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# CHAPTER 4

## A FINITE ELEMENT FORMULATION FOR THE GEOMETRICALLY NON LINEAR ANALYSIS OF SHELLS

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SUMMARY

A finite element formulation for the analysis of geometrically non linear shell problems is presented. Degenerated 3D elasticity elements are used for the finite element discretization of the shell. Large rotation effects are taken into account in the non linear large displacement kinematic description. The shell formulation is derived in detail using a Total Lagrangian approach and explicit forms of the relevant finite element matrices are given. In the last part of the chapter the obtention of the alternative Updated Lagrangian formulation is presented.

### 1. INTRODUCTION

The geometrically non linear analysis of shells involving large displacements and rotations is of considerable interest in many areas of engineering. Typical examples of such problems are the computation of buckling and failure loads in concrete and steel shell structures and the study of instability and post-buckling response of thin steel shell-like components in mechanical, aeronautical and ship engineering, amongst others. Of special significance by its complexity, are the problems related with sheet metal deformation analysis [1] of big interest in metal and automobile industries. All these problems have increased the need for reliable non linear shell formulations which could handle such potentially difficult cases in a simple and efficient manner. This also explains the extensive work reported in recent years on the obtention of different finite element formulations for the non linear analysis of shells [2] - [16].

In this chapter a geometrically non linear finite element shell formulation is presented. The formulation uses degenerated 3D elasticity elements [18] and it allows for large displacement-large rotation effects. The incremental formulation is derived using a Total Lagrangian description. However, details

of the obtention of the alternative Updated Lagrangian formulation are also given.

The layout of the chapter is the following. In Section 2 some introductory concepts are presented. Section 3 and 4 deal respectively with the geometric and the non linear Kinematic description including the treatment of the large rotations. In Sections 5 and 6 the strain and constitutive matrices for the Total Lagrangian approach are respectively treated. Section 7 deals with the obtention of tangent stiffness matrix. The Updated Lagrangian formulation is discussed in Section 8. Finally, in the section 9 some remarks about the practical use of the formulation are given.

## 2. BASIC CONCEPTS AND NOTATION

The shell theory used is based on a degeneration of 3-D elasticity using the following three assumptions

a) Normals to the shell middle surface before deformation remain straight, but not necessarily normal to the middle surface after deformation.
b) The normal stress, in the thickness direction, is zero.
c) The shell thickness remains constant during the deformation.

Assumption a) is equivalent to taking into account the effect of transverse shear deformation and it has been widely used in recent years both in the context of linear and non-linear analysis of shells [4]-[16]. On the other hand, assumption b) is the typical "plane stress" condition, traditionally used in plate and shell theories. Finally, assumption c) implies that the lenght of the normal vector at each point does not change in the different deformed configurations of the shell. This last assumption is essential for an adequate kinematic description of the large rotation of the normals.

For an effective numerical treatment of non linear problems it is necessary to employ the so-called incremental formulation. We consider the motion (deformation) of the shell referred to global, fixed cartesian axes, with unit vectors $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ (see Fig. 1). The objective of the analysis is to evaluate the equilibrium positions of the shell, at the discrete time points (or load levels) $\Delta t$, $2\Delta t$, ..., t, $t+\Delta t$. It will be assumed that the solution for the kinematic and static variables for all time steps from time zero (initial configuration) to time t, inclusive, have been obtained, and that the solution for time $t+\Delta t$ is required next. The Total Lagrangian (TL) description uses the initial, undeformed shell geometry as "reference configuration" for all variables. This description will be used for deriving the incremental formulation in the next sections. On the other hand, the Updated Lagrangian (UL) description uses as reference configuration the last known configuration at time t. The formulation for the UL case will be treated in Section 8.

Geometrical variables in the initial configuration and in the configuration at time t will be denoted by capital and lower case letters, respectively. Lower case subscripts refer to the coordinate axes, whereas superscripts refer to a node of the finite element mesh. (i.e. $x_i^k$ and $x_i^k$ denote the coordinate along the ith global axe of node k in the initial configuration and the configuration at time t, respectively).



## 3. GEOMETRIC DESCRIPTION

In Fig. 1 the geometry of a standard "degenerated" thick shell element in the initial configuration is shown. At each point of the middle surface of the shell in the initial configuration a set of orthogonal local vectors $\vec{V}_r$, $\vec{V}_s$ and $\vec{V}_t$ is defined (the corresponding rotated vectors in the configuration at time t are denoted by $\vec{v}_r$, $\vec{v}_s$ and $\vec{v}_t$, respectively). $\vec{V}_t$ is defined as a unit vector normal to the shell middle surface. On the other hand, vectors $\vec{V}_r$ and $\vec{V}_s$ are contained in the tangent plane to the shell middle surface and can be defined in several ways [18]. Here we have taken $\vec{V}_r$ to be parallel to the global plane $X_1$-$X_3$ and also tangent to the shell middle surface (i.e. $\vec{V}_r = \vec{e}_2 \times \vec{V}_t$). Finally, vector $\vec{V}_s$ is computed from the cross product of vectors $\vec{V}_t$ and $\vec{V}_r$. Note that this definition implies that $\vec{V}_r$ and $\vec{V}_s$ are not unit vectors and that their modulus is

$$|\vec{V}_r| = |\vec{V}_s| = [1 - (V_{t_2})^2]^{1/2} = e \qquad (1)$$

where $\vec{V}_{t2}$ is the component of $\vec{V}_t$ along the $\vec{e}_2$ axe. (For the special case $\vec{V}_t$ is parallel to $\vec{e}_2$ it may be simply taken $\vec{V}_r = \vec{e}_3$ and e=1 in eq.(1)).

