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ABSTRACT

This work investigates the existence, uniqueness, and stability in the sense of Hyers-Ulam for a class of proportional fractional Itô-Doob stochastic integral equations (PFIDSIE). To establish these properties, we employ the Banach fixed point theorem (BFPT) in combination with several fundamental mathematical inequalities that provide insight into the structure of PFIDSIEs. The approach is structured to demonstrate not only the theoretical foundation of the existence and uniqueness of solutions but also the stability of these solutions in the Hyers-Ulam sense, which ensures that approximate solutions remain close to the exact solution under small perturbations. The results contribute to the broader field of fractional stochastic differential equations, particularly in situations where fractional dynamics and stochastic processes intersect. Furthermore, the findings are illustrated through three examples, showcasing the applicability and utility of the developed theory in practical settings.

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1 Introduction

Fractional calculus aims to extend the concept of derivatives from integer orders (as in classical calculus) to non-integer orders (see [1–4]). This theory of fractional-order derivatives is not new; its roots trace back to the late 17th century, when Leibniz and Newton laid the groundwork for differential and integral calculus. Leibniz introduced the notation $\frac{d^k}{ds^k}f(s)$ to represent the kth derivative of a function f. In response to a query from L'Hôpital about the case when $k = 0.5$, Leibniz famously referred to it as a paradox.

Fractional derivatives and integrals offer significant advantages for modeling the mechanical and electrical properties of real materials, as well as in various other fields. While some may view the study of fractional derivatives and integrals as purely theoretical and of little relevance to engineers, a straightforward example from fluid mechanics illustrates their practical application. Specifically, the

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$k = 0.5$ derivative emerges naturally when analyzing the lateral heat flow from a fluid concerning the temporal changes of an internal source (see [5–7]).

One of the key categories of fractional derivatives and integrals includes the Caputo-Proportional fractional derivatives (CPFD) and the Proportional Fractional Integrals (PFI) [8]. Numerous researchers have explored the equations with CPFD and the PFI (see [9–11]). For instance, authors in [9] studied the existence, uniqueness and stability for neutral stochastic differential equations involved CPFD. Authors in [11] studied the averaging principle for proportional fractional Itô-Doob stochastic integral equations (PFIDSIE).

In recent decades, the Hyers-Ulam stability (HUS) theory has garnered considerable attention from researchers due to its applications in mechanics, automation, and biology, where finding explicit solutions can be quite challenging. Numerous studies in this area are referenced in [12–14]. For instance, reference [12] examines the HUS of solutions for a class of stochastic differential equations with fractional Brownian motion. Meanwhile, reference [13] explores the HUS for a class of stochastic functional differential equations. Additionally, reference [14] presents novel results on HUS using the inequality of Gronwall. However, there is currently no existing research on the HUS of PFIDSIE. In this context, our article is a generalization of the work [14] for the proportional fractional case by employing the Banach fixed point theorem (BFPT) and the Gronwall inequality. Here are the key highlights of our article:

- Establish the existence and uniqueness of the solution of PFIDSIE.
- Prove the HUS of PFIDSIE.

The structure of the paper is outlined as follows: Section 2 covers some preliminaries notations and concepts. Section 3 examines the existence, uniqueness and the HUS of solutions for PFIDSIE using the BFPT and the inequality of Gronwall. Finally, Section 4 presents applications of our findings through three examples.

2 Basic Notations

Set $\Pi > 0$ and $\mathcal{R} = \{\mathbb{X}, \widetilde{\mathfrak{G}}, \widetilde{\mathbb{M}} = (\widetilde{\mathfrak{G}}_{\varkappa})_{0\leq\varkappa\leq\Pi}, \mathcal{P}\}$ be a complete probability space and $\mathbb{B}(\varkappa)$ be a standard Brownian motion.

