mathbf{w}^T)(\mathbf{g} + \frac{1}{3}\mathbf{w}\theta) = \bar{\mathbf{E}}\mathbf{g} + \mathbf{w}K\theta = \mathbf{s} + \mathbf{w}p. \quad (34)$$

Identifying gives  $\mathbf{s} = \bar{\mathbf{E}}\mathbf{g}$ . Summarizing, the split constitutive equations for finite  $K$  are

$$\mathbf{g} = \bar{\mathbf{C}}\mathbf{s} = \bar{\mathbf{C}}\boldsymbol{\sigma}, \quad \mathbf{s} = \bar{\mathbf{E}}\mathbf{g} = \bar{\mathbf{E}}\mathbf{e}, \quad \theta = K^{-1}p, \quad p = K\theta. \quad (35)$$

and in terms of total strains and stresses

$$\mathbf{e} = \bar{\mathbf{C}}\mathbf{s} + \frac{1}{3}\mathbf{w}K^{-1}p = \bar{\mathbf{C}}\boldsymbol{\sigma} + \frac{1}{3}\mathbf{w}K^{-1}p, \quad \boldsymbol{\sigma} = \bar{\mathbf{E}}\mathbf{g} + \mathbf{w}K\theta = \bar{\mathbf{E}}\mathbf{e} + \mathbf{w}K\theta. \quad (36)$$

If the material is exactly rigidotropic,  $K \rightarrow \infty$ , in which case (35) collapse to  $\mathbf{e} = \bar{\mathbf{C}}\mathbf{s}$ ,  $\mathbf{s} = \bar{\mathbf{E}}\mathbf{e}$ ,  $\theta = 0$ , and the effective pressure  $p$  is completely decoupled from deformations.

We can now show that  $K^{-1} = \mathbf{w}^T\mathbf{C}\mathbf{w}$  as follows. For the projection treatment of a QRT model:

$$K = \frac{p}{\theta} = \frac{\frac{1}{3}\mathbf{w}^T\boldsymbol{\sigma}}{\mathbf{w}^T\mathbf{e}} = \frac{\frac{1}{3}\mathbf{w}^T\boldsymbol{\sigma}}{\mathbf{w}^T\mathbf{C}\boldsymbol{\sigma}} = \frac{\frac{1}{3}\mathbf{w}^T(\mathbf{s} + \mathbf{w}p)}{\mathbf{w}^T(\bar{\mathbf{C}} + \frac{1}{3}\gamma_1\mathbf{w}\mathbf{w}^T)(\mathbf{s} + \mathbf{w}p)} = \frac{p}{3\gamma_1 p} = \frac{1}{3\gamma_1} = \frac{1}{\mathbf{w}^T\mathbf{C}\mathbf{w}}. \quad (37)$$

If  $\bar{\mathbf{C}}$  is obtained by scaling,  $K^{-1} = \mathbf{w}^T\mathbf{C}\mathbf{w}$  is simply a definition suggested by the Rayleigh quotient approximation properties. It is easily verified that for an isotropic material of elastic modulus  $E$  and Poisson's ratio  $\nu$ ,  $K$  reduces to the ordinary bulk modulus  $\frac{1}{3}E/(1 - 2\nu)$  for any  $\nu$ .

## 2.11. Energy Splitting

Insertion of (29) into the internal energy density  $\mathcal{U} = \frac{1}{2}\boldsymbol{\sigma}^T\mathbf{e}$  produces an exact split into deviatoric and volumetric energies

$$2\mathcal{U} = 2\mathcal{U}_d + 2\mathcal{U}_v = \mathbf{s}^T\mathbf{g} + p\theta = \mathbf{s}^T\mathbf{e} + p\theta = \boldsymbol{\sigma}^T\mathbf{g} + p\theta, \quad (38)$$

in which the last two transformations follow from (32). Substitution of the constitutive equations (35) or (36) and use of orthogonality conditions permits selective eliminations. For example

$$2\mathcal{U}_d = \mathbf{s}^T\mathbf{e} = \mathbf{s}^T\bar{\mathbf{C}}\mathbf{s} = \boldsymbol{\sigma}^T\bar{\mathbf{C}}\boldsymbol{\sigma} = \mathbf{g}^T\bar{\mathbf{E}}\mathbf{g} = \mathbf{e}^T\bar{\mathbf{E}}\mathbf{e}, \quad 2\mathcal{U}_v = p\theta = \frac{p^2}{K} = K\theta^2. \quad (39)$$

