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# A RIGOROUS VARIANT OF THE SHEAR STRENGTH REDUCTION METHOD AND ITS GEOTECHNICAL APPLICATIONS

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**Abstract.** This paper is focused on a new optimization variant of the shear strength reduction (OPT-SSR) in non-associated Mohr-Coulomb plasticity. The OPT-SSR method mimics the limit analysis problem and enables to compute the factor of safety without performing an elasto-plastic analysis. It is shown that this optimization problem is well-defined and closely related to recently developed Davis approaches used in combination with the standard SSR method. Next, the duality between the static and kinematic principles of OPT-SSR is introduced. For the numerical solution, a regularization method is suggested.

## 1 INTRODUCTION

The SSR method [22, 1, 5, 6] has been suggested mainly for the slope stability assessment. It is a conventional method arising from elastic-perfectly plastic models containing (mainly) the Mohr-Coulomb yield criterion. For the numerical solution, a displacement variant of the finite element method (FEM) is mostly used and its implementation is a part of some commercial softwares like Plaxis [2].

The non-associated plastic flow rule is frequently used in geotechnical practice to control the inelastic volume changes of soils subjected to shearing. On the other hand, mathematical theory of non-associated elastic-plastic problems is not fully completed and numerical oscillations depending on mesh density are sometimes observed, especially if the problem is discretized in time by the standard implicit Euler method. In [14, 19, 20], a modified Davis approach has been suggested within the SSR method in order to compute the factor of safety (FoS) more rigorously. The modification leads to the approximation of the non-associated plastic flow rule by the associated one and is based on the limit analysis approach. We also refer to recent papers [21, 10] for comparison of the standard and the modified SSR method.

This paper summarizes selected results presented in [18] and arises from the ideas suggested in [19, 20]. The aim is to introduce an optimization variant of the SSR method, the so-called OPT-SSR method. The OPT-SSR method mimics the limit analysis problem and enables to compute the factor of safety without performing an elasto-plastic analysis. It can be completed by rigorous theory and variational principles.

## 2 THE MODIFIED STRENTGH REDUCTION TECHNIQUE

The Mohr-Coulomb linear elastic-perfectly plastic model contains the following strength parameters: the effective cohesion ($c'$), the effective friction angle ($\phi'$), and the dilatancy angle ($\psi'$). It is assumed that $\psi' \leq \phi'$. In case of $\psi' = \phi'$, we arrive at the associated model, otherwise the non-associated model is considered. The SSR method is based on the reduction of $c'$, $\phi'$ and $\psi'$:

$$c_\lambda := \frac{c'}{\lambda}, \quad \tan\phi_\lambda := \frac{\tan\phi'}{\lambda}, \quad \tan\psi_\lambda := \frac{\tan\psi'}{\lambda}, \tag{1}$$

where $\lambda > 0$ is the reduction parameter. Alternatively, one can use the following formula for $\psi_\lambda$ (see also [20]):

$$\psi_\lambda := \psi' \;\text{ until } \psi' < \phi_\lambda, \;\text{ then } \psi_\lambda := \phi_\lambda. \tag{2}$$

FoS for the SSR method is defined as a maximum of $\lambda$ for which the elastic-perfectly plastic problem has a solution with respect to the parameters $c_\lambda$, $\phi_\lambda$, and $\psi_\lambda$.

We propose to approximate the SSR for the non-associated model with the associated model and the following reduction of the parameter $c'$ and $\phi'$:

$$\tilde{c}_\lambda := \frac{c'}{q(\lambda;\phi',\psi')}, \quad \tan\tilde{\phi}_\lambda := \frac{\tan\phi'}{q(\lambda;\phi',\psi')}, \tag{3}$$

where $q$ is a function with the following general properties:

(A1) $q$ is positive for any $\lambda > 0$, continuous and $q(\lambda;\phi',\psi') \geq \lambda$;

(A2) $q$ is increasing with respect to the variable $\lambda \geq 0$;

(A3) $q$ is non-increasing with respect to the variable $\psi' \geq 0$;

(A4) if $\psi' = \phi'$ then $q(\lambda;\phi',\psi') = \lambda$.

FoS for the approximated SSR method is defined as a maximum of $\lambda$ for which the associatied elastic-perfectly plastic problem has a solution with respect to the parameters $\tilde{c}_\lambda$ and $\tilde{\phi}_\lambda$.

If $\psi' = \phi'$ then the factors of safety for the SSR method and its approximation coincides as follows from the assumption (A4). The assumptions (A1) and (A2) ensure that the strength parameters are reduced. The assumption (A3) enables to include the influence of the difference $\phi' - \psi'$ on FoS. The larger the difference is, the lower the values of FoS are expected.