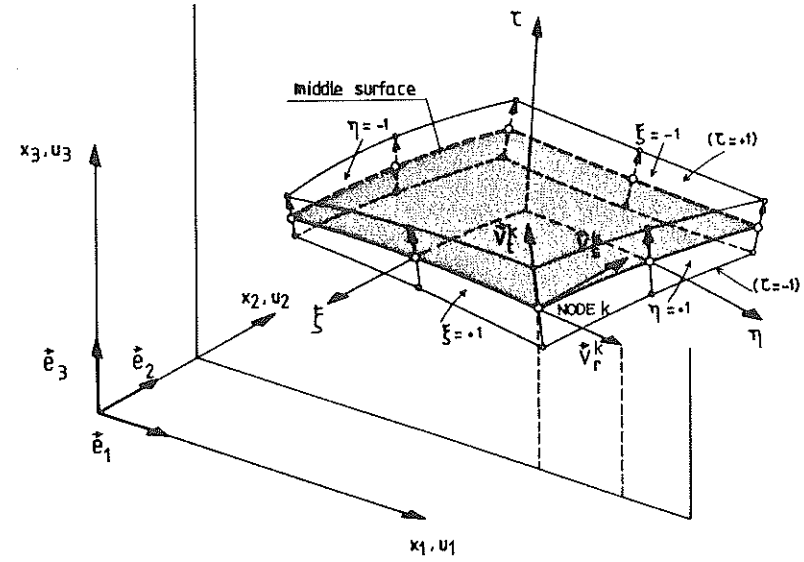


Fig. 1. Geometric description of a thick shell element.

It will be assumed that the geometry of the shell element is defined by the coordinates and thickness of each node. Thus, the position vector of a point in the initial configuration can be interpolated in an isoparametric form [20] as

$$\underset{\sim}{X} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \sum_{k=1}^{n} \underset{\sim}{N}^k (\underset{\sim}{X}^k + \tau \frac{h^k}{2} \underset{\sim}{V}_t^k) \tag{2a}$$

with

$$\underset{\sim}{X}^k = [X_1^k,\ X_2^k,\ X_3^k]^T$$

$$\underset{\sim}{N}^k = \begin{bmatrix} N^k & 0 & 0 \\ 0 & N^k & 0 \\ 0 & 0 & N^k \end{bmatrix} \quad : \quad \underset{\sim}{V}_t^k = [V_{t_1}^k, V_{t_2}^k, V_{t_3}^k]^T \tag{2b}$$

where $h^k$, $X_i^k$ and $N^k(\xi,\eta)$ are the nodal thickness, ith global nodal coordinate and shape function of node k, respectively, $V_{t_i}^k$ is the component of the normal vector, $\underset{\sim}{V}_t^k$ along the ith global axe at node K, $\tau$ the normalized coordinate across the thickness ($\tau=\pm1$ for the upper and lower shell surfaces, respectively) and n the number of nodes per element. It is worth noting that $\underset{\sim}{V}_t^k$ can be obtained as the effective normal vector to the isoparametric shell surface at each node or alternatively, it could be a given thickness direction, not necessarily orthogonal to the shell middle surface, specified for each node. This latter definition is useful for avoiding geometry discontinuities in folded shell situations. In this work $\underset{\sim}{V}_t^k$ will be indistinctly referred to as the "normal vector".

## 4. KINEMATIC DESCRIPTION

The displacement vector is defined as

$$\underset{\sim}{u} = [u_1,\ u_2,\ u_3]^T = \underset{\sim}{x} - \underset{\sim}{X} \tag{3}$$

where $\underset{\sim}{X}$ and $\underset{\sim}{x}$ are the position vectors of a point in the initial configuration and the configuration at time t, respectively. From assumption a) of Section 2 and eq.(2a) it can be obtained

$$\underset{\sim}{u} = \sum_{k=1}^{n} \underset{\sim}{N}^k [\underset{\sim}{u}^k + \tau \frac{h^k}{2} (\underset{\sim}{V}_t^k - \underset{\sim}{V}_t^k)] \tag{4}$$

where

$$u^k = [u_1^k,\ u_2^k,\ u_3^k]^T \tag{5}$$

and $\underset{\sim}{V}_t^k$ and $\underset{\sim}{v}_t^k$ contain the global components of the normal vector in the initial configuration and those of its corresponding rotated vector in the configuration at time t, respectively, (see Fig. 2).

From assumption c) of Section 3 it can be deduced that



$$\vec{v}_t^k = \frac{\text{sen } \alpha^k}{\alpha^k e^k} \theta_s^k \vec{V}_r^k - \frac{\text{sen } \alpha^k}{\alpha^k e^k} \theta_s^k \vec{V}_s^k + \cos \alpha^k \vec{V}_t^k \tag{6}$$

where $\alpha^k$ is the angle rotated by the normal at node k and $e^k$ is obtained from eq.(1).

We can defined now vector $\hat{\underset{\sim}{v}}_t^k$ as

$$\hat{\underset{\sim}{v}}_t^k = \underset{\sim}{v}_t^k - \underset{\sim}{V}_t^k \tag{7}$$

where all components are referred to the local system $\frac{1}{e^k}\vec{V}_r^k$, $\frac{1}{e^k}\vec{V}_s^k$, $\vec{V}_t^k$. From eqs.(6) and (7)

$$\hat{\underset{\sim}{v}}_t^k = [\frac{\text{sen } \alpha^k}{\alpha^k} \theta_s^k,\ -\frac{\text{sen } \alpha^k}{\alpha^k} \theta_r^k,\ \cos \alpha^k - 1]^T \tag{8}$$

In eqs.(6) and (8) $\theta_r^k$ and $\theta_s^k$ are the rotations of the normal vector about vectors $\vec{V}_r^k$ and $\vec{V}_s^k$, respectively, from the initial configuration to the configuration at time t. For sign convention see Fig. 3b.

From eq. (7) and (4) it can be obtained

$$\underset{\sim}{u} = \sum_{k=1}^{n} \underset{\sim}{N}^k [\underset{\sim}{u}^k + \tau \frac{h^k}{2} \underset{\sim}{T}^k \hat{\underset{\sim}{v}}_t^k] \tag{9}$$

where

$$\underset{\sim}{T}^k = [\frac{1}{e^k} \underset{\sim}{V}_r^k,\ \frac{1}{e^k} \underset{\sim}{V}_s^k,\ \underset{\sim}{V}_t^k] \tag{10}$$

The vector of displacement increments from the configuration t to the t+$\Delta$t is defined as

$$\Delta\underset{\sim}{u} = [\Delta u_1, \Delta u_2, \Delta u_3]^T = (\underset{\sim}{x}+\Delta\underset{\sim}{x}) - \underset{\sim}{x} = \Delta\underset{\sim}{x} \tag{11}$$

and from eq. (4)

$$\Delta\underset{\sim}{u} = \sum_{k=1}^{n} \underset{\sim}{N}^k (\Delta\underset{\sim}{u}^k + \tau \frac{h^k}{2} \Delta\underset{\sim}{v}_t^k) \tag{12}$$