In  $\mathcal{U}$ , deviators  $\mathbf{g}$  may be replaced by  $\mathbf{e}$  as in  $\mathbf{s}^T\mathbf{g} = \mathbf{s}^T\mathbf{e}$ , etc., as per (32). Likewise  $\mathbf{s}$  can be replaced by  $\boldsymbol{\sigma}$  as in  $\mathbf{s}^T\mathbf{g} = \boldsymbol{\sigma}^T\mathbf{g}$ , etc. In displacement FEM formulations it is often convenient to use total strains.

In exact rigidtrophy the volumetric energy vanishes.

### 3. Isotropic Material

The foregoing derivations are specialized to isotropic linear elasticity to illustrate connections to conventional splitting methods. The strain-stress relations are

$$\mathbf{e} = \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 2(1+\nu) & 0 & 0 \\ & & & & 2(1+\nu) & 0 \\ \text{symm} & & & & & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \mathbf{C}\boldsymbol{\sigma}, \quad (40)$$

where  $E$  is the elastic modulus and  $\nu$  is Poisson's ratio. The determinant of  $\mathbf{C}$  is  $8(1+\nu)^5(1-2\nu)/E^6$ . For nonnegative  $\nu$  this vanishes for  $\nu = \frac{1}{2}$ . Ordinary inversion of (40) yields

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = E_\nu \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & \frac{1}{2}(1-2\nu) & 0 & 0 \\ & & & & \frac{1}{2}(1-2\nu) & 0 \\ \text{symm} & & & & & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ 2e_{23} \\ 2e_{31} \\ 2e_{12} \end{bmatrix} = \mathbf{E}\mathbf{e}. \quad (41)$$

where  $E_\nu = E/((1-2\nu)(1+\nu))$ . Coefficient  $E_\nu$  "blows up" if  $\nu \rightarrow \frac{1}{2}$ . This is a well known problem in the numerical analysis of such models by displacement-based finite element methods. For this application it has been addressed by splittings into deviatoric stresses and pressure since the mid 1960s [5].

Let  $\bar{\mathbf{C}}$  be  $\mathbf{C}$  evaluated at  $\nu = \frac{1}{2}$ . This matrix projects any  $\boldsymbol{\sigma}$  onto the space of traceless (divergence free) strain tensors. The null eigenvector of  $\bar{\mathbf{C}}$  is

$$\mathbf{w} = [1 \ 1 \ 1 \ 0 \ 0 \ 0]^T \quad (42)$$

This represents hydrostatic pressure; consequently rigidotropic and incompressible models coalesce. In fact, (42) is an eigenvector of the compliance matrix (40) for any  $\nu$ . It is easily verified that

$$\mathbf{w}^T \mathbf{C} \mathbf{w} = \frac{3(1-2\nu)}{E} = K^{-1} \quad (43)$$

where  $K$  is the conventional bulk modulus. Hence  $\gamma_1 = \frac{1}{3}K^{-1}$  exactly for any  $\nu$ . The orthogonal projector  $\mathbf{P}$  is

$$\mathbf{P} = \mathbf{I} - \frac{1}{3}\mathbf{w}\mathbf{w}^T = \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 & -1 & -1 & -1 \\ -1 & 2 & -1 & -1 & -1 & -1 \\ -1 & -1 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 2 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 \end{bmatrix} \quad (44)$$

The generalized inverse of  $\bar{\mathbf{C}}$  defined by (22) becomes

$$\bar{\mathbf{E}} = \mathbf{P}(\bar{\mathbf{C}} + \mathbf{w}\mathbf{w}^T)^{-1} = \frac{E}{9} \begin{bmatrix} 4 & -2 & -2 & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & 0 & 0 \\ -2 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} = \frac{2(1+\nu)}{3} \mathbf{P}\mathbf{E} = \frac{2(1+\nu)}{3} \mathbf{E}\mathbf{P}, \quad (45)$$