For $q \geq 2$, set $\mathbb{X}^q_{\varkappa} = L^q(\mathbb{X}, \mathfrak{M}_{\varkappa}, \mathcal{P})$ the space of all $\mathbb{M}_{\varkappa}$-measurable and q-th integrable functions $f : \mathbb{X} \to \mathbb{R}$ with:

$$\|f\|_q = \left(\mathbb{E}\left(|f|^q\right)\right)^{\frac{1}{q}}.$$

Definition 1. [8] For $0 < \nu \leq 1$, $\beta > 0$, the proportional fractional integral of g is

$$\mathcal{I}^{\beta,\nu}_{a} g(\varkappa) = \frac{1}{\nu^{\beta}\Gamma(\beta)} \int_a^{\varkappa} \exp\left(\frac{\nu-1}{\nu}(\varkappa - l)\right)(\varkappa - l)^{\beta-1} g(l)\, dl.$$

Consider the following integral equations:

$$\zeta(\varkappa) = \psi + \int_0^{\varkappa} F_1\big(l, \zeta(l)\big)dl + \sum_{i=2}^{n} \int_0^{\varkappa} (\varkappa - l)^{\beta_i - 1} \exp\left(\frac{\nu_i - 1}{\nu_i}(\varkappa - l)\right) F_i\big(l, \zeta(l)\big)dl + \int_0^{\varkappa} g\big(l, \zeta(l)\big)d\mathbb{B}(l), \tag{1}$$

where $\psi \in \mathbb{R}$, $0 < \beta_i < 1$, $0 < \nu_i \leq 1$ for $2 \leq i \leq n$, $\varkappa \in [0, \Pi]$ and $g, \digamma_i : [0, \Pi] \times \mathbb{R} \longrightarrow \mathbb{R}$ for $1 \leq i \leq n$ are measurables.

Let's consider the following hypothesis:

$\mathcal{H}_1$: There is $\bar{C} > 0$ such that

$$|g(\varkappa, \gamma_1) - g(\varkappa, \gamma_2)| \vee |\digamma_1(\varkappa, \gamma_1) - \digamma_1(\varkappa, \gamma_2)| \vee \ldots \vee |\digamma_n(\varkappa, \gamma_1) - \digamma_n(\varkappa, \gamma_2)| \leq \bar{C}|\gamma_1 - \gamma_2|, \tag{2}$$

for all $(\varkappa, \gamma_1, \gamma_2) \in [0, \Pi] \times \mathbb{R} \times \mathbb{R}$.

$\mathcal{H}_2$: There is $d > 0$ such that

$\operatorname{ess}\sup_{\varkappa\in[0,\Pi]} |g(\varkappa, 0)| \leq d$, $\operatorname{ess}\sup_{\varkappa\in[0,\Pi]} |\digamma_i(\varkappa, 0)| \leq d$ for $i \in \{1, 2, \ldots, n\}$.

In the rest of the paper, we suppose that $q > \max_{1\leq j\leq n}\{\frac{1}{\beta_j}\}$.

3 Main Results

Denote by $\mathcal{R}^q([0, \Pi])$ the set of all measurable processes ζ that are $\widetilde{\mathbb{M}}$ adapted and satisfy $\sup_{\varkappa\in[0,\Pi]} \|\zeta(\varkappa)\|_q < \infty$. Let the norm $\|\cdot\|_{\mathcal{R}^q}$ on $\mathcal{R}^q([0, \Pi])$ defined by

$$\|\zeta\|_{\mathcal{R}^q([0,\Pi])} = \sup_{\varkappa\in[0,\Pi]} \|\zeta(\varkappa)\|_q.$$

We have $(\mathcal{R}^q([0, \Pi]), \|\cdot\|_{\mathcal{R}^q([0,\Pi])})$ is a Banach space.

Let $\mathcal{K}_\psi : \mathcal{R}^q([0, \Pi]) \to \mathcal{R}^q([0, \Pi])$ given by

$$(\mathcal{K}_\psi\zeta)(\varkappa) = \psi + \int_0^\varkappa \digamma_1\big(s, \zeta(s)\big)ds + \int_0^\varkappa g\big(s, \zeta(s)\big)d\mathbb{B}(s) + \sum_{i=2}^{n} \int_0^\varkappa (\varkappa - s)^{\beta_i - 1} \exp\left(\frac{\nu_i - 1}{\nu_i}(\varkappa - s)\right) \digamma_i\big(s, \zeta(s)\big)ds, \tag{3}$$

for every $\varkappa \in [0, \Pi]$.

