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Abstract

In recent years, Domain Decomposition Methods (DDM) have emerged as advanced solvers in several areas of computational mechanics. In particular, during the last decade, in the area of solid and structural mechanics, they reached a considerable level of advancement and were shown to be more efficient than popular solvers, like advanced sparse direct solvers. The present paper explores the extent of application of the general concept of force-displacement duality in DDM. A general framework for the definition of DDM is set up and it is shown that if the definition of a DDM meets some requirements, then it can lead to one primal and one dual formulation. A number of DDM are included in this setting and its particular implications for each one of them is researched.

1 Introduction – Historical background

In the last decade Domain Decomposition Methods (DDM) have progressed significantly leading to a large number of methods and techniques, capable of giving solution to various problems of computational mechanics. In the field of solid and structural mechanics, in particular, this fruitful period has led to the extensive parallel development of two large families of methods: (a) the Finite Element Tearing and Interconnecting (FETI) methods and (b) the Balancing Domain Decomposition (BDD) methods. Both introduced at the beginning of the 90s [1,2], these two categories of methods today include a large number of variants. Even though these two categories had many differences, it was gradually becoming apparent that there should exist links between them. Thus, in the present decade two studies [3,4] have attempted to determine the relations between the two methods.

In particular, the study [4] and its extention [5] explained a process used to transform a dual method into a primal one. This process was applied to basic FETI variants, like FETI-1, FETI-2 and FETI-DP, thus creating the primal methods P-FETI1, P-FETI2 and P-FETIDP, respectively. Studing the relations between the original FETI and P-FETI variants, it was observed in numerical experiments that FETI-1 and P-FETI1 had practically the same eigenspectrum. This observation was further stengthened by numerical experiments that showed similar convergence properties for the original methods and their primal offspring. These findings thus led reasonably to the conjecture that these two variants and possible other pairs of "compatible" dual and primal methods have practically the same eigenvalues ("practically" here meaning the exclusion of eigenvalues that would have small importance to the convergence of the methods). The equality of the eigenvalues was then proved in [6] for the methods FETI-DP and P-FETIDP (the later also known as BDDC from the work of [7]) and this proof was further simplified in [8]. Recently, another study [9] extended the concept of the relations between primal and dual methods to the case of the lumped preconditioned methods.

In the present paper, we attempt to encompase parts of the above knowledge in a general framework. We start by setting up a general description for the definition of DDM. This description starts by defining an operator that estimates globally subdomain displacements, from subdomain forces. It is then shown that if this operator meets one condition, it can define one primal and one dual method (section 2). It is also shown that if the initial operator meets another condition, then the primal and dual methods will have practically the same eigenspectrum (section 3). Defining the displacement estimate operator for several DDM, we show that they all satisfy the two conditions and thus inherit a number of properties proven in previous studies (section 4). Finally, we discuss the mechanical interpretation of the two employed conditions (section 5) and provide numerical evidence (section 6) in order to verify the conclusions (section 7) of this study.

2 General description of primal and dual DDM

The problem that we are interested in solving can be written as[1]

$$L^{\mathrm{T}}KLu_g = f_g \tag{1}$$

where

$$K = \begin{bmatrix} K^{(1)} & & \\ & \ddots & \\ & & K^{(n_s)} \end{bmatrix} \tag{2}$$

is the block-diagonal assemblage of the stiffness matrices $K^{(s)}$ of the subdomains $s = 1,\ldots, n_s$ and L is a Boolean matrix that maps the global displacements u_g and forces f_g to the corresponding subdomain variables.

If the internal d.o.f. of the subdomains are condensed eq. (1) becomes

$$L_b^{\mathrm{T}}SL_bu_{g_b} = \hat{f}_{g_b} \tag{3}$$

where
$$\hat{f}_{g_b} = f_{g_b} - L_b^{\mathrm{T}}K_{bi}K_{ii}^{-1}f_i \tag{4}$$

In eqs. (3) and (4), subscripts b and i denote restriction to interface and internal d.o.f. of the subdomains, respectively, while $S = K_{bb} - K_{bi}K_{ii}^{-1}K_{ib}$ is the block-diagonal assemblage of the subdomain Schur complements.

Assume that given some subdomain forces, an estimate for the subdomain displacements is given by

$$u = \tilde{K}^{+}f \tag{5}$$

or

$$u_b = \tilde{S}^{+}f_b \tag{6}$$

where

$$u = \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(n_s)} \end{bmatrix} \quad \text{and} \quad f = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(n_s)} \end{bmatrix} \tag{7}$$

are the vector block assemblage of the displacements and forces of the subdomains, respectively. Based on eq. (5), a general description for primal and dual DDM will be defined in the following two subsections.

2.1 *Primal methods*

Based on eqs. (5) and (6), we define a primal method as: apply the PCG method for solving the system

$$L^{\mathrm{T}}KLu_g = f_g \tag{8}$$

or

$$L_b^{\mathrm{T}}SL_bu_{g_b} = \hat{f}_{g_b} \tag{9}$$

[1] The present paper follows the notation used in recent papers [4,9,etc] with the main difference that superscripts s are removed from block-diagonal matrices

with preconditioners

(a) For eq. (8) $$L_p^{\mathrm{T}}\tilde{K}^{+}L_p \qquad (10)$$

(b) For eq. (9) $$L_{p_b}^{\mathrm{T}}\tilde{S}^{+}L_{p_b} \qquad (11)$$

In the above equations, L_p is a scaled variant of L, satisfying equation $L_p^{\mathrm{T}}L = I$ (see [4] and references cited there). Furthermore, for the method to be valid it is required that $\tilde{K}^{+}$ (or $\tilde{S}^{+}$) is such that the preconditioner is positive definite.

2.2 Dual methods

In order to define a dual method, based on the displacement estimates of eqs. (5) and (6), we first need some intermediate analysis. We start assuming that $\tilde{K}^{+}$ (or $\tilde{S}^{+}$) satisfy the condition

$$\tilde{K}^{+}KL = L \quad (\text{or} \quad \tilde{S}^{+}SL_b = L_b) \qquad (12)$$

We can then prove the following theorems:

Lemma 1. *If $\tilde{K}^{+}$ (or $\tilde{S}^{+}$) satisfies condition* (12)*, then if vector λ_1 satisfies relation*

$$L_p f_g - B^{\mathrm{T}}\lambda_1 = KLu_g \quad (\text{or} \quad L_{p_b}\hat{f}_{g_b} - B_b^{\mathrm{T}}\lambda_1 = SL_b u_{g_b}\ ) \qquad (13)$$

where B is a mapping matrix such that $\mathrm{null}(B) = \mathrm{range}(L)$*, then we have*

$$B\tilde{K}^{+}\left(L_p f_g - B^{\mathrm{T}}\lambda_1\right) = 0 \quad (\text{or} \quad B_b\tilde{S}^{+}\left(L_{p_b}\hat{f}_{g_b} - B_b^{\mathrm{T}}\lambda_1\right) = 0\ ) \qquad (14)$$

Proof. Eq. (14) follows from eq. (13), by multiplication by $B\tilde{K}^{+}$ (or $B_b\tilde{S}^{+}$) from the left, while taking into account eq. (12).

