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LEAST-SQUARES METHODS FOR LINEAR ELASTICITY: REFINED ERROR ESTIMATES

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Abstract. We consider the linear elasticity problems and compare the approximations obtained by the Least-Squares finite element method with the approximations obtained by the standard conforming finite element method and the mixed finite element method. The main result is that the H^1-conforming displacement approximations (least-squares finite element and standard finite element) as well as the $H(\text{div})$-conforming stress approximations are higher-order pertubations of each other. This leads to refined a priori bounds and superconvergence results. Numerical experiments illustrate the theory.

1 Introduction

Let $\Omega \in \mathbb{R}^d$ $(d = 2,3)$ be a polytopal convex domain with boundary $\partial\Omega$ divided into two parts Γ_D and Γ_N, i.e. $\partial\Omega = \overline{\Gamma_D \cup \Gamma_N}, \Gamma_D \cap \Gamma_N = \emptyset, \Gamma_D \neq \emptyset$. For given data $f \in \left(L^2(\Omega)\right)^2$, the linear elasticity problem is modeled as

$$\begin{aligned} &\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\varepsilon}(\mathbf{u}) = 0 && \text{in } \Omega \\ &\operatorname{div}\boldsymbol{\sigma} = -\mathbf{f} && \text{in } \Omega \\ &\mathbf{u} = \mathbf{0} && \text{on } \Gamma_D \\ &\boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{0} && \text{on } \Gamma_N, \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}$ is a symmetric d-by-d stress tensor, $\mathbf{u}$ the displacement vector field, $\mathcal{A}$ is the inverse of the elastic material law, defined in terms of the Lamé constants μ and λ by

$$\mathcal{A}\boldsymbol{\tau} = \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{\lambda}{2\mu + d\lambda}\operatorname{tr}(\boldsymbol{\tau})\mathbf{I}\right),$$

the symmetric gradient is defined as

$$\boldsymbol{\varepsilon}(\mathbf{v}) = \frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{v} + (\boldsymbol{\nabla}\mathbf{v})^\top\right),$$

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$\mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{2} \tau_{ii}$ denotes the trace of a vector and $\mathbf{n}$ the outward unit normal vector to Γ_N. The presence of $\mathcal{A}$ instead of the usual stress-strain relation $C = 2\mu\boldsymbol{\varepsilon}\mathbf{u} + \lambda(\mathrm{div}\,\mathbf{u})\mathbf{I}$ allows the formulations to be robust in the incompressible limit (as λ goes to infinity).

Finite element methods are the most widely used tools for computing the deformations of an elastic body subject to forces. In the framework of the (non-robust) standard conforming theory, the variational problem (see e.g. [10, Chapter 11.]) is to minimize the energy $(\mathcal{A}^{-1}\mathbf{v}, \mathbf{v})$ under all $\mathbf{v} \in \mathbf{V} = \mathbf{H}^1_D(\Omega)$. An accurate approximation of the stress tensor, which is often of crucial interest, can be obtained with stress-based variational formulations where the stress is directly seek in

$$\boldsymbol{\Sigma}_N = \begin{cases} \left\{\boldsymbol{\tau} \in H(\mathrm{div};\Omega)^2 : \boldsymbol{\tau}\cdot\mathbf{n} = 0 \text{ on } \Gamma_N\right\} & \text{if } \Gamma_N \neq \emptyset \\ \left\{\boldsymbol{\tau} \in H(\mathrm{div};\Omega)^2 : \int_\Omega \mathrm{tr}(\boldsymbol{\tau})\, d\mathbf{x} = 0\right\} & \text{if } \Gamma_N = \emptyset \end{cases},$$

where each component of the (column) vector divergence operator div is acting on the corresponding row of $\mathbf{H}(\mathrm{div},\Omega) = H(\mathrm{div};\Omega)^2$. Those methods can either lead to a saddle-point formulation (see e.g. [8]) or of Least-Squares type (see e.g. [7]). A comparison of the H^1-conforming approximations (least-squares finite element and standard finite element) as well as the $H(div)$-conforming approximations are was performed in [9] for the Poisson equation, proving that they are higher-order perturbations of each other. This leads to refined a priori bounds and superconvergence results. The purpose of this paper is to extend these results to the linear elasticity problem. The next section will recall the formulations while section 3 presents the discretisations. The direct comparison will be performed in section 4 while section 5 is dedicated to the numerical experiment.

2 Variational formulations

The standard non-robust displacement formulation according tho the energy principle introduced in the introduction reads: find $u \in \mathbf{V}$ such that

$$a_S(\mathbf{u},\mathbf{v}) = 2\mu(\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{L^2(\Omega)} + \lambda(\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v})_{L^2(\Omega)} = (\mathbf{f},\mathbf{v})_{L^2(\Omega)} \quad \forall \mathbf{v} \in \mathbf{V}. \tag{S}$$

The stress-based mixed method maximizes the energy $(\mathcal{A}\boldsymbol{\tau},\boldsymbol{\tau})$ under all $\boldsymbol{\tau}$ satisfying the divergence constraint

$$(\mathrm{div}(\boldsymbol{\sigma}),\mathbf{w}) = (\mathbf{f},\mathbf{v}) \quad \forall \mathbf{w} \in L^2(\Omega)^2 \tag{2}$$

as well as the symmetry condition

$$(\mathrm{skew}\,\boldsymbol{\sigma},\gamma\boldsymbol{\chi}) = 0 \quad \forall \boldsymbol{\gamma} \in L^2(\Omega) \tag{3}$$

where the skew-symmetric part is defined as

$$\mathrm{skew}(\boldsymbol{\tau}) = \frac{1}{2}\left\{\boldsymbol{\tau} - \boldsymbol{\tau}^\top\right\} \text{ and } \boldsymbol{\chi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

This gives rises to the following stress-based mixed method: find $(\boldsymbol{\sigma},\mathbf{u},\omega)$ such that

$$(\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\tau}) + (\mathrm{div}(\boldsymbol{\tau}),\mathbf{u}) + (\mathrm{skew}\,\boldsymbol{\tau},\boldsymbol{\omega}) = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma} \tag{4a}$$

$$(\mathrm{div}(\boldsymbol{\sigma}),\mathbf{w}) = (\mathbf{f},\mathbf{w}) \qquad \forall \mathbf{w} \in L^2(\Omega)^2 \tag{4b}$$

$$(\mathrm{skew}\,\boldsymbol{\sigma},\gamma\boldsymbol{\chi}) = 0 \qquad \forall \gamma \in L^2(\Omega). \tag{4c}$$

Note that this system can be rewritten as

$$(\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\tau}) - b_m(\boldsymbol{\tau},(\mathbf{u},\omega)) = 0 \qquad \forall\boldsymbol{\tau}\in\boldsymbol{\Sigma} \tag{5a}$$

$$b_m(\boldsymbol{\sigma},(\mathbf{w},\boldsymbol{\gamma})) = (\mathbf{f},\mathbf{v}) \qquad \forall(\mathbf{w},\gamma)\in L^2(\Omega)^3. \tag{5b}$$

with the bilinearform $b_m(\boldsymbol{\tau},(\mathbf{w},\boldsymbol{\gamma})) = -(\operatorname{div}(\boldsymbol{\tau}),\mathbf{w}) - (\operatorname{skew}\boldsymbol{\tau},\gamma\boldsymbol{\chi})$ and that the first term $(\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\tau})$ correspond to a symmetric bilinear form, i.e.