Lemma 1. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ hold, then $\mathcal{K}_\psi$ is well defined for every $\psi \in \mathbb{R}$.*

Proof. Let $\zeta \in \mathcal{R}^q([0, \Pi])$, we get for $\varkappa \in [0, \Pi]$,

$$||(\mathcal{K}_\psi\zeta)(\varkappa)||_q^q \leq (n+2)^{q-1}\Big[||\psi||_q^q + \|\int_0^\varkappa \digamma_1\big(s, \zeta(s)\big)ds\|_q^q + \|\int_0^\varkappa g\big(s, \zeta(s)\big)d\mathbb{B}(s)\|_q^q + \sum_{i=2}^{n} \|\int_0^\varkappa (\varkappa - s)^{\beta_i - 1} \exp\left(\frac{\nu_i - 1}{\nu_i}(\varkappa - s)\right) \digamma_i\big(s, \zeta(s)\big)ds\|_q^q. \tag{4}$$

By the Hölder inequality, we obtain

$$\left\|\int_0^\varkappa \digamma_1\big(s, \zeta(s)\big)ds\right\|_q^q \leq \mathbb{E}\left(\int_0^\varkappa |\digamma_1\big(s, \zeta(s)\big)|ds\right)^q \leq \varkappa^{q-1}\mathbb{E}\int_0^\varkappa |\digamma_1\big(s, \zeta(s)\big)|^q ds$$

$$
\begin{aligned}
&\leq \Pi^{q-1}\mathbb{E}\int_0^{\varkappa} |F_1(s,\zeta(s))|^q ds \\
&\leq \Pi^{q-1}\mathbb{E}\int_0^{\varkappa} |F_1(s,\zeta(s)) - F_1(s,0) + F_1(s,0)|^q ds \\
&\leq (2\Pi)^{q-1}\mathbb{E}\int_0^{\varkappa} \left(\bar{C}^q|\zeta(s)|^q + |F_1(s,0)|^q\right) ds \\
&\leq (2\Pi)^{q-1}\bar{C}^q\Pi||\zeta||^q_{\mathcal{R}^q([0,\Pi])} + (2\Pi)^{q-1}\int_0^{\Pi} |F_1(s,0)|^q ds,
\end{aligned} \tag{5}
$$

and

$$
\begin{aligned}
&\| \int_0^{\varkappa} (\varkappa - s)^{\beta_i-1} \exp\left(\frac{\nu_i - 1}{\nu_i}(\varkappa - s)\right) F_i(s,\zeta(s)) ds\|_q^q \\
&\leq \mathbb{E}\left(\int_0^{\varkappa} (\varkappa - r)^{\beta_i-1} \exp\left(\frac{\nu_i - 1}{\nu_i}(\varkappa - r)\right) |F_i(r,\zeta(r))| dr\right)^q \\
&\leq \mathbb{E}\left(\left(\int_0^{\varkappa} (\varkappa - r)^{\frac{(\beta_i-1)q}{q-1}} dr\right)^{q-1} \int_0^{\varkappa} |F_i(r,\zeta(r))|^q dr\right) \\
&\leq \frac{\Pi^{\beta_i q-1}(q-1)^{q-1}}{(\beta_i q-1)^{q-1}} \int_0^{\varkappa} \|F_i(s,\zeta(s))\|_q^q ds \\
&\leq \frac{\Pi^{\beta_i q-1}(2q-2)^{q-1}}{(\beta_i q-1)^{q-1}} \left(\bar{C}^q\Pi\|\zeta\|^q_{\mathcal{R}^q([0,\Pi])} + \int_0^{\Pi} |F_i(s,0)|^q ds\right).
\end{aligned}
$$

It follows from Theorem 7.1 in [15] and the Hölder inequality that

$$
\begin{aligned}
\left\|\int_0^{\varkappa} g(s,\zeta(s)) d\mathbb{B}(s)\right\|_q^q &\leq M_q\mathbb{E}\left(\int_0^{\varkappa} |g(s,\zeta(s))|^2 ds\right)^{\frac{q}{2}} \\
&\leq M_q\mathbb{E}\left(\int_0^{\varkappa} |g(s,\zeta(s))|^q ds\right)\left(\int_0^{\varkappa} ds\right)^{\frac{q-2}{2}} \\
&\leq M_q\Pi^{\frac{q-2}{2}}\mathbb{E}\left(\int_0^{\varkappa} |g(s,\zeta(s))|^q ds\right) \\
&\leq M_q 2^{q-1}\Pi^{\frac{q}{2}}(\bar{C}^q\|\zeta\|^q_{\mathcal{R}^q([0,\Pi])} + d^q),
\end{aligned} \tag{6}
$$

where

$$
M_q = \left(\frac{(q-1)q}{2}\right)^{\frac{q}{2}} \Pi^{\frac{q-2}{2}}.
$$

Therefore, $||\mathcal{K}_\psi\zeta(\varkappa)||_{\mathcal{R}^q([0,\Pi])} < \infty$. □