First, we note that eq. (13) is an equivalent expression of eq. (1) (or (3)). Therefore, eq. (1) (or (3)) has actually been transformed into eq. (14). However, given that in general eq. (14) has infinite solutions, in order to consider it equivalent to eq. (1) (or (3)), we also need to prove the following:

Lemma 2. *If $\tilde{K}^{+}$ (or $\tilde{S}^{+}$) satisfies condition (12) and λ_2 is a solution of eqs. (14) then the solution of eq. (1) (or (3)) is*

$$u_g = L_p^{\mathrm{T}}\tilde{K}^{+}\left(L_p f_g - B^{T}\lambda_2\right) \quad (\text{or} \quad u_{g_b} = L_{p_b}^{\mathrm{T}}\tilde{S}^{+}\left(L_{p_b}\hat{f}_{g_b} - B_b^{\mathrm{T}}\lambda_2\right) \qquad (15)$$

Proof. Substitute eq. (15) into eq. (1) (or (3)), while taking into account eq. (12), (14) and the fact that due to $L_p^{\mathrm{T}}L = I$, LL_p^{T} is the identity in $\mathrm{null}(B)$.

Therefore, based on eqs. (5) and (6) and using Lemmas 1 and 2, we define a dual method as: apply the PCG method for solving the system

$$B\tilde{K}^{+}B^{T}\lambda = B\tilde{K}^{+}L_{p}f_{g} \tag{16}$$

or
$$B_{b}\tilde{S}^{+}B_{b}^{\mathrm{T}}\lambda = B_{b}\tilde{S}^{+}L_{p_{b}}\hat{f}_{g_{b}} \tag{17}$$

with preconditioners

(a) For eq. (16) (lumped prec.) $$B_{p}KB_{p}^{\mathrm{T}} \tag{18}$$

(b) For eq.(17) (Dirichlet prec.) $$B_{p_{b}}SB_{p_{b}}^{\mathrm{T}} \tag{19}$$

where B_p is a scaled variant of B (see [4] and references cited there). Once eq. (16) (or (17)) is solved to the desired precision, the displacements are obtained from eq. (15). For the method to be valid, it is required that $\tilde{K}^{+}$ (or $\tilde{S}^{+}$) is such that matrix $B\tilde{K}^{+}B^{T}$ (or $B_{b}\tilde{S}^{+}B_{b}^{\mathrm{T}}$) is positive semi-definite. It is also worth noting that in all DDM that we treat later in this paper, $\tilde{K}^{+}$ and $\tilde{S}^{+}$ are chosen as to satisfy relation

$$\tilde{S}^{+} = N_{b}\tilde{K}^{+}N_{b}^{\mathrm{T}} \tag{20}$$

where matrix N_b is a Boolean matrix that extracts subdomain interface d.o.f. from full subdomain d.o.f. vectors, like in equation $u_b = N_b u$. Due to eq. (20), the left-hand side of eqs. (16) and (17) is the same.

In later sections of the paper, it will be shown many BDD, FETI and primal alternative methods belong in this framework, using the appropriate definitions for the displacement estimate operators $\tilde{K}^{+}$ and $\tilde{S}^{+}$.

3 Relations between the eigenspectrums of the primal and dual formulations

As it was mentioned in the introduction, in [6] it was shown that the FETI-DP and BDDC methods[2] have the same non-zero eigenvalues, while this proof was further simplified in [8]. In the present section, it will be proven that the primal and dual methods that have been introduced in a general setting in section 2 will have the same non-zero and non-unit eigenvalues, provided that the estimate $\tilde{K}^{s}$ (or $\tilde{S}^{+}$) is chosen so as to satisfy condition (12) and the following

$$\tilde{K}^{+}K\tilde{K}^{+} = \tilde{K}^{+} \quad (\text{or} \quad \tilde{S}^{+}S\tilde{S}^{+} = \tilde{S}^{+}) \tag{21}$$

[2] The BDDC method was introduced in [7] and it can be shown identical to the PFETI-DP with any vertex, edge or face coarse constraints [10]. In fact, this method was introduced independently in three studies: (a) as a preconditioner based on constrained energy minimization in [7] and later called BDDC, (b) as the primal derivative of the FETI-DP with only vertex constraints [11] or vertex, edge and face constraints [12] in [4] (In fact as primal alternative of the FETI-DP it was first mentioned and tested in an earlier publication [13]) and (c) as a preconditioner inspired from FETI-DP in [14]. In fact, even though the work in [14] is apparently restricted to vertex constraints and homogeneous scaling, that paper probably derived this method in the simplest and most intuitive way.

In particular, if we denote by A_P and $\tilde{A}_P^{-1}$ the matrix operator and preconditioner of the primal method and by A_D and $\tilde{A}_D^{-1}$ the respective operators of the dual method, then this eigenspectrum relations hold (as it will be shown) in the two cases:

1. No internal d.o.f. condensation case: A_P is chosen as the coefficient matrix of system (8) and $\tilde{A}_P^{-1}$ as eq. (10), while A_D as the coefficient matrix of system (16) and $\tilde{A}_D^{-1}$ as eq. (18)
2. Internal d.o.f condensation case: A_P is chosen as the coefficient matrix of system (9) and $\tilde{A}_P^{-1}$ as eq. (11), while A_D as the coefficient matrix of system (17) and $\tilde{A}_D^{-1}$ as eq. (19).

In what follows, we will refer only to the case where internal d.o.f. of the subdomains are condensed. It is quite simple to show that the eigenvalue equivalence proof and all remaining issues treated in this paper also hold in the case where internal d.o.f. are not condensed[3].

The eigenvalue equivalence proof follows ideas from [6]. First, we note that the mapping matrices L_b and B_b satisfy the (some of the most complete studies on these equations can be found in [3,6]):

$$\operatorname{range}(L_b) = \operatorname{null}(B_b) \quad \text{and} \quad \operatorname{range}(L_{p_b}) = \operatorname{null}(B_{p_b}) \tag{22}$$

$$(L_{p_b} L_b^{\mathrm{T}})^2 = L_{p_b} L_b^{\mathrm{T}} \quad \text{and} \quad (B_b^{\mathrm{T}} B_{p_b})^2 = B_b^{\mathrm{T}} B_{p_b} \tag{23}$$

$$L_b^{\mathrm{T}} L_{p_b} = I \quad \text{and} \quad L_{p_b} L_b^{\mathrm{T}} + B_b^{\mathrm{T}} B_{p_b} = I \tag{24}$$

We also need to prove the following relations:

Lemma 3. *If conditions (12) and (21) hold, then we have:*

$$L_{p_b}^{\mathrm{T}} \tilde{S}^{+} S L_b = I \tag{25}$$

$$B_b \tilde{S}^{+} S L_b = 0 \tag{26}$$

$$\tilde{S}^{+} B_b^{\mathrm{T}} B_{p_b} S \tilde{S}^{+} = \tilde{S}^{+} B_b^{\mathrm{T}} B_{p_b} \tag{27}$$

$$\tilde{S}^{+} B_b^{\mathrm{T}} B_{p_b} S \tilde{S}^{+} L_{p_b} = 0 \tag{28}$$

$$B_b \tilde{S}^{+} S B_{p_b}^{\mathrm{T}} \ \textit{is a projection} \tag{29}$$

Proof. Eqs. (25) and (26) follow directly from eq. (12) and eqs. (22) and (24). In order to prove (27), it suffices to use the two assumptions of this lemma, to obtain

[3] All equations that follow still hold if one substitutes all Schur complement matrices S with the full stiffness matrices K and all remaining variables that refer to interface d.o.f. with their alternatives that refer to all d.o.f.

$$\tilde{S}^+B_b^{\mathrm{T}}B_{p_b}S\tilde{S}^+ = \tilde{S}^+\left(I - L_{p_b}L_b^{\mathrm{T}}\right)S\tilde{S}^+ = \tilde{S}^+S\tilde{S}^+ - \tilde{S}^+L_{p_b}L_b^{\mathrm{T}}S\tilde{S}^+$$
$$= \tilde{S}^+ - \tilde{S}^+L_{p_b}L_b^{\mathrm{T}} = \tilde{S}^+\left(I - L_{p_b}L_b^{\mathrm{T}}\right) = \tilde{S}^+B_b^{\mathrm{T}}B_{p_b} \tag{30}$$

In addition, eq. (28) follows from (27) by multiplication by L_{p_b} from the right. Finally, to obtain (29), we multiply (27) by $B_{p_b}S$ from the left and by B_b^{T} from the right and the proof follows by noticing that $B_b^{\mathrm{T}}B_{p_b}B_b^{\mathrm{T}} = (I - L_{p_b}L_b^{\mathrm{T}})B_b^{\mathrm{T}} = B_b^{\mathrm{T}}$.

In order to prove that primal and dual methods have the same non-zero and non-unit eigenvalues, it suffices to prove the following theorem

Theorem 4. *If relations (12) and (21) hold, then we have:*

$$T_P\tilde{A}_P^{-1}A_P = \tilde{A}_D^{-1}A_DT_P \tag{31}$$

$$T_D\tilde{A}_D^{-1}A_D = \tilde{A}_P^{-1}A_PT_D \tag{32}$$

$$\tilde{A}_P^{-1}A_Pu_P = \lambda_Pu_P \quad , \quad u_P \neq 0 \, , \;\; \lambda_P \neq 1 \;\; \Rightarrow \;\; T_Pu_p \neq 0 \tag{33}$$

$$\tilde{A}_D^{-1}A_Du_D = \lambda_Du_D \quad , \quad u_D \neq 0 \, , \;\; \lambda_D \neq 0,1 \;\; \Rightarrow \;\; T_Du_D \neq 0 \tag{34}$$

where $T_P = \tilde{A}_D^{-1}A_DB_{p_b}SL_b$ *and* $T_D = L_{p_b}^{\mathrm{T}}\tilde{S}^+B_b^{\mathrm{T}}$.

Proof. In order to prove (31) we develop its left-hand side using relation (24)

$$\begin{aligned} T_P\tilde{A}_P^{-1}A_P &= \tilde{A}_D^{-1}A_DB_{p_b}SL_bL_{p_b}^{\mathrm{T}}\tilde{S}^+L_{p_b}L_b^{\mathrm{T}}SL_b \\ &= \tilde{A}_D^{-1}A_DB_{p_b}S\left(I - B_{p_b}^{\mathrm{T}}B_b\right)\tilde{S}^+\left(I - B_b^{\mathrm{T}}B_{p_b}\right)SL_b \\ &= \tilde{A}_D^{-1}A_DB_{p_b}S\left(\tilde{S}^+\left(I - B_b^{\mathrm{T}}B_{p_b}\right)SL_b - B_{p_b}^{\mathrm{T}}B_b\tilde{S}^+SL_b\right) + \tilde{A}_D^{-1}A_DB_{p_b}^{\mathrm{T}}SB_{p_b}^{\mathrm{T}}B_b\tilde{S}^+B_b^{\mathrm{T}}B_{p_b}SL_b \end{aligned} \tag{35}$$

In the right-hand side, the 2[nd] term of the parenthesis vanishes due to eq. (26), while the last term of the right-hand side equals $\tilde{A}_D^{-1}A_DT_P$. So, eq. (35) becomes

$$T_P\tilde{A}_P^{-1}A_P = \tilde{A}_D^{-1}B_b\tilde{S}^+B_b^{\mathrm{T}}B_{p_b}S\tilde{S}^+L_{p_b}L_b^{\mathrm{T}}SL_b + \tilde{A}_D^{-1}A_DT_P \tag{36}$$

In the later, note that the 1[st] term of the right-hand side vanishes due to eq. (28), which gives us (31).

In order to prove (32), we follow a similar course

$$\begin{aligned} T_D\tilde{A}_D^{-1}A_D &= L_{p_b}^{\mathrm{T}}\tilde{S}^+B^{\mathrm{T}}B_{p_b}SB_{p_b}^{\mathrm{T}}B_b\tilde{S}^+B_b^{\mathrm{T}} \\ &= L_{p_b}^{\mathrm{T}}\tilde{S}^+\left(I - L_{p_b}L_b^{\mathrm{T}}\right)S\left(I - L_bL_{p_b}^{\mathrm{T}}\right)\tilde{S}^+B_b^{\mathrm{T}} \\ &= L_{p_b}^{\mathrm{T}}\tilde{S}^+S\left(I - L_bL_{p_b}^{\mathrm{T}}\right)\tilde{S}^+B_b^{\mathrm{T}} - L_{p_b}^{\mathrm{T}}\tilde{S}^+L_{p_b}L_b^{\mathrm{T}}S\tilde{S}^+B_b^{\mathrm{T}} + L_{p_b}^{\mathrm{T}}\tilde{S}^+L_{p_b}L_b^{\mathrm{T}}SL_bL_{p_b}^{\mathrm{T}}\tilde{S}^+B_b^{\mathrm{T}} \end{aligned} \tag{37}$$

The first two terms of the right-hand side vanish due to eqs. (26) and (28), respectively. Furthermore, the 3[rd] term equals $\tilde{A}_D^{-1}A_DT_P$, which proves eq. (32).