$$(\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\tau}) = (\mathcal{A}\boldsymbol{\tau},\boldsymbol{\sigma}). \tag{6}$$

A third approach, the *two-field* least-squares formulation of (1) considered in [11] consists in minimizing the functional

$$\mathcal{F}(\boldsymbol{\tau},\mathbf{v};\mathbf{f}) = \|\mathcal{A}\boldsymbol{\tau}-\boldsymbol{\varepsilon}(\mathbf{v})\|_0^2 + \|\operatorname{div}\boldsymbol{\tau}+\mathbf{f}\|_0^2$$

in $\boldsymbol{\Sigma}_N\times\mathbf{V}$. Other two least-squares formulations have been introduced in [3] which make use of three and four fields, respectively, by the introduction of the vorticity and the pressure as new unknowns. The three-field formulation seeks a minimizer of the functional

$$\mathcal{G}(\boldsymbol{\tau},\mathbf{v},q;\mathbf{f}) = \|\mathcal{A}\boldsymbol{\tau}-\boldsymbol{\nabla}\mathbf{v}+\boldsymbol{\chi}q\|_0^2 + \|\operatorname{div}\boldsymbol{\tau}+\mathbf{f}\|_0^2 + \|\operatorname{skew}\boldsymbol{\tau}\|_0^2$$

in $\boldsymbol{\Sigma}_N\times\mathbf{V}\times\bar{L}^2(\Omega)$ with

$$\bar{L}^2(\Omega) = \begin{cases} L^2(\Omega) & \text{if } \Gamma_N\neq\emptyset \\ \left\{q\in L^2(\Omega) : \int_\Omega q\,d\mathbf{x}=0\right\} & \text{if } \Gamma_N=\emptyset. \end{cases}$$

The minimization of the functional $\mathcal{F}(\boldsymbol{\tau},\mathbf{v};\mathbf{f})$ gives rise to the following variational formulation: find $\boldsymbol{\sigma}\in\boldsymbol{\Sigma}_N$ and $\mathbf{u}\in\mathbf{V}$ such that

$$(\mathcal{A}\boldsymbol{\sigma},\mathcal{A}\boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\sigma},\operatorname{div}\boldsymbol{\tau}) - (\mathcal{A}\boldsymbol{\tau},\boldsymbol{\varepsilon}(\mathbf{u})) = -(\mathbf{f},\operatorname{div}\boldsymbol{\tau}) \qquad \forall\boldsymbol{\tau}\in\boldsymbol{\Sigma}_N \tag{7a}$$

$$-(\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\mathbf{v})) + (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})) = 0 \qquad \forall\mathbf{v}\in\mathbf{V}. \tag{7b}$$

The variational formulation associated with the minimization of the functional $\mathcal{G}$ is obtained by seeking $\boldsymbol{\sigma}\in\boldsymbol{\Sigma}_N$, $\mathbf{u}\in\mathbf{V}$, and $p\in\bar{L}^2(\Omega)$ such that

$$(\mathcal{A}\boldsymbol{\sigma},\mathcal{A}\boldsymbol{\tau}) + (\operatorname{div}\boldsymbol{\sigma},\operatorname{div}\boldsymbol{\tau}) + (\operatorname{skew}(\boldsymbol{\sigma}),\boldsymbol{\tau}) - (\mathcal{A}\boldsymbol{\tau},\boldsymbol{\varepsilon}(\mathbf{u})) + (\operatorname{skew}(A\boldsymbol{\tau}),p) = -(\mathbf{f},\operatorname{div}\boldsymbol{\tau}) \quad \forall\boldsymbol{\tau}\in\boldsymbol{\Sigma}_N \tag{8a}$$

$$-(\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\mathbf{v})) + (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})) - (p,\operatorname{skew}(\boldsymbol{\nabla}\mathbf{v})) = 0 \quad \forall\mathbf{v}\in\mathbf{V} \tag{8b}$$

$$(\operatorname{skew}(A\boldsymbol{\sigma}),q) - (q,\operatorname{skew}(\boldsymbol{\nabla}\mathbf{u})) + 2(p,q) = 0 \quad \forall q\in\bar{L}^2(\Omega). \tag{8c}$$

In order to consider these Least-Squares formulations in a unified setting, we introduce the space $\boldsymbol{\Phi} = \mathbf{V}\times\bar{L}^2(\Omega)$ and reformulate (7) and (8) as follows

$$\mathcal{B}_k((\boldsymbol{\tau},\boldsymbol{\phi})|(\boldsymbol{\tau},\boldsymbol{\psi})) = -(\mathbf{f},\operatorname{div}\boldsymbol{\tau}) \quad k=1,2 \tag{9}$$

with

$$\mathcal{B}_1((\boldsymbol{\sigma},(\mathbf{u},p))|(\boldsymbol{\tau},(\mathbf{v},q))) = (\mathcal{A}\boldsymbol{\sigma}-\boldsymbol{\varepsilon}(\mathbf{u}),\mathcal{A}\boldsymbol{\tau}-\boldsymbol{\varepsilon}(\mathbf{v})) + (\operatorname{div}\boldsymbol{\sigma},\operatorname{div}\boldsymbol{\tau}) \tag{10}$$

and

$$\mathcal{B}_2((\boldsymbol{\sigma},(\mathbf{u},p))|(\boldsymbol{\tau},(\mathbf{v},q))) = (\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\nabla}(\mathbf{u}) + p\chi, \mathcal{A}\boldsymbol{\tau} - \boldsymbol{\nabla}(\mathbf{v}) + q\chi) + 2(p,q) + +(\operatorname{div}\boldsymbol{\sigma}, \operatorname{div}\boldsymbol{\tau}). \tag{11}$$

We also define define the exact solution

$$\boldsymbol{\phi} = (\mathbf{u}, \frac{1}{2}\,\mathbf{curl}\,\mathbf{u}) = (\mathbf{u}, p). \tag{12}$$

A further useful notation splits the bilinear form B_j in terms defining an inner product and terms corresponding to a mixed formulation:

$$B_j((\boldsymbol{\sigma},\boldsymbol{\phi}),(\boldsymbol{\tau},\boldsymbol{\psi})) = ((\boldsymbol{\sigma},\boldsymbol{\phi}),(\boldsymbol{\tau},\boldsymbol{\psi}))_{LS_j} - b_{LS_j}(\boldsymbol{\sigma},\boldsymbol{\psi}) - b_{LS_j}(\boldsymbol{\tau},\boldsymbol{\phi}) \tag{13}$$

with

$$((\boldsymbol{\sigma},\boldsymbol{\phi}),(\boldsymbol{\tau},\boldsymbol{\psi}))_{LS_1} = ((\boldsymbol{\sigma},\boldsymbol{\phi}),(\mathbf{u},\mathbf{v}))_{LS_1} = (\mathcal{A}\boldsymbol{\sigma},\mathcal{A}\boldsymbol{\tau}) + (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})), \tag{14a}$$

$$((\boldsymbol{\sigma},\boldsymbol{\phi}),(\boldsymbol{\tau},\boldsymbol{\psi}))_{LS_2} = ((\boldsymbol{\sigma},\boldsymbol{\phi}),((\mathbf{u},p),(\mathbf{v},q)))_{LS_2} = (\mathcal{A}\boldsymbol{\sigma},\mathcal{A}\boldsymbol{\tau}) + (\boldsymbol{\nabla}\boldsymbol{\sigma},\boldsymbol{\nabla}\boldsymbol{\tau}), +2(p,q) \tag{14b}$$

and

$$b_{LS_1}(\boldsymbol{\sigma},\boldsymbol{\psi}) = b_{LS_1}(\boldsymbol{\sigma},\mathbf{v}) = (\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\mathbf{v})) \tag{15a}$$

$$b_{LS_2}(\boldsymbol{\sigma},\boldsymbol{\psi}) = b_{LS_2}(\boldsymbol{\sigma},(\mathbf{v},q)) = (\mathcal{A}\boldsymbol{\sigma},\boldsymbol{\nabla}\,\mathbf{v} - q\boldsymbol{\chi}) + (\boldsymbol{\nabla}\,\mathbf{v}, q\boldsymbol{\chi})\ . \tag{15b}$$