Theorem 2. *Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ hold, then the Eq. (1) has a unique solution.*

Proof. For $\rho > 0$, we consider $\|.\|_\rho$ the norm on $\mathcal{R}^q([0,\Pi])$ defined by

$$\|\zeta\|_\rho^q = \text{ess}\sup_{\varkappa\in[0,\Pi]}\left(\frac{\mathbb{E}|\zeta(\varkappa)|^q}{\exp(\rho\varkappa)}\right).$$

We have $\|.\|_{\mathcal{R}^q([0,\Pi])}$ and $\|.\|_\rho$ are equivalent.

We will use the BFPT to prove the existence and uniqueness of the solution of Eq. (1).

We have for all $\varkappa \in [0,\Pi]$

$$\begin{aligned}
&\left\|\mathcal{K}_\psi\zeta(\varkappa) - \mathcal{K}_\psi\widehat{\zeta}(\varkappa)\right\|_q^q \\
&\leq (n+1)^{q-1}\Bigg[\left\|\int_0^\varkappa \left(F_1\big(w,\zeta(w)\big) - F_1\big(w,\widehat{\zeta}(w)\big)\right)dw\right\|_q^q \\
&+\left\|\int_0^\varkappa \left(g\big(w,\zeta(w)\big) - g\big(w,\widehat{\zeta}(w)\big)\right)d\mathbb{B}(w)\right\|_q^q \\
&+\sum_{i=2}^n\left\|\int_0^\varkappa (\varkappa-w)^{\beta_i-1}\exp\left(\frac{\nu_i-1}{\nu_i}(\varkappa-w)\right)\left(F_i\big(w,\zeta(w)\big) - F_i\big(w,\widehat{\zeta}(w)\big)\right)dw\right\|_q^q\Bigg].
\end{aligned} \tag{7}$$

It follows from the Hölder inequality and $\mathcal{H}_1$ that

$$\begin{aligned}
\left\|\int_0^\varkappa \left(F_1\big(l,\zeta(l)\big) - F_1\big(l,\widehat{\zeta}(l)\big)\right)dl\right\|_q^q &\leq \varkappa^{q-1}\mathbb{E}\left(\int_0^\varkappa |F_1\big(s,\zeta(s)\big) - F_1\big(s,\widehat{\zeta}(s)\big)|^q ds\right) \\
&\leq \bar{C}^q\Pi^{q-1}\int_0^\varkappa \|\zeta(s)-\widehat{\zeta}(s)\|_q^q ds,
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
&\left\|\int_0^\varkappa (\varkappa-l)^{\beta_i-1}\exp\left(\frac{\nu_i-1}{\nu_i}(\varkappa-l)\right)\left(F_i\big(l,\zeta(l)\big) - F_i\big(l,\widehat{\zeta}(l)\big)\right)dl\right\|_q^q \\
&\leq \mathbb{E}\left(\left(\int_0^\varkappa (\varkappa-s)^{\frac{q(\beta_i-1)}{q-1}}ds\right)^{q-1}\int_0^\varkappa |F_i\big(s,\zeta(s)\big) - F_i\big(s,\widehat{\zeta}(s)\big)|^q ds\right) \\
&\leq \frac{\bar{C}^q\Pi^{\beta_i q-1}(q-1)^{q-1}}{(\beta_i q-1)^{q-1}}\int_0^\varkappa \|\zeta(s)-\widehat{\zeta}(s)\|_q^q ds.
\end{aligned} \tag{9}$$

Using $\mathcal{H}_1$, Theorem 7.1 in [15] and the Hölder inequality, we obtain

$$\begin{aligned}
\left\|\int_0^\varkappa \left(g\big(\varpi,\zeta(\varpi)\big) - g\big(\varpi,\widehat{\zeta}(\varpi)\big)\right)d\mathbb{B}(\varpi)\right\|_q^q &\leq M_q\mathbb{E}\left|\int_0^\varkappa |g\big(\varpi,\zeta(\varpi)\big) - g\big(\varpi,\widehat{\zeta}(\varpi)\big)|^2 d\varpi\right|^{\frac{q}{2}} \\
&\leq M_q\bar{C}^q\Pi^{\frac{q-2}{2}}\int_0^\varkappa \|\zeta(\varpi)-\widehat{\zeta}(\varpi)\|_q^q d\varpi.
\end{aligned} \tag{10}$$