Assuming small increments, it can be obtained using eq. (6)

$$\Delta\underset{\sim}{u} = \delta\underset{\sim}{u}$$
$$\Delta\underset{\sim}{v}_t^k = \delta\underset{\sim}{v}_t^k = \underset{\sim}{T}^k \bar{\underset{\sim}{C}}_t^k \delta\underset{\sim}{\theta}^k \tag{13}$$

where $\underset{\sim}{T}^k$ is given by eq.(10) and

$$\bar{\underset{\sim}{C}}_t^k = \begin{bmatrix} -A^k \theta_r^k \theta_s^k & -A^k (\theta_s^k)^2 + B^k \\ A^k (\theta_r^k)^2 - B^k & A^k \theta_r^k \theta_s^k \\ -B^k \theta_r^k & -B^k \theta_s^k \end{bmatrix} \tag{14}$$

with

$$A^k = \frac{\text{sen } \alpha^k - \alpha^k \cos \alpha^k}{(\alpha^k)^3}$$

$$B^k = \frac{\text{sen } \alpha^k}{\alpha^k} \tag{15}$$

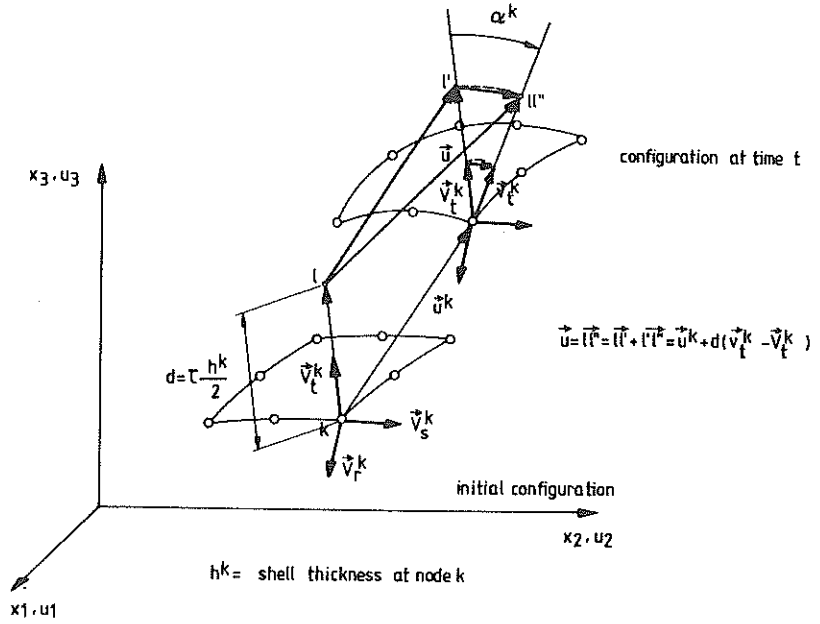


Fig. 2. Kinematic description.

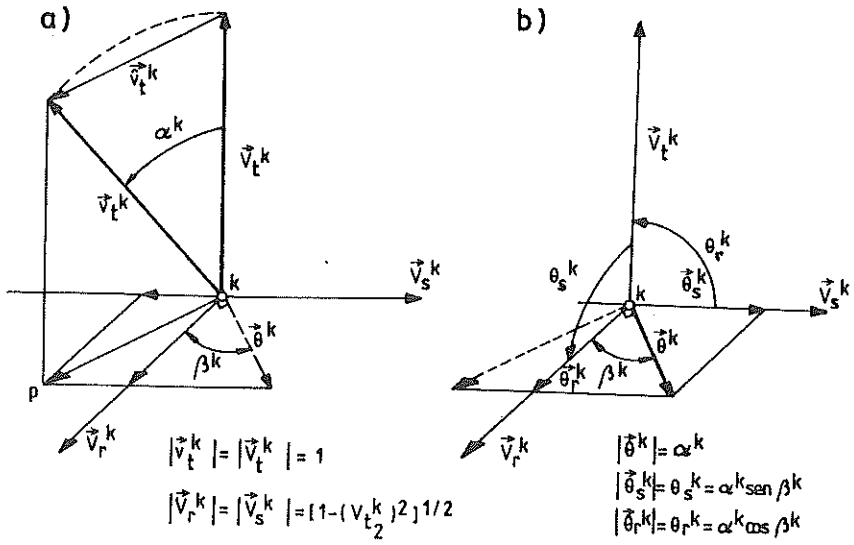


Fig. 3. a) Rotation of the normal.
b) Sign convenion for nodal rotations.



In eq.(13) $\delta\bar{\theta}^k$ is the vector of small local rotation increments of node k given by

$$\delta\underset{\sim}{\bar{\theta}}^k = [\delta\theta_r^k, \delta\theta_s^k]^T \tag{16}$$

Therefore, substituting eq.(13) in (12) yields

$$\delta\underset{\sim}{u} = \sum_{k=1}^{n} \underset{\sim}{N}^k (\delta\underset{\sim}{u}^k + \frac{\tau}{2} h^k \underset{\sim}{T}^k \underset{\sim}{\bar{C}}_t^k \delta\underset{\sim}{\bar{\theta}}^k) \tag{17}$$

For kinked, folded or stiffened shell situation it may be more convenient to work in terms of global rotation increments. This can be easily done, using the transformation

$$\delta\underset{\sim}{\bar{\theta}}^k = \underset{\sim}{\hat{T}}^k \delta\underset{\sim}{\theta}^k \tag{18}$$

with

$$[\underset{\sim}{\hat{T}}^k]^T = \frac{1}{e^k} [\underset{\sim}{V}_r, \underset{\sim}{V}_s] \tag{19}$$

and

$$\delta\underset{\sim}{\theta}^k = [\delta\theta_1^k, \delta\theta_2^k, \delta\theta_3^k]^T \tag{20}$$

where $\delta\theta_i^k$ represents the component along the ith global axe of the rotation increment vector. However, matrix $\underset{\sim}{\bar{C}}_t^k$ remains a function of the total local rotations $\theta_r^k$ and $\theta_s^k$. Thus, if the vector of nodal rotation increments is defined by eq.(20) a transformation is needed for updating the rotations in local axes.