The difference between the energy considered in the mixed formulation and the inner product arising in the Least-Squares method is crucial and we therefore define further

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{LS_\Sigma} = (\mathcal{A}\boldsymbol{\sigma},\mathcal{A}\boldsymbol{\tau}) \text{ and } (\boldsymbol{\phi},\boldsymbol{\psi})_{LS^k_{\boldsymbol{\Phi}}} = ((\mathbf{u},p),(\mathbf{v},q))_{LS^k_{\boldsymbol{\Phi}}} = \begin{cases} (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v})) & k=1 \\ (\boldsymbol{\nabla}(\mathbf{u}),\boldsymbol{\nabla}(\mathbf{v})) + (p,q) & k=2 \end{cases}. \tag{16}$$

We will also drop the index k in the equations where both $k = 1,2$ are allowed.

3 Discretisations

Let Ω_h be a regular triangulation of Ω. The approximation of the formulation presented in the previous section is performed by choosing appropriate subspaces of $\boldsymbol{\Sigma}_N, \mathbf{V}$ and $\bar{L}^2(\Omega)$. For the conforming approximations of the displacement in the standard and Least-Squares formulations, we choose $\mathbf{V}_h \subset \mathbf{V}$ as the conforming Lagrange element of degree k. The discrete version of the standard formulation (S) therefore reads: find $\mathbf{u}_h \in \mathbf{V}_h$ such that

$$(\mathcal{A}^{-1}\varepsilon(\mathbf{u}_h), \nabla\mathbf{v}) = (\mathbf{f},\mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h. \tag{S_h}$$

Recall that the Galerkin orthogonality

$$(\mathcal{A}^{-1}\varepsilon(\mathbf{u} - \mathbf{u}_h), \nabla\mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h. \tag{17}$$

implies the non-robust a priori estimates $\|u - u_h\|_1 \leq C(\mu,\lambda)h^k\|u\|_2$.

For the conforming approximations of the stress tensor in the mixed and Least-Squares formulations, we choose the tensor space $\Sigma_h \subset \Sigma_N$ whose rows consists in the $H(\mathrm{div};\Omega)$-conforming Raviart-Thomas space of degree k. The discrete version of the two-fields formulation (7) therefore reads: find $\mathbf{u}_h^1 \in \mathbf{V}_h$ and $\boldsymbol{\sigma}_h^1 \in \boldsymbol{\Sigma}_h$ such that

$$(\mathcal{A}\boldsymbol{\sigma}_h^1, \mathcal{A}\boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\sigma}_h^1, \mathrm{div}\,\boldsymbol{\tau}) - (\mathcal{A}\boldsymbol{\tau}, \boldsymbol{\varepsilon}(\mathbf{u}_h^1)) = -(\mathbf{f}, \mathrm{div}\,\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \tag{18a}$$

$$-(\mathcal{A}\boldsymbol{\sigma}_h^1, \boldsymbol{\varepsilon}(\mathbf{v})) + (\boldsymbol{\varepsilon}(\mathbf{u}_h^1), \boldsymbol{\varepsilon}(\mathbf{v})) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h. \tag{18b}$$

The three-fields Least-Squares method requires an additional subspace W_h of $L^2(\Omega)$ for the vorticity. As the Least-Squares method does not requires any compatibility condition between the space we choose the space of piecewise discontinuous polynomials of degree $k-1$ in order to obtain corresponding convergence rates for the stress, the displacement and the vorticity. The Galerkin approximation of (8) reads: find $\mathbf{u}_h^2 \in \mathbf{V}_h$, $\boldsymbol{\sigma}_h^2 \in \boldsymbol{\Sigma}_h$ and $p_h \in W_h$ such that

$$(\mathcal{A}\boldsymbol{\sigma}_h^2 - \boldsymbol{\varepsilon}(\mathbf{u}_h^2 + p_h\boldsymbol{\chi}), \mathcal{A}\boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\sigma}_h^2, \mathrm{div}\,\boldsymbol{\tau}) + (\mathrm{skew}(\boldsymbol{\sigma}_h^2), \boldsymbol{\tau}) = -(\mathbf{f}, \mathrm{div}\,\boldsymbol{\tau}) \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \tag{19a}$$

$$(\boldsymbol{\varepsilon}(\mathbf{u}_h^2) - \mathcal{A}\boldsymbol{\sigma}_h^2, \boldsymbol{\varepsilon}(\mathbf{v})), \boldsymbol{\varepsilon}(\mathbf{v})) - (\boldsymbol{\chi} p_h, \nabla \mathbf{v}) = 0 \qquad \forall \mathbf{v} \in \mathbf{V}_h \tag{19b}$$

$$(A\boldsymbol{\sigma}_h^2 - \nabla \mathbf{u}_h^2, q\boldsymbol{\chi}) + 2(p_h, q) = 0 \qquad \forall q \in W_h. \tag{19c}$$

Similarly to the continuous setting we introduce the spaces $\boldsymbol{\Phi}_h = \mathbf{V}_h \times W_h$ and reformulate (18) and (19) as follows

$$\mathcal{B}_k((\boldsymbol{\sigma}_h, \boldsymbol{\phi}_h)|(\boldsymbol{\tau}_h, \boldsymbol{\psi}_h)) = -(\mathbf{f}, \mathrm{div}\,\boldsymbol{\tau}_h) \text{ for all } (\boldsymbol{\tau}_h, \boldsymbol{\psi}_h) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\Phi}_h \quad k = 1,2 \tag{LS_h}$$

as well as

$$\boldsymbol{\phi}_h^k = \begin{cases} (\mathbf{u}_h, \frac{1}{2}\,\mathbf{curl}\,\mathbf{u}_h) & \text{if } k=1 \\ (\mathbf{u}_h, p_h) & \text{if } k=2 \end{cases}. \tag{20}$$

For the discretisation of the mixed method we choose the remaining discrete spaces $\mathbf{W}_h$ for the approximation of the displacement and X_h for the approximation of the vorticity such that the well-posedness of the system is satisfied. According to [8] we can choose $\mathbf{W}_h$ as the space of discontinuous piecewise vector polynomials of degree k and X_h as the space of continuous piecewise polynomials of degree k. The discrete version of (4c) the reads: find $(\boldsymbol{\sigma}_h^m, \mathbf{u}_h^m, \omega_h) \in \boldsymbol{\Sigma}_h \times \mathbf{W}_h \times X_h$ such that

$$(\mathcal{A}\boldsymbol{\sigma}_h^m, \boldsymbol{\tau}_h) + (\mathrm{div}(\boldsymbol{\tau}_h), \mathbf{u}_h^m) + (\mathrm{skew}\,\boldsymbol{\tau}_h, \omega_h) = 0 \qquad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h \tag{21a}$$

$$(\mathrm{div}(\boldsymbol{\sigma}_h^m), \mathbf{w}_h) = (\mathbf{f}, \mathbf{w}_h) \qquad \forall \mathbf{w}_h \in \mathbf{W}_h \tag{21b}$$

$$(\mathrm{skew}\,\boldsymbol{\sigma}_h^m, \gamma_h) = 0 \qquad \forall \gamma_h \in X_h, \tag{21c}$$

i.e.