To prove (33) we make the assumption that $\tilde{A}_P^{-1}A_P u_P = \lambda_P u_P$, $u_P \neq 0$ and $T_P u_p = 0$. It suffices to show that $\lambda_P = 1$. For that, we take equation $T_P u_p = 0$ and multiply by T_D from the left

$$T_D T_P u_p = 0 \Rightarrow T_D \tilde{A}_D^{-1} A_D B_{p_b} S L_b u_p = 0 \tag{38}$$

Using eqs. (32) and (25), eq. (38) becomes

$$\begin{gathered}
\tilde{A}_P^{-1} A_P T_D B_{p_b} S L_b u_p = 0 \Rightarrow \\
\tilde{A}_P^{-1} A_P L_{p_b}^{\mathrm{T}} \tilde{S}^{+} B_b^{\mathrm{T}} B_{p_b} S L_b u_p = 0 \Rightarrow \\
\tilde{A}_P^{-1} A_P L_{p_b}^{\mathrm{T}} \tilde{S}^{+} \left(I - L_{p_b} L_b^{\mathrm{T}}\right) S L_b u_p = 0 \Rightarrow \\
\tilde{A}_P^{-1} A_P L_{p_b}^{\mathrm{T}} \tilde{S}^{+} S L_b u_p - \tilde{A}_P^{-1} A_P L_{p_b}^{\mathrm{T}} \tilde{S}^{+} L_{p_b} L_b^{\mathrm{T}} S L_b u_p = 0 \Rightarrow \\
\tilde{A}_P^{-1} A_P u_p - \tilde{A}_P^{-1} A_P \tilde{A}_P^{-1} A_P u_p = 0
\end{gathered} \tag{39}$$

By simply taking into account that $\tilde{A}_P^{-1}A_P u_P = \lambda_P u_P$, eq. (39) becomes

$$\lambda_P u_P - \lambda_P^2 u_P = 0 \Rightarrow \lambda_P \left(1 - \lambda_P\right) u_P = 0 \tag{40}$$

which gives $\lambda_P = 1$, because the primal methods do not have zero eigenvalues.

Finally, (34) is proven by assuming that $T_D u_D = 0$ and multiply by $B_{p_b} S L_b$ from the left

$$\begin{gathered}
B_{p_b} S L_b L_{p_b}^{\mathrm{T}} \tilde{S}^{+} B_b^{\mathrm{T}} u_D = 0 \Rightarrow B_{p_b} S \left(I - B_{p_b}^{\mathrm{T}} B_b\right) \tilde{S}^{+} B_b^{\mathrm{T}} u_D = 0 \Rightarrow \\
B_{p_b} S \tilde{S}^{+} B_b^{\mathrm{T}} u_D = B_{p_b} S B_{p_b}^{\mathrm{T}} B_b \tilde{S}^{+} B_b^{\mathrm{T}} u_D \Rightarrow B_{p_b} S \tilde{S}^{+} B_b^{\mathrm{T}} u_D = \lambda_D u_D
\end{gathered} \tag{41}$$

The last equation is a contradiction because of (29).

4 Application of the general framework to particular DDM

In the present section, we will prove that conditions (12) and (21) hold for several DDM and therefore the analysis of sections 2 and 3 holds for all of them. All proofs are written only for the case where internal d.o.f. are condensed. The other case follows in a very simple maner[4].

4.1 The one-level FETI method

First, we define the displacement estimate operator that corresponds to the one-level FETI method (FETI-1) [1] and its primal alternative [4] and introduce the corresponding notation. So, for these methods we have:

$$\tilde{K}^{+} = H^{\mathrm{T}} K^{+} H \quad (\text{or} \quad \tilde{S}^{+} = H_b^{\mathrm{T}} S^{+} H_b \ ) \tag{42}$$

where K^{+} (or $\tilde{S}^{+}$) is the block-diagonal assemblage of the generalized inverses of the subdomain stiffness matrices (or subdomain Schur complements), with the property:

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[4] As it was explained in the previous section, in order to obtain the proofs that refer to the case where internat d.o.f. are condenced, in all equations substitute Schur complement matrices S with the full stiffness matrices K and all remaining variables that refer to interface d.o.f. with their alternatives that refer to all d.o.f.

$$\exists Y : K^{+}K = I + RY \quad (\text{or} \quad \exists Y : S^{+}S = I + R_{b}Y \quad) \tag{43}$$

where R is the block-diagonal assemblage of the subdomain zero energy modes:

$$\text{range}(R) = \text{null}(K) \tag{44}$$

In addition, H is a projector given by

$$H = I - B^{\mathrm{T}}QG(G^{\mathrm{T}}QG)^{-1}R^{\mathrm{T}} \quad \text{where} \quad G = BR \tag{45}$$

and Q is a symmetric positive definite matrix used in the FETI-1 coarse projector (see for instance [15]). H_b is the restriction of H to interface d.o.f., which means

$$H_b = I - B_b^{\mathrm{T}}QG(G^{\mathrm{T}}QG)^{-1}R_b^{\mathrm{T}} \tag{46}$$

It is worth noting that if Q is set equal to the Dirichlet preconditioner, then the primal formulation of section 2 becomes the standard formulation of the BDD method [2] (see [4], section 8). Thus, the analysis that follows also holds for this method. We note that H_b satisfies relations

$$R_b^{\mathrm{T}}H_b = 0 \quad , \quad H_b^{\mathrm{T}}L_b = L_b \quad \text{and} \quad H_bS = S \tag{47}$$

We proceed to the proofs of conditions (12) and (21). First, we note that

$$\exists Y : H_b^{\mathrm{T}}S^{+}S = H_b^{\mathrm{T}}\left(I + R_bY\right) = H_b^{\mathrm{T}} \tag{48}$$

and

$$\tilde{S}^{+}S = H_b^{\mathrm{T}}S^{+}H_bS = H_b^{\mathrm{T}}S^{+}S = H_b^{\mathrm{T}} \tag{49}$$

Given eq. (49), conditions (12) and (21) are obtained by

$$\tilde{S}^{+}SL_b = H_b^{\mathrm{T}}L_b = L_b \tag{50}$$

and

$$\tilde{S}^{+}S\tilde{S}^{+} = H_b^{\mathrm{T}}\tilde{S}^{+} = \left(H_b^{\mathrm{T}}\right)^2 S^{+}H_b = H_b^{\mathrm{T}}S^{+}H_b = \tilde{S}^{+} \tag{51}$$

4.2 The two-level FETI method

For the two-level FETI method (FETI-2) [16,17] and its primal alternative [4], we have:

$$\tilde{K}^{+} = H^{\mathrm{T}}\left(K^{+} - K^{+}HB^{\mathrm{T}}C_{\lambda}\left(C_{\lambda}^{\mathrm{T}}BH^{\mathrm{T}}K^{+}HB^{\mathrm{T}}C_{\lambda}\right)^{-1}C_{\lambda}^{\mathrm{T}}BH^{\mathrm{T}}K^{+}\right)H \tag{52}$$

or

$$\tilde{S}^{+} = H_b^{\mathrm{T}}\left(S^{+} - S^{+}H_bB_b^{\mathrm{T}}C_{\lambda}\left(C_{\lambda}^{\mathrm{T}}B_bH_b^{\mathrm{T}}S^{+}H_bB_b^{\mathrm{T}}C_{\lambda}\right)^{-1}C_{\lambda}^{\mathrm{T}}B_bH_b^{\mathrm{T}}S^{+}\right)H_b \tag{53}$$

where C_{λ} is a Boolean matrix that extracts some selected Lagrange multipliers from a full Lagrange multiplier vector. Using eq. (48), we have