$\bar{\mathbf{E}}$  projects any strain  $\mathbf{e}$  onto the space of traceless stress tensors. Since all relations survive if  $\bar{\mathbf{C}}$  and  $\bar{\mathbf{E}}$  are interchanged, they are *dual* of each other. In particular, exchanging  $\bar{\mathbf{C}}$  and  $\bar{\mathbf{E}}$  in (45) recovers  $\bar{\mathbf{C}}$  as expected:

$$\bar{\mathbf{C}} = \mathbf{P}(\bar{\mathbf{E}} + \mathbf{w}\mathbf{w}^T)^{-1} = \frac{1}{2E} \begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} = \frac{3}{2(1+\nu)} \mathbf{P}\mathbf{C} = \frac{3}{2(1+\nu)} \mathbf{C}\mathbf{P}. \quad (46)$$

The mean-deviatoric splitting of strains and stresses is  $\boldsymbol{\sigma} = \mathbf{s} + \mathbf{w} p$  and  $\mathbf{e} = \mathbf{g} + \frac{1}{3} \mathbf{w} \theta$ , with

$$p = \frac{1}{3} \mathbf{w}^T \boldsymbol{\sigma} = \frac{1}{3} (\sigma_{11} + \sigma_{22} + \sigma_{33}), \quad \theta = \mathbf{w}^T \mathbf{e} = e_{11} + e_{22} + e_{33}, \quad p = K\theta. \quad (47)$$

Here  $\mathbf{s}$  and  $\mathbf{g}$  are the deviatoric stresses and strains arranged as per (30),  $p$  is the mean normal stress and  $\theta$  the conventional volumetric strain. The induced decomposition of the constitutive relations reads

$$\mathbf{e} = \mathbf{C}\boldsymbol{\sigma} = \mathbf{C}\mathbf{s} + \mathbf{w} \frac{1}{3} K^{-1} p = \mathbf{C}\mathbf{s} + \frac{1}{3} \mathbf{w} \theta, \quad \boldsymbol{\sigma} = \mathbf{E}\mathbf{e} = \mathbf{E}\mathbf{g} + \mathbf{w} K \theta = \mathbf{E}\mathbf{g} + \mathbf{w} p. \quad (48)$$

The deviatoric constitutive equations reduce to

$$\mathbf{g} = \mathbf{C}\mathbf{s} = \frac{2(1+\nu)}{3} \bar{\mathbf{C}}\boldsymbol{\sigma} = \frac{2(1+\nu)}{3} \bar{\mathbf{C}}\mathbf{s}, \quad \mathbf{s} = \mathbf{E}\mathbf{g} = \frac{3}{2(1+\nu)} \bar{\mathbf{E}}\mathbf{e} = \frac{3}{2(1+\nu)} \bar{\mathbf{E}}\mathbf{g}, \quad (49)$$

where those involving  $\bar{\mathbf{E}}$  stay bounded as  $\nu \rightarrow \frac{1}{2}$ . The strain energy density splits exactly into deviatoric and volumetric parts for any  $\nu$ :

$$2\mathcal{U} = \boldsymbol{\sigma}^T \mathbf{e} = \mathbf{s}^T \mathbf{g} + p\theta = \mathbf{s}^T \mathbf{e} + p\theta = \boldsymbol{\sigma}^T \mathbf{g} + p\theta. \quad (50)$$

This decomposition is the basis of energy methods (in particular finite element methods) for incompressible and near-incompressible materials. For example, in a displacement-pressure formulation

$$2\mathcal{U} = \frac{3}{2(1+\nu)} \mathbf{e}^T \bar{\mathbf{E}} \mathbf{e} + \frac{p^2}{K}. \quad (51)$$

Here  $\mathbf{e}$  comes from displacements. Displacement-pressure coupling terms, such as  $p \operatorname{div} \mathbf{u}$ , are added to establish mixed variational principles [6]. For an incompressible material  $K \rightarrow \infty$  and the  $p^2/K$  term vanishes.

## 4. Orthotropic Materials

In this section we study an orthotropic material with the  $x_i$  aligned with the principal material axes. This case is worth separate consideration on account of the technical importance of such materials and because several important results can be obtained in closed form.