$$(\mathcal{A}\boldsymbol{\sigma}_h^m, \boldsymbol{\tau}_h) + (\mathrm{div}(\boldsymbol{\tau}_h), \mathbf{u}_h^m) + (\mathrm{skew}\,\boldsymbol{\tau}_h, \omega_h) + (\mathrm{div}(\boldsymbol{\sigma}_h^m), \mathbf{w}_h) + (\mathrm{skew}\,\boldsymbol{\sigma}_h^m, \gamma_h) = (\mathbf{f}, \mathbf{w}_h) \tag{M_h}$$

for all $(\boldsymbol{\tau}_h, \mathbf{w}_h, \gamma_h) \in \boldsymbol{\Sigma}_h \times \mathbf{W}_h \times X_h$. Based on the Galerkin orthogonalities

$$(\mathcal{A}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m, \boldsymbol{\tau}_h) + (\mathrm{div}(\boldsymbol{\tau}_h), \mathbf{u} - \mathbf{u}_h^m) + (\mathrm{skew}\,\boldsymbol{\tau}_h, \omega - \omega_h) = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \tag{22a}$$

$$(\mathrm{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m), \mathbf{w}_h) = 0 \qquad \forall \mathbf{w}_h \in \mathbf{W}_h \tag{22b}$$

$$(\mathrm{skew}\,(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m), \gamma_h) = 0 \qquad \forall \gamma_h \in X_h. \tag{22c}$$

we obtain the a priori estimates

$$\|\sigma - \sigma_h^m\|_0 + \|u - u_h\|_0 + \|\omega - \omega_h\|_0 \le Ch^{k+1}(\|\sigma\|_0 + \|u\|_0 + \|\omega\|_0).$$

According to [8] (see also [2] for on general domains), this element combination allows for positive constants C_1 and C_2 independent on h, such that

$$(\mathcal{A}\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \ge C_1 \|\boldsymbol{\tau}_h\|_{\boldsymbol{\Sigma}}^2 \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_N \text{ with } b_m(\boldsymbol{\tau}_h, (\mathbf{v}, \boldsymbol{\omega})) = 0 \ \forall (\mathbf{v}, \boldsymbol{\omega}) \in \mathbf{W}_h \times X_h.$$

and

$$\inf_{(\mathbf{v},\boldsymbol{\omega}) \in \mathbf{W}_h \times X_h} \sup_{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h} \frac{b_m(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\omega})}{|||(\mathbf{v}, \boldsymbol{\omega})||| \|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}} \ge C_2$$

4 Comparison of the approximations

The results of this paper are based on the crucial Galerkin properties of the Least-Squares methods, i.e. for $k = 1, 2$

$$B_k((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^k, \boldsymbol{\phi} - \boldsymbol{\phi}_h^k) | (\boldsymbol{\tau}, \boldsymbol{\psi})) = 0 \tag{23}$$

for all $(\boldsymbol{\tau}, \boldsymbol{\psi}) \in \Sigma_h \times \boldsymbol{\Phi}_h^k$. Since $\boldsymbol{\tau}$ needs to be $H(\text{div};\Omega)$-conforming, we can compare the conforming stress approximations, i.e. the stress approximations of the mixed and of the Least-Squares methods. We therefore define $\boldsymbol{\sigma}_h^{\Delta,1} = \boldsymbol{\sigma}_h^1 - \boldsymbol{\sigma}_h^m$, $\boldsymbol{\sigma}_h^{\Delta,2} = \boldsymbol{\sigma}_h^2 - \boldsymbol{\sigma}_h^m$, and $\boldsymbol{\sigma}_h^{\Delta,0} = \boldsymbol{\sigma}_h^1 - \boldsymbol{\sigma}_h^2$. For the displacement test function, we can insert any conforming displacement, i.e. the displacement approximations of the standard and of the Least-Squares method. We therefore define $\mathbf{u}_h^{\Delta,1} = \mathbf{u}_h^1 - \mathbf{u}_h$, $\mathbf{u}_h^{\Delta,2} = \mathbf{u}_h^2 - \mathbf{u}_h$ as well as $\mathbf{u}_h^{\Delta,0} = \mathbf{u}_h^2 - \mathbf{u}_h^1$. In order to deal with the three-fields formulation we also define $\boldsymbol{\phi}_h^{\Delta,1} = \boldsymbol{\phi}_h^1 - \boldsymbol{\phi}_h$, $\boldsymbol{\phi}_h^{\Delta,2} = \boldsymbol{\phi}_h^2 - \boldsymbol{\phi}_h$ as well as $\boldsymbol{\phi}_h^{\Delta,0} = \boldsymbol{\phi}_h^2 - \boldsymbol{\phi}_h^1$.

Choosing $\boldsymbol{\tau} = \boldsymbol{\sigma}_h^{\Delta,j}$ and $\boldsymbol{\psi} = \boldsymbol{\phi}_h^{\Delta,j}$ in (23) leads to

$$B_j((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^j, \boldsymbol{\phi} - \boldsymbol{\phi}_h^j) | (\boldsymbol{\tau}_h^{\Delta,j}, \boldsymbol{\phi}_h^{\Delta,j})) = 0 \tag{24}$$

for $j = 0, 1, 2$. This immediately leads to

$$\begin{aligned} B_j((\boldsymbol{\sigma}_h^{\Delta,j}, \boldsymbol{\phi}_h^{\Delta,j}) | (\boldsymbol{\sigma}_h^{\Delta,j}, \boldsymbol{\phi}_h^{\Delta,j})) &= B_j((\boldsymbol{\sigma}_h^j - \boldsymbol{\sigma}_h^m, \boldsymbol{\phi}_h^j - \boldsymbol{\phi}_h) | (\boldsymbol{\sigma}_h^{\Delta,j}, \boldsymbol{\phi}_h^{\Delta,j})) \\ &= B_j((\boldsymbol{\sigma}_h^j - \boldsymbol{\sigma} + \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m, \boldsymbol{\phi}_h^j - \boldsymbol{\phi} + \boldsymbol{\phi} - \boldsymbol{\phi}_h) | (\boldsymbol{\sigma}_h^{\Delta,j}, \boldsymbol{\phi}_h^{\Delta,j})) \\ &= B_j((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m, \boldsymbol{\phi} - \boldsymbol{\phi}_h) | (\boldsymbol{\sigma}_h^{\Delta,j}, \boldsymbol{\phi}_h^{\Delta,j})) \\ &= B_j((\boldsymbol{\sigma}^\Delta, \boldsymbol{\phi}^\Delta) | (\boldsymbol{\sigma}_h^{\Delta,j}, \boldsymbol{\phi}_h^{\Delta,j})), \end{aligned} \tag{25}$$

where we denote $\mathbf{u}^\Delta = \mathbf{u} - \mathbf{u}_h$, $\boldsymbol{\phi}^\Delta = \boldsymbol{\phi} - \boldsymbol{\phi}_h$ and $\boldsymbol{\sigma}^\Delta = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m$.