Eq.(12) can be rewritten in general form as

$$\delta\underset{\sim}{u} = \sum_{k=1}^{n} N^k [\underset{\sim}{I}_3, \frac{\tau}{2} h^k \underset{\sim}{C}_t^k] \delta\underset{\sim}{a}^k \tag{21}$$

where $\underset{\sim}{I}_3$ is the 3 x 3 unit matrix and

$$\underset{\sim}{C}_t^k = \underset{\sim}{T}^k \underset{\sim}{\bar{C}}_t^k \quad , \qquad \delta\underset{\sim}{a}^k = \begin{Bmatrix} \delta\underset{\sim}{u}^k \\ \delta\underset{\sim}{\bar{\theta}}^k \end{Bmatrix} \tag{22}$$

for smooth shells problems, for which the local rotation increments can be used as nodal variables, or

$$\underset{\sim}{C}_t^k = \underset{\sim}{T}^k \underset{\sim}{\bar{C}}_t^k \underset{\sim}{\hat{T}}^k \quad ; \qquad \delta\underset{\sim}{a}^k = \begin{Bmatrix} \delta\underset{\sim}{u}^k \\ \delta\underset{\sim}{\theta}^k \end{Bmatrix} \tag{23}$$

for kinked shells, or similar type of problems, where the assembly of the rotation increments has to be performed in the global coordinate system [20].

It is worth pointing out that if eq.(22) is used for the definition of $\underset{\sim}{C}_t^k$, the number of nodal variables is equal to five (three global displacements increments and two local rotation increments). On the other hand, the use of eq.(23) implies that the number of degrees of freedom per node increases to six, since the three global rotation increments are now needed. This has to be properly taken into account in the practical coding

of the formulation.

## 5. STRAIN - DISPLACEMENT RELATIONSHIPS

The assumption of zero normal stress (see Section 2) implies that the constitutive equations relating stresses and strains must be established in the local coordinate system. As a consequence, several alternatives for obtaining the tangent stiffness matrix in terms of the global or local strain matrices arise. The three more obvious options are:

- Option 1: The local strain matrix (relating local strains and nodal displacements increments) is directly obtained from the local displacement gradients. The stiffness matrix is then computed using the constitutive equation expressed in local axes.
- Option 2: The local strain matrix is obtained via tensor transformations of the global strain matrix. The stiffness matrix is computed as in option 1.
- Option 3: The stiffness matrix is computed using the global strain matrix and the global constitutive matrix. (The latter is obtained by transformation of its expression in local axes).

Option 1 has been used by the authors who have derived the explicit form of the derivatives of the local displacements with respect to the local axes, from which the local strain matrix can be directly obtained [14]. However, the simplicity of this alternative is very much dependent on the definition of the local axes. It can be shown, that relatively simple expressions for the local displacement derivatives and the local strain matrix can be obtained if the local axes $\vec{V}_r$ and $\vec{V}_s$ are defined to be tangent to the shell principal curvature directions. Details about this procedures can be found in references [13] - [16].

Both options 2 and 3 involve the computation of the global strain matrix. The main difference between these options lays in the fact that in option 2 the strain matrix is transformed from local to global axes, whereas in option 3 such a transformation is performed on the constitutive equation.

In the next sections details of the obtention of the global and local strain matrices and of the computation of the tangent stiffness matrix using options 2 and 3 are given.

### 5.1. Computation of the global strain matrix.

The Green-Lagrange strain tensor associated to the Total Lagrangian description is defined in global axes as [19]

$$\varepsilon_{ij} = \frac{1}{2}\,(u_{i,j} + u_{j,i} + u_{k,i}\,u_{k,j}) \tag{24}$$

where

$$u_{i,j} = \frac{\partial u_i}{\partial X_j} \tag{25}$$

are the derivatives of the global displacements $u_i$ with respect to the global coordinate $X_j$ in the initial configuration.



From eq.(24) it can be easily deduced that

$$\delta\varepsilon_{ij} = \frac{1}{2}\,(\delta u_{i,j} + \delta u_{j,i} + u_{k,i}\,\delta u_{k,j} + \delta u_{k,i}\,u_{k,j}) \tag{26}$$

with

$$\delta u_{i,j} = \frac{\partial u_i}{\partial X_j} = \frac{\partial(\delta u_i)}{\partial X_j} \tag{27}$$

The derivatives $u_{i,j}$ and $\delta u_{i,j}$ can be obtained using eqs.(9) and (21). After some transformations it can be written

$$\delta \underset{\sim}{u}_{,j} = \sum_{k=1}^{n} [\underset{\sim}{N}^k_{,j}\ ,\ \underset{\sim}{G}^k_j]\ \delta \underset{\sim}{a}^k \tag{28}$$

$$\underset{\sim}{u}_{,j} = \sum_{k=1}^{n} [\underset{\sim}{N}^k_{,j}\ \underset{\sim}{u}^k + \frac{h^k}{2}\,(\tau\ \underset{\sim}{N}^k_{,j} + N^k\ \underset{\sim}{J}_3)\underset{\sim}{T}^k\ \underset{\sim}{v}^k_t] \tag{29}$$

where

$$u_{,j} = \frac{\partial}{\partial X_j}\,\underset{\sim}{u} \quad , \quad \delta \underset{\sim}{u}_{,j} = \frac{\partial}{\partial X_j}\,\delta \underset{\sim}{u}$$

$$N^k_{,i} = \frac{\partial N^k}{\partial X_i} = J^{-1}_{i1}\,\frac{\partial N^k}{\partial \xi} + J^{-1}_{i2}\,\frac{\partial N^k}{\partial \eta} \tag{30}$$

$$\underset{\sim}{G}^k_i = \frac{h^k}{2}\,(\tau\ \underset{\sim}{N}^k_{,i} + N^k\ \underset{\sim}{J}_3)\ \underset{\sim}{C}^k_t$$

$$\underset{\sim}{N}^k_{,j} = N^k_{,j}\ \underset{\sim}{I}_3 \quad ; \quad \underset{\sim}{J}_3 = \begin{bmatrix} J^{-1}_{13} & 0 & 0 \\ 0 & J^{-1}_{23} & 0 \\ 0 & 0 & J^{-1}_{33} \end{bmatrix}$$

In eq.(30) $J^{-1}_{ij}$ is the element (i,j) of the inverse of the Jacobian matrix which contains the derivatives of the initial coordinates $X_i$ with respect to the natural coordinates $\xi,\eta$ and $\tau$ [20].