$$\begin{aligned}
\tilde{S}^{+}S &= H_b^{\mathrm{T}}\left(S^{+} - S^{+}H_bB_b^{\mathrm{T}}C_{\lambda}\left(C_{\lambda}^{\mathrm{T}}B_bH_b^{\mathrm{T}}S^{+}H_bB_b^{\mathrm{T}}C_{\lambda}\right)^{-1}C_{\lambda}^{\mathrm{T}}B_bH_b^{\mathrm{T}}S^{+}\right)H_bS \\
&= \left(I - H_b^{\mathrm{T}}S^{+}H_bB_b^{\mathrm{T}}C_{\lambda}\left(C_{\lambda}^{\mathrm{T}}B_bH_b^{\mathrm{T}}S^{+}H_bB_b^{\mathrm{T}}C_{\lambda}\right)^{-1}C_{\lambda}^{\mathrm{T}}B_b\right)H_b^{\mathrm{T}}S^{+}S \\
&= \left(I - H_b^{\mathrm{T}}S^{+}H_bB_b^{\mathrm{T}}C_{\lambda}\left(C_{\lambda}^{\mathrm{T}}B_bH_b^{\mathrm{T}}S^{+}H_bB_b^{\mathrm{T}}C_{\lambda}\right)^{-1}C_{\lambda}^{\mathrm{T}}B_b\right)H_b^{\mathrm{T}}
\end{aligned} \tag{54}$$

Condition (12) is obtained by multiplying eq. (54) by L_b from the right and using the 2nd of eqs. (47). Likeways, condition (21) is obtained by multiplying eq. (54) by $\tilde{S}^{+}$ from the right and doing the algebra.

4.3 *The dual-primal FETI method*

In the case of the dual-primal FETI method (FETI-DP) [11,12], when only nodal coarse constraints are considered we have:

$$\tilde{K}^{+} = N_r^{\mathrm{T}} K_{rr}^{-1} N_r + \left(N_c^{\mathrm{T}} - N_r^{\mathrm{T}} K_{rr}^{-1} K_{rc}\right) L_c K_{c_g c_g}^{*-1} L_c^{\mathrm{T}} \left(-K_{cr} K_{rr}^{-1} N_r + N_c\right) \tag{55}$$

where $$K_{c_g c_g}^{*} = L_c^{\mathrm{T}} \hat{S}_{cc} L_c \quad \text{and} \quad \hat{S}_{cc} = K_{cc} - K_{cr} K_{rr}^{-1} K_{rc} \tag{56}$$

In the above equations, subscripts c and r denote the restriction of the matrices to the coarse problem d.o.f. and the remaining d.o.f., respectively. Matrix N_r is a Boolean matrix which extracts the subdomain d.o.f. that do not belong to the coarse problem, from subdomain d.o.f. vectors, like in equation $u_r = N_r u$. Furthermore, matrix N_c is used in order to extract the coarse problem d.o.f. from global d.o.f. vectors, like in equation $u_c = N_c u$. Berore proceeding, it is worth noting, that with definition (55), matrix $B\tilde{K}^{+}B^{\mathrm{T}}$ has a higher dimension than the usual FETI-DP coefficient matrix (see [11]). Our version simply adds redundant information and the corresponding zero eigenvalues to the FETI-DP coefficient matrix.

Operator $\tilde{S}^{+}$ can be obtained from eq. (55), using eq. (20). However, for our proofs, we will use the equivalent expression

$$\tilde{S}^{+} = N_{r_b}^{\mathrm{T}} S_{r_b r_b}^{-1} N_{r_b} + \left(N_c^{\mathrm{T}} - N_{r_b}^{\mathrm{T}} S_{r_b r_b}^{-1} S_{r_b c}\right) L_c S_{c_g c_g}^{*-1} L_c^{\mathrm{T}} \left(-S_{c r_b} S_{r_b r_b}^{-1} N_{r_b} + N_c\right) \tag{57}$$

where $$S_{c_g c_g}^{*} = L_c^{\mathrm{T}} \hat{S}_{cc} L_c \quad \text{and} \quad \hat{S}_{cc} = S_{cc} - S_{c r_b} S_{r_b r_b}^{-1} S_{r_b c} \tag{58}$$

In eq. (57) instead of decomposing the subdomain stiffness matrices K to submatrices pertaining to coarse and remaining d.o.f., we have decomposed in the same way the subdomain Schur complements S.

Before proceeding to the proofs, we will generalize expressions (55) and (57) in order to include the edge and face constraints that are often used in FETI-DP and its primal alternative. For this reason, we will redefine coarse and remaining d.o.f. as follows: Perform the following change of basis in subdomain d.o.f.:

$$u = \begin{bmatrix} N_c^{\mathrm{T}} & N_r^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} u_c \\ u_r \end{bmatrix} \tag{59}$$

where matrix $\begin{bmatrix} N_c^{\mathrm{T}} & N_r^{\mathrm{T}} \end{bmatrix}$ is orthogonal and its subblocks satisfy equations

$$range(N_c L) = range(L_c) \Leftrightarrow \exists X : N_c L = L_c X \tag{60}$$

and $$range(N_r L) = range(L_r) \Leftrightarrow \exists X : N_r L = L_r X \tag{61}$$

where L_r has the same sence as matrix L_c, with the difference that it refers to the remaining d.o.f. rather than the coarse ones. Note that the orthogonality condition implies

$$N_r N_r^{\mathrm{T}} = I \quad , \quad N_c N_c^{\mathrm{T}} = I \quad , \quad N_c N_r^{\mathrm{T}} = 0 \quad , \quad N_r^{\mathrm{T}} N_r + N_c^{\mathrm{T}} N_c = I \tag{62}$$

and
$$\begin{bmatrix} u_c \\ u_r \end{bmatrix} = \begin{bmatrix} N_c \\ N_r \end{bmatrix} u \tag{63}$$

Furthermore, we note that K may be decomposed in the form:

$$\begin{aligned} K &= \begin{bmatrix} N_r^{\mathrm{T}} & N_c^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} K_{rr} & K_{rc} \\ K_{cr} & K_{cc} \end{bmatrix} \begin{bmatrix} N_c \\ N_r \end{bmatrix} \\ &= N_r^{\mathrm{T}} K_{rr} N_r + N_r^{\mathrm{T}} K_{rc} N_c + N_c^{\mathrm{T}} K_{cr} N_r + N_c^{\mathrm{T}} K_{cc} N_c \end{aligned} \tag{64}$$

where
$$K_{rr} = N_r K N_r^{\mathrm{T}} \quad , \quad K_{rc} = N_r K N_c^{\mathrm{T}} \quad \text{and} \quad K_{cc} = N_c K N_c^{\mathrm{T}} \tag{65}$$

With the above framework, N_c can be selected in a way that the relation

$$u_c = L_c u_{c_g} \Leftrightarrow N_c u = L_c u_{c_g} \Leftrightarrow B_c N_c u = 0 \tag{66}$$

can express whatever vertex, edge-averaged or face-averaged coarse-space continuity conditions are usually used in FETI-DP, PFETI-PD and BDDC literature. For instance, in order to express the continuity condition for an average displacement of an interface edge with n_e nodes, it suffices to create for each of the subdomains of this edge a column of N_c with values $1/\sqrt{n_e}$ to each entry that corresponds to the displacements of the edge's nodes. Then, the columns of N_r that correspond to this edge, should be selected orthogonal to the N_c columns. This change of basis would probably not be favorable for an implementation – because for instance matrix K_{rr} of eq. (65) would be much denser than normally – but as it will be shown it simplifies the proofs of conditions (12) and (21) to a matter of doing the algebra in the left-hand side of these equations. It is also worth noting that this change of basis is similar to the one used to define coarse d.o.f. in [8]. The difference in our case is that the change of basis is orthogonal.