Considering the difference $\boldsymbol{\sigma}_h^\Delta \in \boldsymbol{\Sigma}_h$ of the least-squares problem and the mixed method we now state the following auxiliary problem: find $(\boldsymbol{\eta}, \boldsymbol{\xi}, \zeta) \in \boldsymbol{\Sigma}_N \times \mathbf{V} \times L^2(\Omega)$ such that

$$\mathcal{A}\boldsymbol{\eta} - \boldsymbol{\varepsilon}(\boldsymbol{\xi}) = 0 \qquad \text{in } \Omega \tag{26a}$$

$$\operatorname{div}\boldsymbol{\eta} = \operatorname{div}\boldsymbol{\sigma}_h^\Delta \qquad \text{in } \Omega \tag{26b}$$

$$\operatorname{skew}(\boldsymbol{\eta}) = \operatorname{skew}(\boldsymbol{\sigma}_h^\Delta) \qquad \text{in } \Omega. \tag{26c}$$

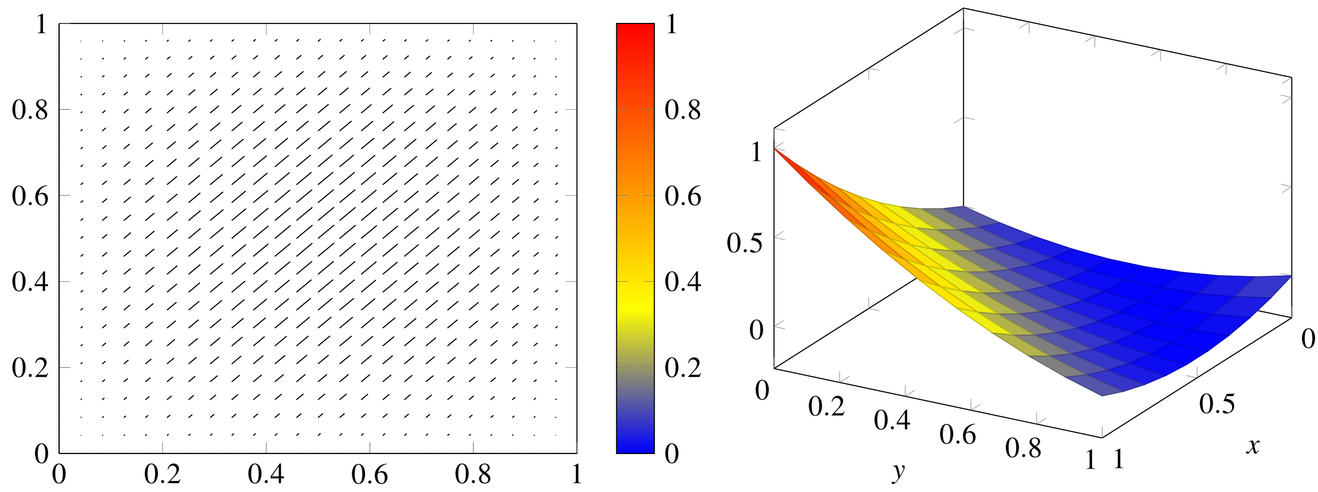

Figure 1: Exact solution for displacement **u** (left) and vorticity p (right)

The corresponding mixed formulation reads

$$(\mathcal{A}\boldsymbol{\eta},\boldsymbol{\tau}) + (\operatorname{div}(\boldsymbol{\tau}),\boldsymbol{\xi}) + (\operatorname{skew}\boldsymbol{\tau},\zeta) = 0 \qquad \forall\boldsymbol{\tau}\in\boldsymbol{\Sigma}_N \tag{27a}$$

$$(\operatorname{div}(\boldsymbol{\eta}),\mathbf{w}) = (\operatorname{div}(\boldsymbol{\sigma}_h^\Delta),\mathbf{w}) \qquad \forall\mathbf{w}\in\mathbf{W} \tag{27b}$$

$$(\operatorname{skew}(\boldsymbol{\eta}-\boldsymbol{\sigma}_h^\Delta),\gamma) = 0 \qquad \forall\gamma\in L^2(\Omega). \tag{27c}$$

The discretisation of this problem using the mixed method introduced in the previous section reads: find $(\boldsymbol{\eta}_h,\boldsymbol{\xi}_h,\zeta_h)\in\boldsymbol{\Sigma}_h\times\mathbf{W}_h\times X_h$ such that

$$(\mathcal{A}\boldsymbol{\eta}_h,\boldsymbol{\tau}_h) + (\operatorname{div}(\boldsymbol{\tau}_h),\boldsymbol{\xi}_h) + (\operatorname{skew}\boldsymbol{\tau}_h,\zeta_h) = 0 \qquad \forall\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h \tag{28a}$$

$$(\operatorname{div}(\boldsymbol{\eta}_h),\mathbf{w}_h) = (\operatorname{div}(\boldsymbol{\sigma}_h^\Delta),\mathbf{w}_h) \qquad \forall\mathbf{w}_h\in\mathbf{W}_h \tag{28b}$$

$$(\operatorname{skew}(\boldsymbol{\eta}_h-\boldsymbol{\sigma}_h^\Delta),\gamma_h) = 0 \qquad \forall\gamma_h\in X_h. \tag{28c}$$

The crucial relation $\operatorname{div}(\boldsymbol{\eta}_h-\boldsymbol{\sigma}_h^\Delta)=0$ together with the weakly symmetric condition implies

$$b_m(\boldsymbol{\sigma}_h^\Delta-\boldsymbol{\eta},(\mathbf{u}-\mathbf{u}_h^m,\omega-\omega_h)) = 0. \tag{29}$$

Inserting this in equation (22a) we obtain

$$\begin{aligned}(\mathcal{A}\boldsymbol{\sigma}^\Delta,\boldsymbol{\sigma}_h^m) &= -(\operatorname{div}(\boldsymbol{\sigma}_h^\Delta),\mathbf{u}-\mathbf{u}_h^m) - (\operatorname{skew}\boldsymbol{\sigma}_h^\Delta,\omega-\omega_h)\\ &= -b(\boldsymbol{\sigma}_h^\Delta,(\mathbf{u}-\mathbf{u}_h^m,\omega-\omega_h))\\ &= -b(\boldsymbol{\eta}_h,(\mathbf{u}-\mathbf{u}_h^m,\omega-\omega_h)) = (\mathcal{A}\boldsymbol{\sigma}^\Delta,\boldsymbol{\eta}_h)\end{aligned}$$

This leads to

$$(\mathcal{A}\boldsymbol{\sigma}^\Delta,\boldsymbol{\sigma}_h^m) = (\mathcal{A}\boldsymbol{\sigma}^\Delta,\boldsymbol{\eta}_h-\boldsymbol{\eta}) + (\mathcal{A}\boldsymbol{\sigma}^\Delta,\boldsymbol{\eta}) \tag{30}$$

as well as

$$(\boldsymbol{\sigma}^\Delta,\boldsymbol{\sigma}_h^m)_{LS_\Sigma} = (\mathcal{A}\boldsymbol{\sigma}^\Delta,\mathcal{A}\boldsymbol{\sigma}_h^m) = (\mathcal{A}\boldsymbol{\sigma}^\Delta,\mathcal{A}(\boldsymbol{\eta}_h-\boldsymbol{\eta})) + (\mathcal{A}\boldsymbol{\sigma}^\Delta,\mathcal{A}\boldsymbol{\eta}) \tag{31}$$