### 4.1. Rigidropy and Incompressibility Conditions

The principal elastic moduli are  $E_1$ ,  $E_2$  and  $E_3$ , whereas the principal shear moduli are  $G_1$ ,  $G_2$  and  $G_3$ . The six moduli are assumed to be positive. Three Poisson ratios are defined symmetrically with respect to the geometric means  $\sqrt{E_1 E_2}$ ,  $\sqrt{E_2 E_3}$  and  $\sqrt{E_3 E_1}$ , a device that circumvents the widespread befuddling practice of carrying six Poisson ratios linked by three symmetry constraints. Accordingly the compliance matrix is written

$$\mathbf{C} = \begin{bmatrix} 1/E_1 & -\nu_{12}/\sqrt{E_1 E_2} & -\nu_{13}/\sqrt{E_1 E_3} & 0 & 0 & 0 \\ & 1/E_2 & -\nu_{23}/\sqrt{E_2 E_3} & 0 & 0 & 0 \\ & & 1/E_3 & 0 & 0 & 0 \\ & & & 1/G_1 & 0 & 0 \\ & & & & 1/G_2 & 0 \\ \text{symm} & & & & & 1/G_3 \end{bmatrix} \quad (52)$$

The determinant of  $\mathbf{C}$  is  $(1 - (\nu_{12}^2 + \nu_{23}^2 + \nu_{13}^2) + 2\nu_{12}\nu_{23}\nu_{13})/(E_1 E_2 E_3 G_1 G_2 G_3)$ . This vanishes if

$$\nu_{12}^2 + \nu_{23}^2 + \nu_{13}^2 + 2\nu_{12}\nu_{23}\nu_{13} = 1. \quad (53)$$

If  $\nu_{12} = \nu_{23} = \nu_{13} = \nu$  condition (53) reduces to  $(\nu - \frac{1}{2})(\nu + 1)^2 = 0$ , which has the positive root  $\nu = \frac{1}{2}$  and the negative double root  $-1$ . This result is more general than the isotropic material studied in Section 3 because it holds for arbitrary  $E_1$ ,  $E_2$  and  $E_3$ . Many rational solutions of (53) exist. The simplest one with three different positive ratios is  $\nu_{12} = 9/16$ ,  $\nu_{23} = 3/4$ ,  $\nu_{13} = 1/8$ , and cyclic permutations thereof; also the signs of two ratios may be simultaneously switched.

Equation (53) is only a necessary condition for rigidropy. It remains to check stability. One eigenvalue of  $\mathbf{C}$  is 0 and three others are simply  $1/G_i > 0$ . Two roots remain. An inertia analysis of the uppermost  $3 \times 3$  principal minor shows that both roots are positive if and only if

$$-1 < \nu_{12} < 1, \quad -1 < \nu_{13} < 1, \quad -1 < \nu_{23} < 1, \quad (54)$$

for any  $E_1 > 0$ ,  $E_2 > 0$  and  $E_3 > 0$ . Consequently (53) and (54) are necessary and sufficient conditions for rigidropy. Poisson's ratios over that range may be realized in composite materials.

The incompressibility conditions are far more restrictive:  $C_{11} + C_{12} + C_{13} = C_{12} + C_{22} + C_{23} = C_{13} + C_{23} + C_{33} = 0$ . Given  $E_1, E_2$ , and  $E_3$  as data, this can only be satisfied if the Poisson ratios become  $\nu_{12} = \frac{1}{2}(E_1^{-1} + E_2^{-1} - E_3^{-1})\sqrt{E_1 E_2}$ ,  $\nu_{13} = \frac{1}{2}(E_3^{-1} + E_1^{-1} - E_2^{-1})\sqrt{E_1 E_3}$  and  $\nu_{23} = \frac{1}{2}(E_2^{-1} + E_3^{-1} - E_1^{-1})\sqrt{E_2 E_3}$ . If  $E_1 = E_2 = E_3 = E$  this gives  $\nu_{12} = \nu_{13} = \nu_{23} = \frac{1}{2}$  as can be expected. But if the moduli are widely different, at least one of the Poisson ratios may stray out of the stable region (54). As a consequence the compliance matrix becomes unstable, meaning that incompressible orthotropic models are physically impossible if the ratios  $E_1/E_2$  and  $E_1/E_3$  depart sufficiently away from unity. A detailed analysis along with a stability chart is provided in [2].