Thus, with the change of basis of eq. (59) and the corresponding decomposition of matrix K in the submatrices of eqs. (65), expression (55) gives the FETI-DP estimator of displacements for any coarse constraints. The same holds for expression (57), in which we use the corresponding decomposition

$$\begin{aligned} S &= \begin{bmatrix} N_{r_b}^{\mathrm{T}} & N_c^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} S_{r_b r_b} & S_{r_b c} \\ S_{c r_b} & S_{cc} \end{bmatrix} \begin{bmatrix} N_c \\ N_{r_b} \end{bmatrix} \\ &= N_{r_b}^{\mathrm{T}} S_{r_b r_b} N_{r_b} + N_{r_b}^{\mathrm{T}} S_{r_b c} N_c + N_c^{\mathrm{T}} S_{c r_b} N_{r_b} + N_c^{\mathrm{T}} S_{cc} N_c \end{aligned} \tag{67}$$

where
$$S_{r_b r_b} = N_{r_b} S N_{r_b}^{\mathrm{T}} \quad , \quad S_{r_b c} = N_{r_b} S N_c^{\mathrm{T}} \quad \text{and} \quad S_{cc} = N_c S N_c^{\mathrm{T}} \tag{68}$$

With the above definition, we are ready to proceed to the proofs of conditions (12) and (21). We note that in the case where internal d.o.f. are not condensed these proofs were done in [9], while in the case where internal d.o.f. are condensed, these two conditions follow from [6, Lemma 28]. Here, we will perform a different and simpler proof, that covers both cases.

First, we note that substituting from eqs. (57) and (67) and doing the matrix algebra, we obtain:

$$\tilde{S}^{+}S = N_{r_b}^{\mathrm{T}}N_{r_b} + N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}N_c + \left(N_c^{\mathrm{T}} - N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}\right)L_cS_{cc}^{*-1}L_c^{\mathrm{T}}\hat{S}_{cc}N_c^s \tag{69}$$

Then, multiplying by L_b from the right and using eq. (60), we conclude that $\exists X$:

$$\begin{aligned}
\tilde{S}^{+}SL_b &= N_{r_b}^{\mathrm{T}}N_{r_b}L_b + N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}N_cL_b + \left(N_c^{\mathrm{T}} - N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}\right)L_cS_{cc}^{*-1}L_c^{\mathrm{T}}\hat{S}_{cc}N_cL_b \\
&= N_{r_b}^{\mathrm{T}}N_{r_b}L_b + N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}N_cL_b + \left(N_c^{\mathrm{T}} - N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}\right)L_cS_{cc}^{*-1}L_c^{\mathrm{T}}\hat{S}_{cc}L_cX \\
&= N_{r_b}^{\mathrm{T}}N_{r_b}L_b + N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}N_cL_b + \left(N_c^{\mathrm{T}} - N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}\right)L_cS_{cc}^{*-1}S_{cc}^{*}X \\
&= N_{r_b}^{\mathrm{T}}N_{r_b}L_b + N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}N_cL_b + N_c^{\mathrm{T}}N_cL_b - N_{r_b}^{\mathrm{T}}S_{r_br_b}^{-1}S_{r_bc}N_c^sL_b \\
&= \left(N_{r_b}^{\mathrm{T}}N_{r_b} + N_c^{\mathrm{T}}N_c\right)L_b = L_b
\end{aligned} \tag{70}$$

Finally, it suffices substituting from eq. (69) and definition (57) into the left-hand side of condition (21), in order to prove it, after doing the necessary matrix algebra.

5 Mechanical interpretation of the conditions that are imposed to the displacement estimators

In the previous sections, conditions (12) and (21) were imposed on the subdomain displacement estimators of eqs. (5) and (6) and it was shown first, that if the first condition is satisfied then the primal problem can be transformed ot a dual one and second, that if both conditions hold, the primal and dual formulations have the same non-zero and non-unit eigenvalues. In this section, we will discuss the mechanical interpretation of these conditions.

From the point of view of mechanics, these conditions can be interpreted either in terms of displacements, or in terms of forces. In our opinion, the interpretation with respect to displacements is more enlightening for their importance and we prefer it. We start by assuming that a displacement field u_g is applied to the left-hand side of condition (12) giving $\left\{\forall u_g : \tilde{K}^{+}KLu_g = Lu_g\right\} \Leftrightarrow \left\{\forall u \text{ , } Bu = 0 : \tilde{K}^{+}Ku = u\right\}$. So, this condition means that by applying any subdomain forces f that are a result of continuous across the interface displacements u ($f = Ku$), the estimate $\tilde{K}^{+}f$ must give the same displacements u. In order to obtain this, the estimator $\tilde{K}^{+}$ needs to deliver one of the infinite displacement solutions for floating subdomains and balance the zero energy modes of the subdomains using a coarse corrector, as in all DDM that we have discussed above. Condition (21) means that $\forall u \in range(\tilde{K}^{+}) : \tilde{K}^{+}Ku = u$, which means that if the forces $f = Ku$ are given as input to the estimate operator $\tilde{K}^{+}$, the same displacements must again be retrieved. Under this point of view, both conditions appear to be reasonable to expect from some displacement estimator $\tilde{K}^{+}$.