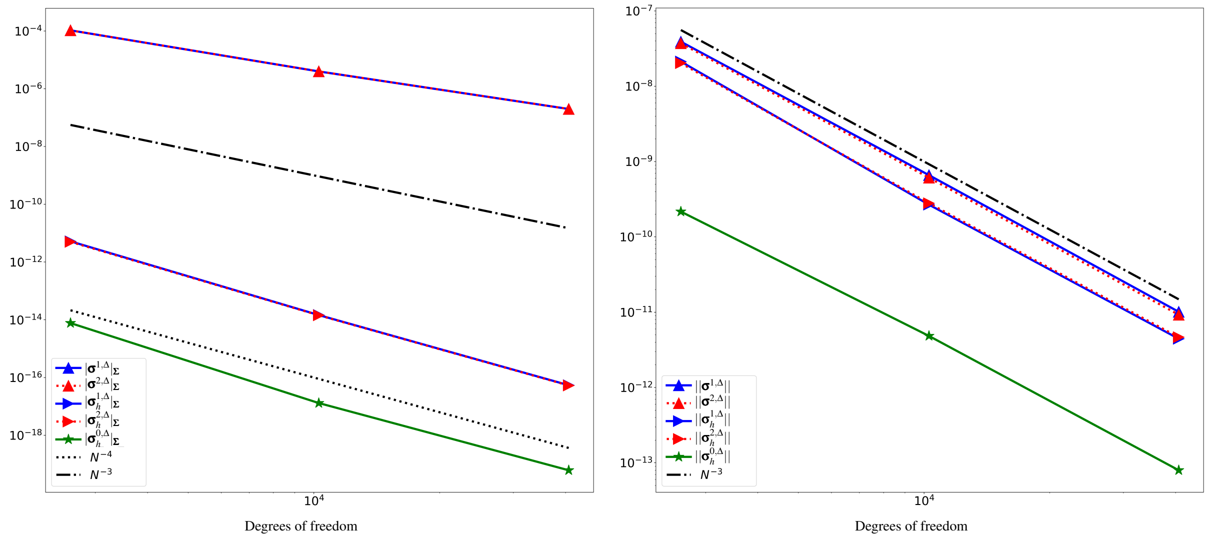

Figure 2: Difference of the approximations of the stress tensor the $H(\mathrm{div};\Omega)$-seminorm and in the $L^2(\Omega)$-norm

Moreover, the symmetry (6) together with (27a) implies

$$(\mathcal{A}\boldsymbol{\sigma}^{\Delta},\boldsymbol{\eta}) = (\mathcal{A}\boldsymbol{\eta},\boldsymbol{\sigma}^{\Delta}) = -(\mathrm{div}(\boldsymbol{\sigma}^{\Delta}),\boldsymbol{\xi}) - (\mathrm{skew}\,(\boldsymbol{\sigma}^{\Delta}),\zeta) = -b_m(\boldsymbol{\sigma}^{\Delta},(\boldsymbol{\xi},\zeta)) \tag{32}$$

Combining this with (31) leads to

$$(\mathcal{A}\boldsymbol{\sigma}^{\Delta},\boldsymbol{\sigma}_h^m) = (\mathcal{A}\boldsymbol{\sigma}^{\Delta},\boldsymbol{\eta}_h-\boldsymbol{\eta}) - b_m(\boldsymbol{\sigma}^{\Delta},(\boldsymbol{\xi},\zeta)) \tag{33}$$

Using (22b) and (22c) we have for any $(\mathbf{w}_h,\gamma_h)\in\mathbf{W}_h\times X_h$

$$(\mathcal{A}\boldsymbol{\sigma}^{\Delta},\boldsymbol{\sigma}_h^m) = (\mathcal{A}\boldsymbol{\sigma}^{\Delta},\boldsymbol{\eta}_h-\boldsymbol{\eta}) - b_m(\boldsymbol{\sigma}^{\Delta},(\boldsymbol{\xi}-\mathbf{w}_h,\zeta-\gamma_h)). \tag{34}$$

On the other hand, (29) and integrating by parts allow

$$\begin{aligned} b_m(\boldsymbol{\sigma}_h^{\Delta},(\mathbf{u}-\mathbf{u}_h^s,\frac{1}{2}\,\mathbf{curl}(\mathbf{u}-\mathbf{u}_h^s))) &= b_m(\boldsymbol{\eta},(\mathbf{u}-\mathbf{u}_h^s,\frac{1}{2}\,\mathbf{curl}(\mathbf{u}-\mathbf{u}_h^s))) \\ &= (\mathrm{div}(\boldsymbol{\eta}),(\mathbf{u}-\mathbf{u}_h^s)) + (\mathrm{skew}\,\boldsymbol{\eta},\frac{1}{2}\,\mathbf{curl}(\mathbf{u}-\mathbf{u}_h^s))\boldsymbol{\chi}) \\ &= (\boldsymbol{\eta},\boldsymbol{\nabla}(\mathbf{u}-\mathbf{u}_h^s)) + (\mathrm{skew}\,\boldsymbol{\eta},\frac{1}{2}\,\mathbf{curl}(\mathbf{u}-\mathbf{u}_h^s))\boldsymbol{\chi}) \\ &= (\boldsymbol{\eta},\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h^s)) = (\mathcal{A}^{-1}\boldsymbol{\varepsilon}(\boldsymbol{\xi}),\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h^s)) \\ &= (\boldsymbol{\varepsilon}(\boldsymbol{\xi}),\mathcal{A}^{-1}\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h^s)) \end{aligned}$$

The Galerkin orthogonality (17) now implies

$$b_m(\boldsymbol{\sigma}_h^{\Delta},(\mathbf{u}-\mathbf{u}_h^s,\frac{1}{2}\,\mathbf{curl}(\mathbf{u}-\mathbf{u}_h^s))) = (\boldsymbol{\varepsilon}(\boldsymbol{\xi}-\mathbf{v}_h),\mathcal{A}^{-1}\boldsymbol{\varepsilon}(\mathbf{u}-\mathbf{u}_h^s)) \tag{35}$$

for any $\mathbf{v}_h\in\mathbf{V}_h$. Both results (34) and (35) leads to the following supercloseness theorem.