Eqs.(28) and (29) can be used to obtain the following relations

$$\delta \underset{\sim}{\varepsilon} = \underset{\sim}{B}_L\ \delta \underset{\sim}{a} \tag{31}$$

where

$$\delta \underset{\sim}{\varepsilon} = [\delta\varepsilon_{11}, \delta\varepsilon_{22}, \delta\varepsilon_{33}, 2\delta\varepsilon_{12}, 2\delta\varepsilon_{13}, 2\delta\varepsilon_{23}]^T \tag{32}$$

are the first and second order strain increment vectors

$$\underset{\sim}{B}_L = \underset{\sim}{B}_{L_0} + \underset{\sim}{B}_{L_1} \tag{33}$$

The explicit forms of $B_{L_0}$ and $B_{L_1}$ can be seen in Table 1, where the terms $l_{ij}$ are directly obtained from eq.(29).

It is worth noting that the sixth column of matrices $\underset{\sim}{B}^k_{L_0}$ and $\underset{\sim}{B}^k_{L_1}$ of Table 1 will only be necessary if the rotation increments are defined in global axes. On the other hand, if the

$$\underset{\sim}{B}_{L_0} = [\underset{\sim}{B}^1_{L_0}, \ldots, \underset{\sim}{B}^k_{L_0}, \ldots, \underset{\sim}{B}^n_{L_0}]$$

$$\underset{\sim}{B}_{L_1} = [\underset{\sim}{B}^1_{L_1}, \ldots, \underset{\sim}{B}^k_{L_1}, \ldots, \underset{\sim}{B}^n_{L_1}]$$

$$\underset{\sim}{B}^k_{L_0} = \begin{bmatrix} N_{k,1} & 0 & 0 & (\underset{\sim}{G}^k_1)_{11} & (\underset{\sim}{G}^k_1)_{12} & (\underset{\sim}{G}^k_1)_{13} \\ 0 & N_{k,2} & 0 & (\underset{\sim}{G}^k_2)_{21} & (\underset{\sim}{G}^k_2)_{22} & (\underset{\sim}{G}^k_2)_{23} \\ 0 & 0 & N_{k,3} & (\underset{\sim}{G}^k_3)_{31} & (\underset{\sim}{G}^k_3)_{32} & (\underset{\sim}{G}^k_3)_{33} \\ N_{k,2} & N_{k,1} & 0 & [(\underset{\sim}{G}^k_2)_{11}+(\underset{\sim}{G}^k_1)_{21}] & [(\underset{\sim}{G}^k_2)_{12}+(\underset{\sim}{G}^k_1)_{22}] & [(\underset{\sim}{G}^k_2)_{13}+(\underset{\sim}{G}^k_1)_{23}] \\ N_{k,3} & 0 & N_{k,1} & [(\underset{\sim}{G}^k_3)_{11}+(\underset{\sim}{G}^k_1)_{31}] & [(\underset{\sim}{G}^k_3)_{12}+(\underset{\sim}{G}^k_1)_{32}] & [(\underset{\sim}{G}^k_3)_{13}+(\underset{\sim}{G}^k_1)_{33}] \\ 0 & N_{k,3} & N_{k,2} & [(\underset{\sim}{G}^k_3)_{21}+(\underset{\sim}{G}^k_2)_{31}] & [(\underset{\sim}{G}^k_3)_{22}+(\underset{\sim}{G}^k_2)_{32}] & [(\underset{\sim}{G}^k_3)_{23}+(\underset{\sim}{G}^k_2)_{33}] \end{bmatrix}$$

$$\underset{\sim}{B}^k_{L_1} = \begin{bmatrix} l_{11}N_{k,1} & l_{21}N_{k,1} & l_{31}N_{k,1} & (\phi^k_1)_{11} & (\phi^k_1)_{12} & (\phi^k_1)_{13} \\ l_{12}N_{k,2} & l_{22}N_{k,2} & l_{32}N_{k,2} & (\phi^k_2)_{21} & (\phi^k_2)_{22} & (\phi^k_2)_{23} \\ l_{13}N_{k,3} & l_{23}N_{k,3} & l_{33}N_{k,3} & (\phi^k_3)_{31} & (\phi^k_3)_{32} & (\phi^k_3)_{33} \\ (l_{11}N_{k,2}+l_{12}N_{k,1}) & (l_{21}N_{k,2}+l_{22}N_{k,1}) & (l_{31}N_{k,2}+l_{32}N_{k,1}) & [(\phi^k_1)_{21}+(\phi^k_2)_{11}] & [(\phi^k_1)_{22}+(\phi^k_2)_{12}] & [(\phi^k_1)_{23}+(\phi^k_2)_{13}] \\ (l_{11}N_{k,3}+l_{13}N_{k,1}) & (l_{21}N_{k,3}+l_{23}N_{k,1}) & (l_{31}N_{k,3}+l_{33}N_{k,2}) & [(\phi^k_1)_{31}+(\phi^k_3)_{11}] & [(\phi^k_1)_{32}+(\phi^k_3)_{12}] & [(\phi^k_1)_{33}+(\phi^k_3)_{13}] \\ (l_{12}N_{k,3}+l_{13}N_{k,2}) & (l_{22}N_{k,3}+l_{23}N_{k,2}) & (l_{32}N_{k,3}+l_{33}N_{k,2}) & [(\phi^k_2)_{31}+(\phi^k_3)_{21}] & [(\phi^k_2)_{32}+(\phi^k_3)_{22}] & [(\phi^k_2)_{33}+(\phi^k_3)_{23}] \end{bmatrix}$$

with $l_{ij} = \dfrac{\partial u_i}{\partial X_j}$ ; $(\phi^k_m)_{jn} = \sum_{i=1}^{3} l_{ij}(\underset{\sim}{G}^k_m)_{in}$

TABLE 1. Matrices $\underset{\sim}{B}_{L_0}$ and $\underset{\sim}{B}_{L_1}$.



are needed. This can be deduced from the definition of matrix $\underset{\sim}{C}^k_t$ in eqs.(22) and (23).