6 Numerical tests

In this section, we run some numerical tests, in order to verify the proof of eigenvalues' equality and to compare the numerical performance of the methods. In various small examples we computed the entire eigenspectrum of several dual and primal formulations and we found a complete equivalence in all non-zero and non-unit eigenvalues, excluding of course round-off errors. Two large-scale tests were also performed and their results are

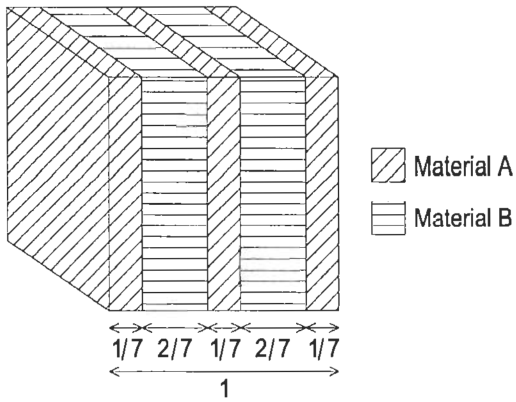
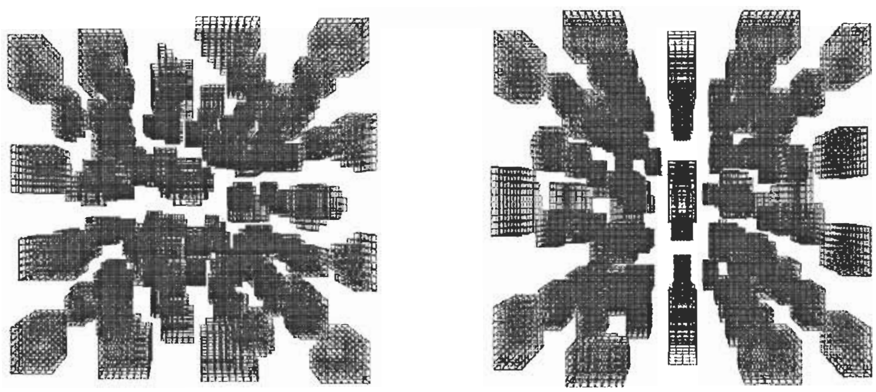


Figure 1. A cubic structure composed of two materials



(a) (b)

Figure 2. Two decompositions of the cubic test problem in 100 subdomains:
(a) Optimal aspect ratio partitioning (Decomposition P1),
(b) Layered partitioning (Decomposition P2)

reported in the following two subsections. In both examples, the condition number and number of iterations are compared for various DDM in primal and dual formulation.

6.1 Second-order example

Here, we consider the 3-D elasticity problem of Fig. 1. This cubic structure is composed of five layers of two different materials and is discretized with $28\times28\times28$ 8-node brick elements. Additionally, it is pinned at the four corners of its left surface. Various ratios E_A/E_B of the Young modulus and ρ_A/ρ_B of the density of the two materials are considered, while their Poisson ratio is set equal to $\nu_A=\nu_B=0.30$. An optimal decomposition of this heterogeneous problem must generate subdomains with good aspect ratios, while preserving the material interfaces when partitioning the model [15].

Hence, two decompositions of this heterogeneous model of 73,155 d.o.f. in 100 subdomains are considered: In the first decomposition (Fig. 2a), the model has been partitioned in subdomains with good aspect ratios without taking into account the material interfaces (Decomposition P1). In the second decomposition (Fig. 2b), the five layers of different materials have been partitioned independently, thus generating a decomposition

Table 1

Condition nr. and nr. of iterations (Tolerance: 10^{-3}) of the FETI-1 and P-FETI1 methods (case of no internal d.o.f. condensation) for the solution of the example of Fig. 1

Ratio of Young moduli	Type of decomposition	Condition. nr. of both formulations	Nr. of iterations	
			Primal formulation	Dual formulation
10^0	P1	1.1E+2	24	25
10^3	P1	7.2E+1	41	44
10^3	P2	6.4E+1	24	25
10^6	P1	7.9E+1	47	53
10^6	P2	6.3E+1	26	30

Table 2

Condition nr. and nr. of iterations (Tolerance: 10^{-3}) of the FETI-1 and P-FETI1 methods (case of internal d.o.f. condensation) for the solution of the example of Fig. 1

Ratio of Young moduli	Type of decomposition	Condition. nr. of both formulations	Nr. of iterations	
			Primal formulation	Dual formulation
10^0	P1	4.0E+1	21	21
10^3	P1	4.0E+1	31	33
10^3	P2	2.6E+1	20	21
10^6	P1	4.4E+1	34	38
10^6	P2	2.6E+1	24	26

which preserves the material interfaces but produces subdomains of suboptimal aspect ratio in the thin layers (Decomposition P2).

In this example, we have implemented the one-level FETI method in both primal and dual formulations and both with and without condensation of the internal d.o.f. of the subdomains. In the homogeneous ($E_{\mathrm{A}} = E_{\mathrm{B}}$) configuration of this problem we used homogenous scaling in the mapping matrices of displacements and forces, while matrix Q of the FETI-1 projector was set equal to the unit matrix. In addition, in the heterogeneous configuration ($E_{\mathrm{A}} \neq E_{\mathrm{B}}$), we used heterogeneous scaling and Q was set equal to the superlumped choice [15] $Q = B_{p_b} K_{bb_d} B_{p_b}^{\mathrm{T}}$, where K_{bb_d} is a diagonal matrix, whose main diagonal is equal to the main diagonal of matrix K_{bb}. The obtained results are shown in Tables 1 and 2. The condition numbers were computed equal for both primal and dual formulations, while their iteration counts are also quite close. In addition, it is worth noting that the heterogeneous scaling and the superlumped matrix Q succeed in keeping the condition numbers of the heterogeneous configurations of this problem almost the same as those of the homogeneous configuration. However, the interation counts increase quite importantly in the cases where material interfaces do not coincide with subdomain interfaces. Finally, the condensation of the internal d.o.f. lowers the condition numbers and iteration counts, as it was reasonable to expect.

6.2 Fourth-order example

As a fourth-order example, we adopt the semi-cylindrical panel of Fig. 3. This shell problem has a radius of ,0.5 a length of 1.6 and a thickness of $t = 10^{-3}$ or $t = 10^{-4}$. Moreover, the Young modulus is 1×10^6 and the Poisson ratio 0.30. The panel is modeled with a structured mesh of 131×131 nodes and is discretized with triangular TRIC shell elements [18]. Furthermore, it is fixed on 16 nodes along its two linear edges as shown in Fig. 3. This model of 102,870 d.o.f. is decomposed in 130 subdomains (Fig. 4).

For the analysis of this problem, we have implemented the FETI-1, FETI-2 and FETI-DP methods, while the internal d.o.f. of the subdomains are condensed, because this choice is known to be favourable for fourth-order problems. All methods have been implemented with homogeneous scaling, while for FETI-1, matrix Q has been set equal to the Diriclet preconditioner $Q = B_{p_b} S B_{p_b}^{\mathrm{T}}$. In addition, the coarse spaces of FETI-2 and FETI-DP have been formed from d.o.f. of nodes that coincide with the ends of the interface edges. For the coarse space of FETI-DP, all d.o.f. of these nodes have been included, while only the translational d.o.f. have been used for FETI-2.

The condition numbers and iteration counts of the primal and dual formulations are shown in Tables 3 and 4. Again, the computed condition numbers are equal between related primal and dual variants, while iteration counts are quite close. It is also worth noting that in both tables the condition numbers vary considerably and they are particularly poor in the case of FETI-1, which is known not to be scalable for fourth-order problems. Furthermore, the performance of all methods deteriorates importantly in the more ill-conditioned configuration of the shell, while the dual formulation of the two-level FETI fails to reach the threshold accuracy of 10^{-3}.