## 4.2. Spectral Analysis

Suppose the rigidropy conditions (53)-(54) exactly holds so that  $\mathbf{C}$  is renamed  $\bar{\mathbf{C}}$ . The null eigenvector of squared-length 3 is as usual called  $\mathbf{w}$ , which has the configuration

$$\mathbf{w} = [w_1 \ w_2 \ w_3 \ 0 \ 0 \ 0]^T, \quad \mathbf{w}^T \mathbf{w} = w_1^2 + w_2^2 + w_3^2 = 3. \quad (55)$$

A closed form solution for the first 3 entries is:

$$\begin{aligned} w_1 &= v_1 \sqrt{3E_1/S}, & w_2 &= v_2 \sqrt{3E_2/S}, & w_3 &= v_3 \sqrt{3E_3/S}, \\ v_1 &= (1 + v_{23})(1 + v_{12} + v_{13} - v_{23}), & v_2 &= (1 + v_{13})(1 + v_{12} - v_{13} + v_{23}), \\ v_3 &= (1 + v_{12})(1 - v_{12} + v_{13} + v_{23}), & S &= E_1 v_1^2 + E_2 v_2^2 + E_3 v_3^2. \end{aligned} \quad (56)$$

The effective pressure and volumetric strain are defined as explained in Section 2.8:  $p = \frac{1}{3} \mathbf{w}^T \boldsymbol{\sigma} = \frac{1}{3}(w_1 \sigma_{11} + w_2 \sigma_{22} + w_3 \sigma_{33})$  and  $\theta = \mathbf{w}^T \mathbf{e} = w_1 e_{11} + w_2 e_{22} + w_3 e_{33}$ . The projected elasticity matrix, split constitute equations and energy are obtained in the usual manner. where the normal stresses and strains are weighted by the first 3 entries of  $\mathbf{w}$ .

For the quasi-rigidropic (QRT) case the reference model  $\bar{\mathbf{C}}$  can be obtained by projection or scaling. Since the last 3 components of the eigenvector  $\mathbf{v}_1$  should be zeros, both methods preserve the off-diagonal zero entries and thus the orthotropy pattern. Hence as noted in Section 2.6 the projection method is preferable.

## 4.3. Example

As an example, consider the QRT orthotropic complaine:

$$\mathbf{C} = \frac{1}{144} \begin{bmatrix} 144.00396 & -53.99340 & -26.99208 & 0 & 0 & 0 \\ -53.99340 & 36.01100 & -2.98680 & 0 & 0 & 0 \\ -26.99208 & -2.98680 & 16.01584 & 0 & 0 & 0 \\ 0 & 0 & 0 & 288 & 0 & 0 \\ 0 & 0 & 0 & 0 & 720 & 0 \\ 0 & 0 & 0 & 0 & 0 & 432 \end{bmatrix} \quad (57)$$

The reference model is obtained by the projection method. The eigenvalues of  $\mathbf{C}$  are  $\gamma_1 = 0.03080$ ,  $\gamma_2 = 25.93059$ ,  $\gamma_3 = 170.06941$ ,  $\gamma_4 = 288$ ,  $\gamma_5 = 432$  and  $\gamma_6 = 720$ . The eigenvector corresponding to  $\gamma_1$ , normalized to squared-length 3 is

$$\mathbf{w} = \mathbf{v}_1 = [0.621059 \ 1.035098 \ 1.242118 \ 0 \ 0 \ 0]^T. \quad (58)$$

from which  $\bar{\mathbf{C}} = \mathbf{C} - \frac{1}{3} \gamma_1 \mathbf{w} \mathbf{w}^T$  giving

$$\bar{\mathbf{C}} = \frac{1}{144} \begin{bmatrix} 144 & -54 & -27 & 0 & 0 & 0 \\ -54 & 36 & -3 & 0 & 0 & 0 \\ -27 & -4 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 288 & 0 & 0 \\ 0 & 0 & 0 & 0 & 720 & 0 \\ 0 & 0 & 0 & 0 & 0 & 432 \end{bmatrix} = \begin{bmatrix} 1 & -3/8 & -3/16 & 0 & 0 & 0 \\ -3/8 & 1/4 & -1/48 & 0 & 0 & 0 \\ -3/16 & -1/48 & 1/9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \quad (59)$$