5.2. Computation of the local strain matrix.

The local strain increments can be directly obtained from their global expressions by simple tensor transformations as

$$\delta\underset{\sim}{\varepsilon}' = \underset{\sim}{H}\ \delta\underset{\sim}{\varepsilon} \tag{34}$$

where

$$\delta\underset{\sim}{\varepsilon} = [\delta\varepsilon_{rr}, \delta\varepsilon_{ss}, 2\delta\varepsilon_{rs}, 2\delta\varepsilon_{rt}, 2\delta\varepsilon_{st}]^T \tag{35}$$

and matrix $\underset{\sim}{H}$ is given by

$$\underset{\sim}{H} = \frac{1}{e^2}\begin{bmatrix} V_{r_1}^2 & V_{r_2}^2 & V_{r_3}^2 & V_{r_1}V_{r_2} & V_{r_1}V_{r_3} & V_{r_2}V_{r_3} \\ V_{s_1}^2 & V_{s_2}^2 & V_{s_3}^2 & V_{s_1}V_{s_2} & V_{s_1}V_{s_3} & V_{s_2}V_{s_3} \\ 2V_{r_1}V_{s_1} & 2V_{r_2}V_{s_2} & 2V_{r_3}V_{s_3} & (V_{r_1}V_{s_2}+V_{r_2}V_{s_1}) & (V_{r_1}V_{s_3}+V_{r_3}V_{s_1}) & (V_{r_2}V_{s_3}+V_{r_3}V_{s_2}) \\ 2eV_{r_1}V_{t_1} & 2eV_{r_2}V_{t_2} & 2eV_{r_3}V_{t_3} & e(V_{r_1}V_{t_2}+V_{r_2}V_{t_1}) & e(V_{r_1}V_{t_3}+V_{r_3}V_{t_1}) & e(V_{r_2}V_{t_3}+V_{r_3}V_{t_2}) \\ 2eV_{s_1}V_{t_1} & 2eV_{s_2}V_{t_2} & 2eV_{s_3}V_{t_3} & e(V_{s_1}V_{t_2}+V_{s_2}V_{t_1}) & e(V_{s_1}V_{t_3}+V_{s_3}V_{t_1}) & e(V_{s_2}V_{t_3}+V_{s_3}V_{t_2}) \end{bmatrix} \tag{36}$$

where e is the modulus of vectors $\vec{V}_r$ and $\vec{V}_s$ (see eq.(1)).

Substituting eq.(37) in (33) it can be obtained

$$\delta\underset{\sim}{\varepsilon}' = \bar{\underset{\sim}{B}}_L\ \delta\underset{\sim}{a} \tag{37}$$

where the local strain matrix is given by

$$\bar{\underset{\sim}{B}}_L = \underset{\sim}{H}\ \underset{\sim}{B}_L \tag{38}$$

6. CONSTITUTIVE EQUATIONS

Assuming linear isotropic behaviour the incremental stress-strain relationship can be written in local axes as

$$\delta\underset{\sim}{\sigma}' = \bar{\underset{\sim}{D}}\ \delta\underset{\sim}{\varepsilon}' \tag{39}$$

where

$$\begin{aligned} \delta\underset{\sim}{\varepsilon}' &= [\delta\varepsilon_{rr}, \delta\varepsilon_{ss}, 2\delta\varepsilon_{rs}, 2\delta\varepsilon_{rt}, 2\delta\varepsilon_{st}]^T \\ \delta\underset{\sim}{\sigma}' &= [\delta\sigma_{rr}, \delta\sigma_{ss}, \delta\sigma_{rs}, \delta\sigma_{rt}, \delta\sigma_{st}]^T \end{aligned} \tag{40}$$

are the increment of the Green-Lagrange strain and the 2° Piola-Kirchhoff stress vectors referred to local axes. Matrix $\bar{\underset{\sim}{D}}$ can be found by simplification of the general constitutive matrix for 3D linear elasticity using the assumption of zero normal stress ($\sigma_{tt} = 0$) as

$$\bar{\underset{\sim}{D}} = \frac{E}{1-\nu^2} = \begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1-\nu}{2}k & 0 \\ 0 & 0 & 0 & 0 & \frac{1-\nu}{2}k \end{bmatrix} \tag{41}$$

where E and $\nu$ are the Young modulus and Poisson's ratio, respectively, and k is a factor to take into account transverse shear warping (usually $k=5/6$) [21].

For non linear material behaviour the appropiate non linear constitutive matrix $\bar{\underset{\sim}{D}}$ should be used. If the total strains are small, it is quite usual to assume that the constitutive equations obtained for infinitesimal displacement analysis hold for large displacement / large rotation problems. [19]

The local constitutive matrix of eq.(44) can be transformed to global axes as

$$\underset{\sim}{D} = \underset{\sim}{H}^T \bar{\underset{\sim}{D}} \underset{\sim}{H} \tag{42}$$

where $\underset{\sim}{H}$ was given in eq.(36).

Matrix $\underset{\sim}{D}$ relates the global stress and strain increments as

$$\delta\underset{\sim}{\sigma} = \underset{\sim}{D}\ \delta\underset{\sim}{\varepsilon} \tag{43}$$

where

$$\begin{aligned} \delta\underset{\sim}{\sigma} &= [\delta\sigma_{11}, \delta\sigma_{22}, \delta\sigma_{33}, \delta\sigma_{12}, \delta\sigma_{13}, \delta\sigma_{23}]^T \\ \delta\underset{\sim}{\varepsilon} &= [\delta\varepsilon_{11}, \delta\varepsilon_{22}, \delta\varepsilon_{33}, 2\delta\varepsilon_{12}, 2\delta\varepsilon_{13}, 2\delta\varepsilon_{23}]^T \end{aligned} \tag{44}$$

are the 2° Piola-Kirchhoff stress vector and the Green-Lagrange strain vector, respectively, referred to the global stationary axes.

## 7. DISCRETIZED EQUILIBRIUM EQUATIONS

The discretized equilibrium equations are derived via the virtual work expression which in the Total Lagrangian description is

$$\Psi(\underset{\sim}{u}) = \int_V \delta\underset{\sim}{\varepsilon}^T\ \underset{\sim}{\sigma}\ dV - \int_V \delta\underset{\sim}{u}^T\ \underset{\sim}{b}\ dV - \int_S \delta\underset{\sim}{u}^T\ \underset{\sim}{t}\ dS = 0 \tag{45}$$

where $\underset{\sim}{\sigma}$ and $\delta\underset{\sim}{\varepsilon}$ are the 2° Piola-Kirchhoff stress and the virtual Green-Lagrange deformation vectors, respectively, $\underset{\sim}{b}$ and $\underset{\sim}{t}$ are the body force and surface traction loads, respectively, and V and S are the volume and surface of the structure in the initial configuration.