Therefore, we get

$$\left\|\mathcal{K}_\psi\zeta(\varkappa) - \mathcal{K}_\psi\widehat{\zeta}(\varkappa)\right\|_q^q \leq \lambda\int_0^\varkappa \|\zeta(s)-\widehat{\zeta}(s)\|_q^q ds, \tag{11}$$

where

$$\lambda = (n+1)^{q-1}\left[\bar{C}^q\Pi^{q-1} + \sum_{i=2}^{n}\bar{C}^q\Pi^{\beta_i q-1}\left(\frac{q-1}{\beta_i q-1}\right)^{q-1} + \bar{C}^q M_q \Pi^{\frac{q-2}{2}}\right].$$

Then,

$$\begin{aligned}\left\|\mathcal{K}_\psi\zeta(\varkappa) - \mathcal{K}_\psi\widehat{\zeta}(\varkappa)\right\|_q^q &\leq \lambda\int_0^\varkappa \|\zeta(s)-\widehat{\zeta}(s)\|_q^q ds\\ &\leq \lambda\int_0^\varkappa \frac{\|\zeta(s)-\widehat{\zeta}(s)\|_q^q}{\exp(\rho s)}\exp(\rho s)ds\\ &\leq \lambda\|\zeta-\widehat{\zeta}\|_\rho^q\int_0^\varkappa \exp(\rho s)ds\\ &\leq \frac{\lambda\exp(\rho\varkappa)}{\rho}\|\zeta-\widehat{\zeta}\|_\rho^q. \end{aligned} \tag{12}$$

Thus,

$$\left\|\mathcal{K}_\psi\zeta - \mathcal{K}_\psi\widehat{\zeta}\right\|_\rho \leq \left(\frac{\lambda}{\rho}\right)^{\frac{1}{q}}\|\nu-\widehat{\nu}\|_\rho. \tag{13}$$

Therefore, $\mathcal{K}_\psi$ is contractive for some $\rho > 0$. Hence, the Eq. (1) has a unique solution.

Now we will define the Ulam-Hyers stability. □

Definition 2. Let

$$S(\varkappa) = y(0) + \int_0^\varkappa F_1\big(s, y(s)\big)ds + \sum_{i=2}^{n}\int_0^\varkappa (\varkappa-s)^{\beta_i-1}\exp\left(\frac{\nu_i-1}{\nu_i}(\varkappa-s)\right)F_i\big(s,y(s)\big)ds$$
$$+\int_0^\varkappa g\big(s,y(s)\big)d\mathbb{B}(s),\ \varkappa\in[0,\Pi],$$

with $0<\beta_i<1$, $0<\nu_i\leq 1$ for $2\leq i\leq n$, $y(0)=\psi\in\mathbb{R}$ and the functions $F_i, g:[0,\Pi]\times\mathbb{R}\longrightarrow\mathbb{R}$, $1\leq i\leq n$, are measurables.

Eq. (1) is Ulam-Hyers stable if there is $\Delta > 0$ such that, for every $\epsilon > 0$ and for every $y\in\mathcal{R}^q([0,\Pi])$, with

$$\|y(\varkappa)-S(\varkappa)\|_q^q \leq \epsilon,\ \forall\,\varkappa\in[0,\Pi], \tag{14}$$

there is a solution $\varphi\in\mathcal{R}^q([0,\Pi])$ of (1), with $\varphi(0)=y(0)$, and

$$\|y(\varkappa)-\varphi(\varkappa)\|_q^q \leq \Delta\epsilon,\ \forall\,\varkappa\in[0,\Pi].$$

Theorem 3. *Under $\mathcal{H}_1$ and $\mathcal{H}_2$, the Eq. (1) is UHS.*

Proof. We will use the Gronwall inequality to prove the UHS of Eq. (1).