The eigenvalues are the same as  $\mathbf{C}$  except for  $\gamma_1 = 0$ . The bulk modulus is  $K = 1/(\mathbf{w}^T \bar{\mathbf{C}} \mathbf{w}) = 1/(3\gamma_1) = 10.8225$ .

The nice result (59) should come as no surprise since (57) was “cooked” by a spectral shift of  $\gamma_1 = 77/2500$  of the  $\bar{\mathbf{C}}$  matrix fabricated with  $E_1 = 1$ ,  $E_2 = 4$ ,  $E_3 = 9$ ,  $G_1 = 1/2$ ,  $G_2 = 1/5$ ,  $G_3 = 1/3$ ,  $\nu_{12} = 3/4$ ,  $\nu_{13} = 9/16$  and  $\nu_{23} = 1/8$ . The exact null eigenvector is  $\mathbf{w}^T = [3 \ 5 \ 6 \ 0 \ 0 \ 0] \sqrt{3/70}$ .

The projected elasticity matrix is

$$\bar{\mathbf{E}} = \mathbf{P}(\bar{\mathbf{C}} + \frac{1}{3}\mathbf{w}\mathbf{w}^T)^{-1} = \frac{1}{7350} \begin{bmatrix} 6432 & 2880 & -5616 & 0 & 0 & 0 \\ 2880 & 21600 & -19440 & 0 & 0 & 0 \\ -5616 & -19440 & 19008 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3675 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1470 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2450 \end{bmatrix}. \quad (60)$$

The eigenvalues of  $\bar{\mathbf{E}}$  are  $\mu_1 = 0$ ,  $\mu_2 = 5.55329$ ,  $\mu_3 = 0.846713$ ,  $\mu_4 = 0.5$ ,  $\mu_5 = 0.333333$ , and  $\mu_6 = 0.2$ . These are the reciprocals of the corresponding eigenvalues of  $\bar{\mathbf{c}}$  except for  $\gamma_1$  which becomes  $\mu_1 = 0$ .

The effective pressure and volumetric strain are

$$p = \frac{1}{3}\mathbf{w}^T \boldsymbol{\sigma} = \sqrt{\frac{1}{210}}(3\sigma_{11} + 5\sigma_{22} + 6\sigma_{33}), \quad \theta = \mathbf{w}^T \mathbf{e} = \sqrt{\frac{3}{70}}(3e_{11} + 5e_{22} + 6e_{33}). \quad (61)$$

The calculation of a similar  $\bar{\mathbf{C}}$  by scaling is left as an exercise.

## 5. Conclusions

This article considers the case of volumetric constraints on linearly-elastic anisotropic solids. The objective is to obtain splittings appropriate for numerical treatment. The constraints are subsumed under the umbrella rigidotropic behavior, in which the material does not deform under a stress pattern characterized by the null eigenvector of the compliance matrix. This model includes incompressible behavior as special case when that pattern is hydrostatic stress. Anisotropic incompressibility, however, is relatively rare in comparison to the isotropic case. Further, if enforced for highly anisotropic materials incompressibility may lead to unstable models [2].

The main findings of the paper are:

1. Rigidotropic behavior can be characterized by a spectral analysis of the compliance matrix. The incompressible subset is defined by a specialization of the null eigenvector.
2. Quasi-rigidotropic (QRT) models, which represent the extension of the quasi-incompressible models of isotropic materials, are handled by decomposition into a exactly rigidotropic reference model and the bulk behavior. Two methods: projection and scaling, for effecting this decomposition are described. These methods can also be used for fixing slightly unstable compliances due to noisy experimental data.
3. Correct splittings of the constitutive equations and internal energy require a redefinition of pressure and volumetric strain as effective quantities.

Using the splittings derived here, mixed variational principles appropriate for constructing finite element models have been derived in a separate document [7].

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