Eq.(45) is the basis for obtaining the finite element incremental equations of equilibrium. If direct linearization is used via the standard Newton-Raphson method [17], [20], the following system of equations for the iterative values of the displacement increments is obtained

$$\underset{\sim}{K}^{(i)}\ \delta\underset{\sim}{a}^{(i)} = -\ \underset{\sim}{\Psi}^{(i)} \tag{46}$$

where index i refers to values for the ith iteration, $\underset{\sim}{K}$ is the tangent stiffness matrix given by

$$\underset{\sim}{K}(\underset{\sim}{a}) = \frac{\partial \underset{\sim}{\Psi}}{\partial \underset{\sim}{a}} \tag{47}$$

and $\delta a$ and $\Psi$ are the displacement increment and the residual force vectors, respectively. From eq.(46), the value of $\delta\underset{\sim}{a}^{(i)}$ can be found as

$$\delta\underset{\sim}{a}^{(i)} = -\ [\underset{\sim}{K}^{(i)}]^{-1}\ \underset{\sim}{\Psi}^{(i)} \tag{48}$$

and the next "improved" solution is obtained by

$$\underset{\sim}{a}^{(i+1)} = \underset{\sim}{a}^{(i)} + \delta\underset{\sim}{a}^{(i)} \tag{49}$$

The solution for the next displacement increment $\delta a^{(i+1)}$ can then be started once the values of $\underset{\sim}{K}^{(i+1)}$ and $\underset{\sim}{\Psi}^{(i+1)}$ have been appropiately computed. The process stops when a prescribed error norm for the displacement increments or the residual forces is satisfied. A norm widely used is the quadratic norm for the residual forces given by

$$\frac{\sum_j [\Psi_j^{(i+1)} - \Psi_j^{(i)}]^2}{\sum_j [f_j]^2} \leq \varepsilon^2 \tag{50}$$

where $f_j$ is the jth component of the external force vector and $\varepsilon$ a prescribed convergence error value.

### 7.1. Computation of the tangent matrix using the local strain matrix.

Matrix $\underset{\sim}{K}^{(i)}$ can be obtained using the local definition for the strain and constitutive matrices as

$$\underset{\sim}{K}^{(i)} = \underset{\sim}{K}_L^{(i)} + \underset{\sim}{K}_S^{(i)} \tag{51}$$

where

$$\underset{\sim}{K}_L = \int_V \bar{\underset{\sim}{B}}_L^T\ \bar{\underset{\sim}{D}}\ \bar{\underset{\sim}{B}}_L\ dV$$

$$\underset{\sim}{K}_S = \int_V \bar{\underset{\sim}{B}}_{N_L}^T\ \bar{\underset{\sim}{S}}\ \bar{\underset{\sim}{B}}_{N_L}\ dV \tag{52}$$

where $\bar{\underset{\sim}{B}}_L$ and $\bar{\underset{\sim}{D}}$ have been defined in eqs.(38) and (41), respectively, and $\bar{\underset{\sim}{B}}_{N_L}$ and $\bar{\underset{\sim}{S}}$ are shown in Table 2. It is worth pointing out that the sixth column of $\underset{\sim}{B}_{NL}^K$ is only necessary if a global definition of the nodal rotation increments is used.

Matrix $\underset{\sim}{K}_L$ can be written using eq.(33) as

$$\underset{\sim}{K}_L = \underset{\sim}{K}_{L_0} + \underset{\sim}{K}_{L_1} \tag{53}$$

where

$$\underset{\sim}{K}_{L_0} = \int_V \bar{\underset{\sim}{B}}_{L_0}^T\ \bar{\underset{\sim}{D}}\ \bar{\underset{\sim}{B}}_{L_0}\ dV \tag{54}$$

and

$$\underset{\sim}{K}_{L_1} = \int (\bar{\underset{\sim}{B}}_{L_0}^T\ \bar{\underset{\sim}{D}}\ \bar{\underset{\sim}{B}}_{L_1} + \bar{\underset{\sim}{B}}_{L_1}^T\ \bar{\underset{\sim}{D}}\ \bar{\underset{\sim}{B}}_{L_0} + \bar{\underset{\sim}{B}}_{L_1}^T\ \bar{\underset{\sim}{D}}\ \bar{\underset{\sim}{B}}_{L_1}) dV \tag{55}$$

$$\bar{B}_{NL} = F^T \, B_{NL}$$

where

$$F = \begin{bmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{bmatrix} \quad ; \quad T = \left[\frac{1}{e} V_r, \frac{1}{e} V_s, V_t\right]$$

$$B_{NL}^K = \begin{bmatrix} N_{k,1} & 0 & 0 & (G_1^k)_{11} & (G_1^k)_{12} & (G_1^k)_{13} \\ N_{k,2} & 0 & 0 & (G_2^k)_{11} & (G_2^k)_{12} & (G_2^k)_{13} \\ N_{k,3} & 0 & 0 & (G_3^k)_{11} & (G_3^k)_{12} & (G_3^k)_{13} \\ 0 & N_{k,1} & 0 & (G_1^k)_{21} & (G_1^k)_{22} & (G_1^k)_{23} \\ 0 & N_{k,2} & 0 & (G_2^k)_{21} & (G_2^k)_{22} & (G_2^k)_{23} \\ 0 & N_{k,3} & 0 & (G_3^k)_{21} & (G_3^k)_{22} & (G_3^k)_{23} \\ 0 & 0 & N_{k,1} & (G_1^k)_{31} & (G_1^k)_{32} & (G_1^k)_{33} \\ 0 & 0 & N_{k,2} & (G_2^k)_{31} & (G_2^k)_{32} & (G_2^k)_{33} \\ 0 & 0 & N_{k,3} & (G_3^k)_{31} & (G_3^k)_{32} & (G_3^k)_{33} \end{bmatrix}$$

$$\bar{S} = \begin{bmatrix} [\bar{\sigma}] & 0 & 0 \\ 0 & [\bar{\sigma}] & 0 \\ 0 & 0 & [\bar{\sigma}] \end{bmatrix} \quad ; \quad [\bar{\sigma}] = \begin{bmatrix} \sigma_{rr} & \sigma_{rs} & \sigma_{rt} \\ \sigma_{rs} & \sigma_{ss} & \sigma_{st} \\ \sigma_{rt} & \sigma_{st} & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} [\sigma] & 0 & 0 \\ 0 & [\sigma] & 0 \\ 0 & 0 & [\sigma] \end{bmatrix} \quad [\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}$$

TABLE 2. Matrices $\bar{B}_{NL}$, $B_{NL}$, $\bar{S}$ and $S$.



depends on the displacement level through matrix $\bar{B}_{L_1}$ and its known as the "initial displacement matrix". Finally, $K_S$ is the "initial stress matrix" depending on the actual stress level on the structure.