Set $\epsilon>0$ and $y\in\mathcal{R}^q([0,\Pi])$ satisfies (14). Let $\varphi\in\mathcal{R}^q([0,\Pi])$ the solution of (1) with $\varphi(0)=y(0)$, then

$$\varphi(\varkappa) = \varphi(0) + \int_0^\varkappa F_1\big(s,\varphi(s)\big)ds + \sum_{i=2}^{n}\int_0^\varkappa(\varkappa-s)^{\beta_i-1}\exp\left(\frac{\nu_i-1}{\nu_i}(\varkappa-s)\right)F_i\big(s,\varphi(s)\big)ds$$

$$+ \int_0^{\varkappa} g\big(s, \varphi(s)\big) d\mathbb{B}(s). \tag{15}$$

Thus,

$$\begin{aligned}
&\mathbb{E}|y(\varkappa) - \varphi(\varkappa)|^q \\
&\leq \mathbb{E}|y(\varkappa) - S(\varkappa) + S(\varkappa) - \varphi(\varkappa)|^q \\
&\leq (n+2)^{q-1}\Bigg[\epsilon + \left\|\int_0^{\varkappa} \big(F_1\big(w, \varphi(w)\big) - F_1\big(w, y(w)\big)\big)\, dw\right\|_q^q \\
&+ \left\|\int_0^{\varkappa} \big(g\big(w, \varphi(w)\big) - g\big(w, y(w)\big)\big)\, d\mathbb{B}(w)\right\|_q^q \\
&+ \sum_{i=2}^{n} \left\|\int_0^{\varkappa} (\varkappa - w)^{\beta_i - 1} \exp\left(\frac{\nu_i - 1}{\nu_i}(\varkappa - w)\right) \big(F_i\big(w, \varphi(w)\big) - F_i\big(w, y(w)\big)\big)\, dw\right\|_q^q\Bigg].
\end{aligned} \tag{16}$$

Following the approach used in the proof of Theorem 1, we obtain

$$\begin{aligned}
&\mathbb{E}|y(\varkappa) - \varphi(\varkappa)|^q \\
&\leq (n+2)^{q-1}\epsilon + (n+2)^{q-1}\tilde{\lambda}\int_0^{\varkappa} \mathbb{E}|y(s) - \varphi(s)|^q ds,
\end{aligned} \tag{17}$$

where $\tilde{\lambda} = \dfrac{\lambda}{(n+1)^{q-1}}$.

Using Gronwall inequality, we get

$$\mathbb{E}|y(\varkappa) - \varphi(\varkappa)|^q \leq (n+2)^{q-1}\epsilon e^{(n+2)^{q-1}\tilde{\lambda}\varkappa}, \ \forall \varkappa \in [0, \Pi]. \tag{18}$$

Hence,

$$\mathbb{E}|y(\varkappa) - \varphi(\varkappa)|^q \leq N\epsilon, \ \forall \varkappa \in [0, \Pi], \tag{19}$$

where $N = (n+2)^{q-1} e^{(n+2)^{q-1}\tilde{\lambda}\Pi}$.

Thus, Eq. (1) is UHS. □

4 Examples

In this section, we present three examples to demonstrate the effectiveness of our results.

Example 1: Let consider the Eq. (1) for $\Pi = 2$, $n = 2$,

$$F_1(\varkappa, \zeta) = \cos(\zeta)$$

$$F_2(\varkappa, \zeta) = \frac{\sin(\zeta)}{\varkappa^6 + 5}$$

$$g(\varkappa, \zeta(\varkappa)) = \varkappa \cos(\zeta).$$

We have

$$|F_1(\varkappa, \zeta_1) - F_1(\varkappa, \zeta_2)| \leq |\zeta_1 - \zeta_2|, \tag{20}$$

$$|F_2(\varkappa,\zeta_1) - F_2(\varkappa,\zeta_2)| \leq |\zeta_1 - \zeta_2|, \tag{21}$$

and

$$|g(\varkappa,\zeta_1) - g(\varkappa,\zeta_2)| \leq 2|\zeta_1 - \zeta_2|. \tag{22}$$

Thus, assumption $\mathcal{H}_1$ holds for $\bar{C} = 2$. Additionally, for $i \in \{1, 2\}$, we get

$$\operatorname*{ess\,sup}_{\varkappa\in[0,2]} |F_i(\varkappa, 0)| \leq 2 \tag{23}$$

and

$$\operatorname*{ess\,sup}_{\varkappa\in[0,2]} |g(\varkappa, 0)| \leq 2. \tag{24}$$

Therefore, assumption $\mathcal{H}_2$ holds for $d = 2$.