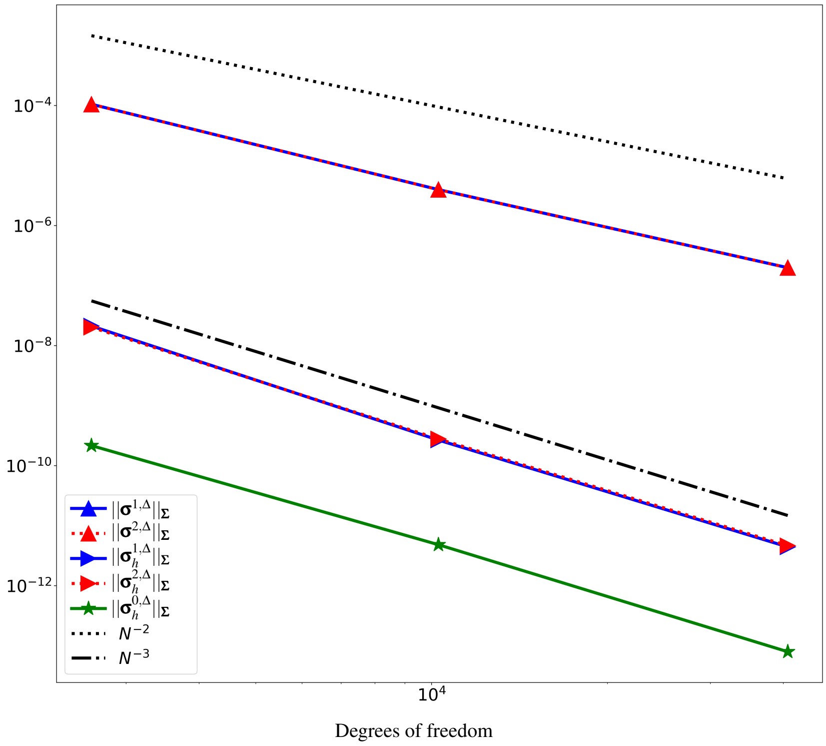

Figure 3: Difference of the approximations of the stress approximations

Theorem 1. *Let* $\mathbf{u} \in \mathbf{V}$ *and* $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_N$ *be the exact solution of the linear elasticity problem* (1)*. Consider the discrete solutions* $\mathbf{u}_h^s$ *of* (S_h)*,* $(\boldsymbol{\sigma}_h^m, \mathbf{u}_h^m, \omega_h) \in \boldsymbol{\Sigma}_h \times \mathbf{W}_h \times X_h$ *of* (M_h) *and* $(\boldsymbol{\sigma}_h, \boldsymbol{\phi}_h)$ *of* (LS_h)*. Define* $\mathbf{u}^\Delta = \mathbf{u} - \mathbf{u}_h$*,* $\boldsymbol{\phi}^\Delta = \boldsymbol{\phi} - \boldsymbol{\phi}_h$ *and* $\boldsymbol{\sigma}^\Delta = \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^m$*. Moreover, let* $(\boldsymbol{\eta}, \boldsymbol{\xi}, \zeta) \in \boldsymbol{\Sigma} \times \mathbf{W} \times X$ *and* $(\boldsymbol{\eta}_h, \boldsymbol{\xi}_h, \zeta_h) \in \boldsymbol{\Sigma}_h \times \mathbf{W}_h \times X_h$ *be the solution of the auxiliary problem defined as in* (27) *and* (28)*. Then, it holds*

$$||(\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}_h^\Delta)||_{\boldsymbol{\Sigma} \times \boldsymbol{\Phi}} \lesssim \|\boldsymbol{\sigma}^\Delta\|_{\mathcal{A}} ||(\mathcal{A}(\boldsymbol{\eta}_h - \boldsymbol{\eta}), \boldsymbol{\xi} - \mathbf{w}_h, \zeta - \gamma_h)||_{\boldsymbol{\Sigma}_h \mathbf{W} \times X} + \|\boldsymbol{\varepsilon}(\boldsymbol{\xi} - \mathbf{v}_h)\| \|\boldsymbol{\varepsilon}(\mathbf{u}^\Delta)\| \,. \tag{36}$$

The coercivity of the Least-Squares bilinearform together with (25) implies

$$\begin{aligned} ||(\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}_h^\Delta)||_{\boldsymbol{\Sigma} \times \boldsymbol{\Phi}} &\lesssim \mathcal{B}((\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}_h^\Delta), (\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}_h^\Delta)) = B((\boldsymbol{\sigma}^\Delta, \boldsymbol{\phi}^\Delta) | (\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}_h^\Delta)) \\ &= ((\boldsymbol{\sigma}^\Delta, \boldsymbol{\phi}^\Delta), (\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}_h^\Delta))_{LS} - b_{LS}(\boldsymbol{\sigma}^\Delta, \boldsymbol{\phi}_h^\Delta) - b_{LS}(\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}^\Delta) \\ &= (\boldsymbol{\sigma}^\Delta, \boldsymbol{\sigma}_h^\Delta)_{LS_{\boldsymbol{\Sigma}}} + (\boldsymbol{\phi}^\Delta, \boldsymbol{\phi}_h^\Delta)_{LS_{\mathbf{V}}} - b_{LS}(\boldsymbol{\sigma}^\Delta, \boldsymbol{\phi}_h^\Delta) - b_{LS}(\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}^\Delta) \end{aligned}$$

The first term can be replaced by (33) for arbitrary function $(\mathbf{w}_h, \gamma_h) \in \mathbf{W}_h \times X_h$ while the Galerkin orthogonality (17) allows the second term and the fourth term to vanish. For the third term, simple computations (in both Least-Squares cases) show that

$$b_{LS}(\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}_h^\Delta) = (\boldsymbol{\varepsilon}(\boldsymbol{\xi} - \mathbf{v}_h), \mathcal{A}^{-1} \boldsymbol{\varepsilon}(\mathbf{u} - \mathbf{u}_h^s)) \quad \forall \mathbf{v}_h \in \mathbf{V}_h \tag{37}$$

follows from (35). Altogether we obtain

$$\begin{aligned} ||(\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}_h^\Delta)||_{\boldsymbol{\Sigma} \times \boldsymbol{\Phi}} &\lesssim (\mathcal{A}\boldsymbol{\sigma}^\Delta, \mathcal{A}(\boldsymbol{\eta}_h - \boldsymbol{\eta})) - b_m(\boldsymbol{\sigma}^\Delta, (\boldsymbol{\xi} - \mathbf{w}_h, \zeta - \gamma_h)) - (\boldsymbol{\varepsilon}(\boldsymbol{\xi} - \mathbf{v}_h), \mathcal{A}^{-1} \boldsymbol{\varepsilon}(\mathbf{u}^\Delta)) \\ &\lesssim ||\boldsymbol{\sigma}^\Delta||_{\mathcal{A}} ||(\mathcal{A}(\boldsymbol{\eta}_h - \boldsymbol{\eta}), \boldsymbol{\xi} - \mathbf{w}_h, \zeta - \gamma_h)||_{\boldsymbol{\Sigma}_h \mathbf{W} \times X} + \|\boldsymbol{\varepsilon}(\boldsymbol{\xi} - \mathbf{v}_h)\| \|\mathcal{A}^{-1} \boldsymbol{\varepsilon}(\mathbf{u}^\Delta)\| \,. \end{aligned}$$

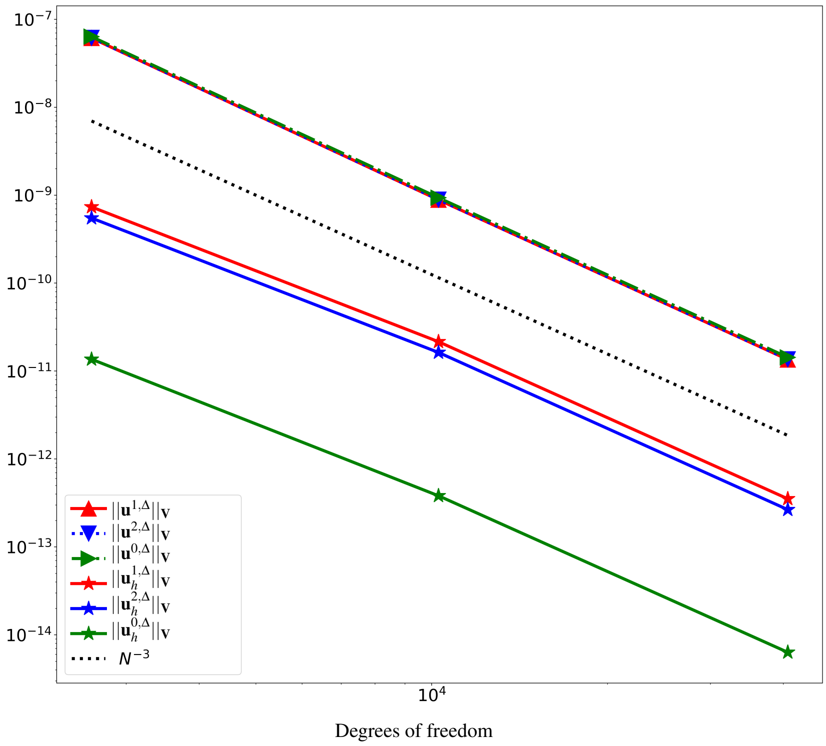

Figure 4: Difference of the approximations of the displacements

This immediately leads to refined a priori bounds for the Least-Squares method. For this, we now assume that the problem is H^2 regular. For all $f \in L^2(\Omega)$, the solution u of the elasticity problem fulfills

$$\|u\|_2 \lesssim \|f\|,$$

and it follows

$$\|\boldsymbol{\eta}\|_1 \leq C\|\boldsymbol{\xi}\|_2 \leq C\|\operatorname{div}\boldsymbol{\sigma}_h\| \leq C\|\boldsymbol{\sigma}_h\|_{H(\mathrm{div};\Omega)}. \tag{38}$$