On the other hand, the residual force vector is given by

$$\Psi = \left(\int_V B_L^T \, \sigma' \, dV\right) - f \tag{56}$$

where $\sigma'$ is the local second Piola-Kirchhoff stress vector

$$f = \int_V N^T \, b \, dV - \int_S N^T \, t \, dS \tag{57}$$

is the equivalent external nodal forces vector and $N$ is the shape function matrix [20].

All matrices and vectors of eqs.(52)-(57) are computed by assembling the contributions of the different elements in an standard manner [20].

7.2. Computation of the tangent matrix using the global strain vector.

The tangent matrix can also be obtained using the global strain, stress and constitutive matrices, as mentioned in Section 5. The expression of $K_L$ and $K_S$ are in this case

$$K_L = \int_V B_L^T \, D \, B_L dV$$

$$K_S = \int_V B_{NL}^T \, S \, B_{NL} \, dV \tag{58}$$

where $B_L$, $D$, $B_{NL}$ and $S$ have been given in Table 1, eq.(42) and Table 2, respectively.

The residual force vector is obtained now by

$$\Psi = \int_V B_L^T \, \sigma \, dV - f \tag{59}$$

where $\sigma$ is the global second Piola-Kirchhoff stress vector.

## 8. UPDATED LAGRANGIAN FORMULATION

In the Updated Lagrangian description all static and Kinematic variables are referred to the configuration at time t. This implies that the reference configuration is conveniently updated after each displacement increment solution has been found. However, the basic steps for deriving the finite element formulation do not difer much from those followed for the Total Lagrangian description. Moreover, all the relevant finite element matrices can be easily deduced from those obtained in

previous sections simply taking into account the following:

1) All cartesian derivatives appearing in the expressions of the strain matrices are now with respect to the coordinates at time t, i.e.

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} \quad , \quad u_{i,j} = \frac{\partial (\partial u_i)}{\partial x_j} \tag{60}$$

and

$$N_{i,j} = \frac{\partial N_i}{\partial x_j} \tag{61}$$

2) The Jacobian matrix $\underset{\sim}{J}$ contains the derivatives of the coordinates of the configuration at time t with respect to the natural coordinates $\xi$, $\eta$ and $\tau$.

3) The initial displacement effect is automatically taken into account by updating the reference configuration. This implies that $u = 0$ and, therefore, $u_{i,j} = l_{ij} = 0$ and, consequently, $\underset{\sim}{B}_{L_1} = \underset{\sim}{0}$. Thus, matrix $\underset{\sim}{K}_L$ is simply given by

$$\underset{\sim}{K}_L = \underset{\sim}{K}_{L_0} \tag{62}$$

4) The volume and surface of the shell correspond to the actual values in the configuration t.

5) The 2° Piola-Kirchhoff stresses are referred to the configuration at time t and, thus, become identical to the Cauchy (true) stresses. The Cauchy stresses are updated after each new displacement increment has been found, and then they are appropiately transformed to the "next reference configuration".

A final point should be said about the constitutive equation to be used. It can be shown that both the Updated and Total Lagrangian formulation give the same results if the coefficients of the constitutive matrix in both formulation are related by the expressions [19]

$$D^t_{ijkl} = \frac{\rho^t}{\rho^o} \frac{\partial x_i}{\partial x_m} \frac{\partial x_j}{\partial x_n} D^o_{mnpq} \frac{\partial x_k}{\partial x_p} \frac{\partial x_l}{\partial x_q} \tag{63}$$

where indexes o and t refer to the initial configuration and the configuration at time t, respectively and $\rho$ is the material density. Moreover, it can be proved [19] that for small strain conditions eq.(67) gives approximately

$$D^t_{ijkl} \simeq \underset{\sim}{D}^o_{mnpq} \tag{64}$$

which means that in these situations, and for practical purposes, the same constitutive matrix, $\underset{\sim}{D}$, given in eq.(44) can be effectively used in the Updated Lagrangian formulation.

## 9. FINAL REMARKS

The finite element shell formulation presented in previous sections suffers from the inconvenients of the classical thick shell formulations when used in the context of very thin situations, i.e. the shear stiffness terms dominate the solution and very stiff solutions one encountered [21], [22]. This phenomenon, typically termed as "locking effect" can be overcome in several ways, such as the underevaluation of the shear terms in the element stiffness matrix using reduced or selective numerical integration techniques [20], or the use of specific finite elements developed to avoid locking [21], [22], [23], [24].



It is also worth pointing out that the formulation presented here can be effectively used for the analysis of shells of arbitrary shape. However, when mixed geometry situations involving both smooth and kinked surfaces are treated, two possibilities for the definition of the nodal rotational degrees of freedom are encountered: 1) The two local rotation increments are used as degrees of freedom for nodes laying on smooth surfaces (see eq.(23)), whereas for nodes laying on kinks or folding lines a global definition for the rotation increments is used (see eq.(24)). 2) To work in terms of the three global rotation increments for all nodes. Option 1 implies that the computer program developed has to be able to handle a different number of degrees of freedom (five or six) at each node, and this usually makes option 2 more attractive.

However, extreme case must be taken if option 2 is chosen, to avoid singularity of the tangent stiffness matrix at nodes laying on smooth surfaces, where only two nodal rotational degrees of freedom are strictly needed. This singularity can be avoided by adding to the global stiffness matrix the following "spring" matrix

$$\underset{\sim}{K}'_{ij} = \underset{\sim}{F}_i \begin{bmatrix} \underset{\sim}{0} & 0 \\ 0 & \alpha \end{bmatrix} \underset{\sim}{F}^T_j \tag{65}$$

where $\alpha$ is a large number, $\underset{\sim}{0}$ is a $5 \times 5$ null matrix and

$$\underset{6x6}{F_i} = \begin{bmatrix} \underset{\sim}{0}_{3x3} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{T}^i \end{bmatrix} \tag{66}$$

where $\underset{\sim}{T}^i$ is given by eq.(10).

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