Thus, the equation is UHS.

Example 2: Let consider the Eq. (1) for $\Pi = 5$, $n = 4$,

$$F_1(\varkappa,\zeta) = e^{-\varkappa}\zeta$$

$$F_2(\varkappa,\zeta) = 5\sin(\zeta)$$

$$F_3(\varkappa,\zeta) = \frac{1}{5}\varkappa\sin(\zeta)$$

$$F_4(\varkappa,\zeta) = \frac{\cos(\zeta)}{\varkappa^2 + 4}$$

$$g(\varkappa,\zeta(\varkappa)) = \frac{1}{25}\varkappa^2\arctan(\zeta).$$

We have

$$|F_1(\varkappa,\zeta_1) - F_1(\varkappa,\zeta_2)| \leq |\zeta_1 - \zeta_2|, \tag{25}$$

$$|F_2(\varkappa,\zeta_1) - F_2(\varkappa,\zeta_2)| \leq 5|\zeta_1 - \zeta_2|, \tag{26}$$

$$|F_3(\varkappa,\zeta_1) - F_3(\varkappa,\zeta_2)| \leq |\zeta_1 - \zeta_2|, \tag{27}$$

$$|F_4(\varkappa,\zeta_1) - F_4(\varkappa,\zeta_2)| \leq |\zeta_1 - \zeta_2|, \tag{28}$$

and

$$|g(\varkappa,\zeta_1) - g(\varkappa,\zeta_2)| \leq |\zeta_1 - \zeta_2|. \tag{29}$$

Thus, assumption $\mathcal{H}_1$ holds for $\bar{C} = 5$. Additionally, for $i \in \{1, 2, 3, 4\}$, we get

$$\operatorname*{ess\,sup}_{\varkappa\in[0,5]} |F_i(\varkappa, 0)| \leq 1 \tag{30}$$

and

$$\operatorname*{ess\,sup}_{\varkappa\in[0,5]} |g(\varkappa, 0)| \leq 1. \tag{31}$$

Therefore, assumption $\mathcal{H}_2$ holds for $d = 1$.

Thus, the equation is UHS.

Example 3: Let consider the Eq. (1) for $\Pi = 1, n = 3$,

$F_1(\varkappa,\zeta) = \arctan(\zeta)$

$F_2(\varkappa,\zeta) = 2\cos(\zeta)\sin(\varkappa^2)$

$F_3(\varkappa,\zeta) = 2\zeta$

$g(\varkappa,\zeta(\varkappa)) = \varkappa^2\sin(\zeta)$.

We have

$$|F_1(\varkappa,\zeta_1) - F_1(\varkappa,\zeta_2)| \leq |\zeta_1 - \zeta_2|, \tag{32}$$

$$|F_2(\varkappa,\zeta_1) - F_2(\varkappa,\zeta_2)| \leq 2|\zeta_1 - \zeta_2|, \tag{33}$$

$$|F_3(\varkappa,\zeta_1) - F_3(\varkappa,\zeta_2)| \leq 2|\zeta_1 - \zeta_2|, \tag{34}$$

and

$$|g(\varkappa,\zeta_1) - g(\varkappa,\zeta_2)| \leq |\zeta_1 - \zeta_2|. \tag{35}$$

Thus, assumption $\mathcal{H}_1$ holds for $\bar{C} = 2$. Additionally, for $i \in \{1,2,3\}$, we get

$$\operatorname*{ess\,sup}_{\varkappa\in[0,1]} |F_i(\varkappa,0)| \leq 2, \tag{36}$$

and

$$\operatorname*{ess\,sup}_{\varkappa\in[0,1]} |g(\varkappa,0)| \leq 1. \tag{37}$$

Therefore, assumption $\mathcal{H}_2$ holds for $d = 1$.

Thus, the equation is UHS.

5 Conclusion

In this work, we apply the Banach fixed point theorem to show the existence and uniqueness of solutions for PFIDSIE. We use classical stochastic analysis techniques along with the Gronwall inequality to prove the UHS of such equations. To demonstrate our results, we provide three examples in the concluding section. In future work, we intend to extend our results to the case of PFIDSIE with time delay.

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