We choose $\mathbf{v}_h$ as the orthogonal interpolation of $\boldsymbol{\xi}$ in $\mathbf{V}_h$ such that $\|\boldsymbol{\xi} - \mathbf{v}_h\| \lesssim h^k\|\xi\|_1$ holds. Similarly, $\mathbf{w}_h$ and γ_h are the L^2-orthogonal projections of $\boldsymbol{\xi}$ and ζ on $\mathbf{W}_h$ and X_h such that

$$||(\boldsymbol{\xi} - \mathbf{w}_h, \zeta - \gamma_h)||_{\boldsymbol{\Sigma}\times\mathbf{W}\times X} \lesssim h^k(||\boldsymbol{\xi}_h|| + ||\gamma_h||)\ .$$

This leads to

$$||(\boldsymbol{\sigma}_h^\Delta, \boldsymbol{\phi}_h^\Delta)||_{\boldsymbol{\Sigma}\times\boldsymbol{\Phi}} \lesssim h^k\left(\|\boldsymbol{\sigma}^\Delta\|_{\mathcal{A}} + \|\boldsymbol{\varepsilon}(\mathbf{u}^\Delta)\|\right)\ . \tag{39}$$

By the triangle inequality we obtain similarly to [9] the refined estimate

$$||\boldsymbol{\phi} - \boldsymbol{\phi}_h^\Delta||_{\boldsymbol{\Phi}} \lesssim ||\mathbf{u} - \mathbf{u}_h^s|| + h^k||(\boldsymbol{\sigma}^\Delta, \boldsymbol{\phi}^\Delta)||_{\boldsymbol{\Sigma}\times\boldsymbol{\Phi}}\ . \tag{40}$$

Moreover, if $\mathbf{f}$ is a piece-wise constant the mixed finite element method (M_h) has exact local mass conservation we obtain

$$||\operatorname{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||_0 = ||\operatorname{div}(\boldsymbol{\sigma}^m - \boldsymbol{\sigma}_h)||_0 \lesssim h^k||(\boldsymbol{\sigma}^\Delta, \boldsymbol{\phi}^\Delta)||_{\boldsymbol{\Sigma}\times\boldsymbol{\Phi}}\ , \tag{41}$$

i.e. the mass conservation of the Least-Squares method is of higher-order.

5 Numerical results

Our numerical results confirm the theoretical investigations of the previous sections. A simple polygonal design of an exact displacement with homogeneous boundary conditions on $\partial\Omega$ implies

$$\mathbf{u}(x,y) = \begin{pmatrix} xy(1-x)(1-y) \\ xy(1-x)(1-y) \end{pmatrix} \tag{42}$$

and thus

$$\varepsilon(\mathbf{u}) = \begin{pmatrix} y(y-1)(2x-1) & \frac{1}{2}(y+x-1)(2xy-x-y) \\ \frac{1}{2}(y+x-1)(2xy-x-y) & x(x-1)(2y-1) \end{pmatrix}, \tag{43}$$

$$\operatorname{div}(\mathbf{u}) = (y+x-1)(2xy-x-y) \text{ and } p(x,y) = \frac{1}{2}(2x-x-y+1)(x-y)\ . \tag{44}$$

This leads to

$$\boldsymbol{\sigma}(x,y) = \begin{pmatrix} 2\mu y(y-1)(2x-1) & 0 \\ 0 & 2\mu x(x-1)(2y-1) \end{pmatrix} + (y+x-1)\begin{pmatrix} \lambda(2xy-x-y) & \mu(2xy-x-y) \\ \mu(2xy-x-y) & \lambda(2xy-x-y) \end{pmatrix}$$

and

$$\mathbf{f} = \begin{pmatrix} \mu(2x^2+4xy+4y^2-4x-6y+1) + \lambda(4xy+2y^2-2x-4y+1) \\ \mu(4x^2+4xy+2y^2-6x-4y+1) + \lambda(2x^2+4xy-4x-2y+1) \end{pmatrix} \tag{45}$$

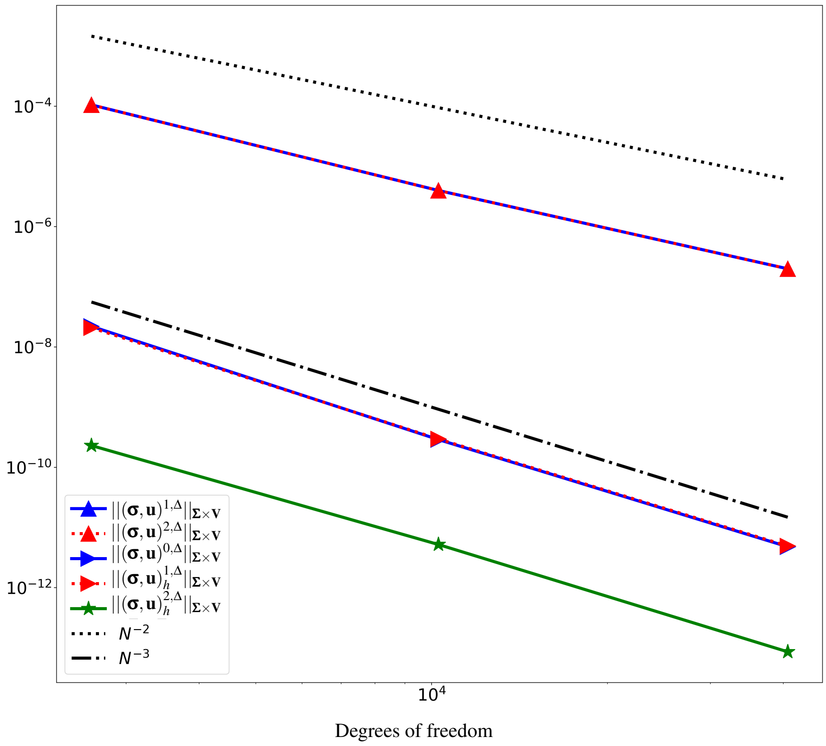

Figure 5: Difference of the approximations

6 Conclusions

For the linear elasticity problems, we compared the approximations obtained by the Least-Squares finite element method with the approximations obtained by the standard conforming finite element method and the mixed finite element method and prove that the H^1-conforming displacement approximations (least-squares finite element and standard finite element) as well as the $H(div)$-conforming stress approximations are higher-order perturbations of each other. Future work will consider domain with curved boundaries in the spirit of [5, 4, 6, 1].

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