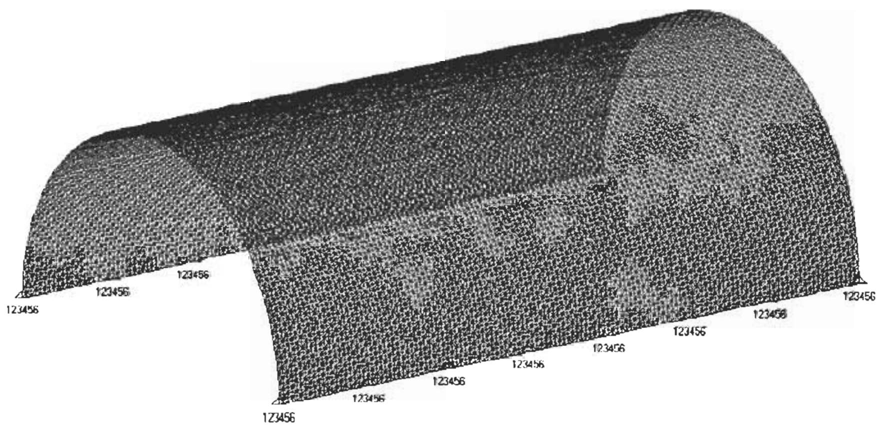


Figure 3. A semi-cylindrical panel, discretized with triangular shell elements

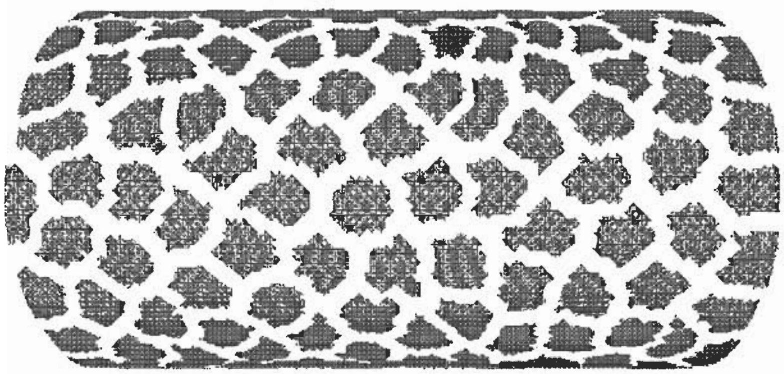


Figure 4. A decomposition of the semi-cylindrical panel in 130 subdomains – Top view

7 Concluding remarks

The final section of this paper aims at summing up what has been shown in this work and then studying the consequences of these findings. The main concept behind this work has been the fact that the definition of an operator that estimates subdomain displacements from subdomain forces can be used to build a primal and a dual DDM, that are strongly connected, provided that the operator satisfies two conditions. In particular, the first condition allows the primal formulation of the problem to be turned into a dual one. Then, a second condition guarantees that the two resulting DDM, the primal and the dual one, will have the same non-zero and non-unit eigenvalues. This suggests that the two formulations will probably have similar convergence properties and iteration counts. Some of the most popular DDM since the beginning of the 90's in structural and solid mechanics, have been inserted in this general framework, by proving that these two conditions hold for them. In particular, this has been proven when applying the methods FETI-1, FETI-2 and FETI-DP or their primal alternatives BDD and BBDC for the static analysis of structural problems. It is also worth noting that it would be simple to extend the proofs performed in this work to the applications of these methods for implicit dynamic structural problems The only things that change are the subdomain stiffness matrices that are substituted with the corresponding matrices of the implicit dynamic analysis and the zero energy mode projections that must be removed.

Hence, after summing up the general theoretical framework that has been set up here, we can investigate its consequences. A first and obvious but not unimportant consequence of this framework is that the setting of section 2 allows a very modular programming of primal and dual methods. If this general setting of primal and dual methods is programmed then by programming separately the estimate operator $\tilde{K}^{+}$ for some DDM, both primal and dual formulations are directly obtained. Furthermore, in the version of this formulations where internal d.o.f. of the subdomains are condensed, recent results show that the primal formulation, while it has similar performance to the dual one in well-conditioned problems, it is statistically faster and more robust in ill-conditioned ones [4,5]. Furthermore, with reference to the case where internal d.o.f. are not condensed, a recent study [9] proves that if the two conditions that have been set in the present paper hold, then the algorithm of the primal formulation can be made to operate on dual variables, instead of primal ones. The

Table 3

Condition nr. and nr. of iterations (Tolerance: 10^{-3}) of some DDM for the solution of the example of Fig. 3 ($t = 10^{-3}$)

Method	Condition. nr. of both formulations	Nr. of iterations	
		Primal formulation	Dual formulation
FETI-1	2.2E+6	131	135
FETI-2	5.4E+1	35	36
FETI-DP	2.0E+3	48	49

Table 4

Condition nr. and nr. of iterations (Tolerance: 10^{-3}) of some DDM for the solution of the example of Fig. 3 ($t = 10^{-4}$)

Method	Condition. nr. of both formulations	Nr. of iterations	
		Primal formulation	Dual formulation
FETI-1	1.9E+8	319	330
FETI-2	5.3E+2	91	–
FETI-DP	7.1E+4	139	145

main gain from this transformation is that the primal algorithm that would be excessively costly because it operates on the full displacement vector of the structure (thus practically inhibiting for instance the process of reorthogonalization in the PCG algorithm), is now converted to an algorithm, which operating on dual variables has comparable computational cost to the pure dual formulation. Hence, when internal d.o.f. are not condensed, the results of [9] show that the primal and dual formulations have comparable efficiency. However, in this case, the dual formulation turns out to be more robust and our tests show that in most problems it will probably be faster than the primal one.

Hence, in order to draw a general conclusion from comparing the primal and dual formulations, it is necessary to discuss when it is favourable to condense the internal d.o.f. of the subdomains. In modern DDM practice it has been noted that usually the condensation of internal d.o.f. leads to higher computational efficiency, while, avoiding the condensation can probably lead to less memory-consuming solutions in large-scale second-order problems. However, the results of [9] also suggest that the primal formulation requires less memory in fourth-order problems and in many second-order problems. Consequently, the general picture at the moment , with respect to computational cost, robustness and memory requirement, seems to be in favour of the primal formulation, at least for the majority of the cases. However, since the beginning of the 90's, the dual methods have been implemented in many other areas beyond the pure static and dynamic analysis of structures. Therefore, for the primal formulation this work will have to be adapted when possible, or simply redone. Thus, the best choice today is probably to be armed with both options.

This paper comes to offer a small piece to the long series of works that have gradually led to today's understanding of the concept of duality in DDM for structural and solid mechanics. Before ending it, it is thus probably suitable to remind some of the most important steps in the efforts that have led here. A large step in this process was made in the beginning of the 90's, with the introduction of the FETI method, which was a dual method that quickly gained a lot of popularity. Since then, the major advances in the dual methods, like the introduction of the FETI-2 and FETI-DP methods were closely followed by similar advances in the area of the primal methods. While more and more advances were appearing that suggested that there were connections missing between primal and dual formulations, the first studies of these connections came forward. Today, the international research community of DDM has gone a long way since the introduction of the first dual methods and it can be said with a lot of certainty there is a lot more to come.

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