From now on, we will not emphasize the dependence of $q$ on $\phi'$ and $\psi'$, for the sake of simplicity, and write simply $q := q(\lambda)$. We introduce three examples of the function $q$ which are related to the DAVIS A, DAVIS B and DAVIS C approaches presented in [20]. These functions will be denoted as $q_A$, $q_B$, and $q_C$, respectively, and are defined as follows:

$$q_A(\lambda) = \lambda\,\frac{1-\sin\psi'\sin\phi'}{\cos\psi'\cos\phi'}, \tag{4}$$

$$q_B(\lambda) = \lambda\,\frac{1-\sin\psi_\lambda\sin\phi_\lambda}{\cos\psi_\lambda\cos\phi_\lambda}, \tag{5}$$

$$q_C(\lambda) = \begin{cases} \lambda\,\frac{1-\sin\psi'\sin\phi_\lambda}{\cos\psi'\cos\phi_\lambda}, & \text{if } \phi_\lambda \geq \psi', \\ \lambda, & \text{if } \phi_\lambda \leq \psi', \end{cases} \tag{6}$$

where

$$\tan\phi_\lambda := \frac{\tan\phi'}{\lambda}, \quad \tan\psi_\lambda := \frac{\tan\psi'}{\lambda}. \tag{7}$$

In [18], it is shown that the assumptions (A1)–(A4) are satisfied for $q_A$, $q_B$, and $q_C$. In addition, the following statements have been proven:

(S1) $q_A(\lambda) = q_B(\lambda) = q_C(\lambda)$ for $\lambda = 1$;

(S2) $q_A(\lambda) \geq q_B(\lambda) \geq q_C(\lambda)$ for any $\lambda \geq 1$;

(S3) $q_C(\lambda) \geq q_B(\lambda) \geq q_A(\lambda)$ for any $\lambda \leq 1$.

It is also important to note that the choice of the function $q$ can differ from the Davis approaches and be optimized, for example, by inverse analysis.

## 3 THE OPT-SSR METHOD

One can solve the approximated SSR problem based on the formula (3) similarly as the standard SSR method. Nevertheless, due to the fact that the approximated problem is built on the associated plasticity one can consider several simplifications. In particular, it suffices to work with the rigid plastic model and eliminate the plastic multiplier as it is usual in limit analysis, see for example [11]. The corresponding optimization problem (OPT-SSR) reads as follows:

$\omega^*$ = *supremum of* $\lambda \geq 0$ *subject to*

$$\left.\begin{aligned} -\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{F} \ \text{ in } \Omega, \quad \boldsymbol{\sigma n} = \boldsymbol{f} \ \text{ on } \partial\Omega_f, \\ \Phi\big(q(\lambda);\boldsymbol{\sigma}\big) \leq 0 \ \text{ in } \Omega, \end{aligned}\right\} \tag{8}$$

Here, $\omega^*$ denotes FoS and the function $\Phi$ represents the Mohr-Coulomb yield criterion for the parameters $\tilde{c}_\lambda$ and $\tilde{\phi}_\lambda$ defined by (3). This function can be arranged to the following form [18]:

$$\Phi(q(\lambda);\boldsymbol{\sigma}) := (\sigma_1-\sigma_3)\sqrt{q^2(\lambda)+\tan^2\phi'}+(\sigma_1+\sigma_3)\tan\phi'-2c', \tag{9}$$

where $\boldsymbol{\sigma}$ is the effective Cauchy stress tensor and $\sigma_1, \sigma_3$ are its maximal and minimal principle stresses (in current mechanical sign convention). Next, $\Omega$ is a bounded domain in 2D or 3D representing an investigated body, $\boldsymbol{F}$ is a volume force (e.g. the weight of the body), $\boldsymbol{f}$ is a prescribed surface force acting on the part $\partial\Omega_f$ of the boundary $\partial\Omega$, $\boldsymbol{n}$ denotes the outward unit normal to the boundary $\partial\Omega$.

From the assumptions (A1)–(A2), it follows that if the constraints (8) are satisfied for some $\lambda := \overline{\lambda} > 0$ then (8) holds for any $\lambda < \overline{\lambda}$, see [18]. Hence, (8) holds for any $\lambda < \omega^*$. Without this basic but crucial property, it would be very difficult to find $\omega^*$.

The OPT-SSR method enables to compare safety factors for the Davis A-C models. Let $\lambda_A^*$, $\lambda_B^*$, and $\lambda_C^*$ denote FoS for the functions $q_A$, $q_B$, and $q_C$, respectively, and let $\lambda_{ass}^*$ denote denote FoS for the associated model with $q(\lambda)=\lambda$. From the statements (S1)–(S3), we have the following results (see [18]) which are in accordance with numerical observations presented in [20, 10]:

1. $\lambda_A^* \leq \lambda_{ass}^*$, $\lambda_B^* \leq \lambda_{ass}^*$, $\lambda_C^* \leq \lambda_{ass}^*$;
2. either $1 \leq \lambda_A^* \leq \lambda_B^* \leq \lambda_C^*$ or $1 \geq \lambda_A^* \geq \lambda_B^* \geq \lambda_C^*$.
3. If one of the values $\lambda_A^*$, $\lambda_B^*$, $\lambda_C^*$ is equal to one then the same holds for the remaining values.

## 4 DUALITY FOR THE OPT-SSR PROBLEM

We introduce the following functional spaces:

$$V = \{\boldsymbol{v} \in [H^1(\Omega)]^3 \mid \boldsymbol{v} = \boldsymbol{0} \text{ on } \partial\Omega_u\}, \tag{10}$$

$$\Sigma = \{\boldsymbol{\sigma} \in [L^2(\Omega)]^{3\times 3} \mid \sigma_{ij} = \sigma_{ji} \text{ in } \Omega\}, \tag{11}$$

where the space $V$ represents admissible velocity fields and $\Sigma$ is used for stress and strain fields. $L^2(\Omega)$ and $H^1(\Omega)$ denote the Lebesgue and Sobolev spaces, respectively.

Using the space $V$ we arrive at the weak form of $(8)_1$:

$$\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V, \tag{12}$$

where $\boldsymbol{\varepsilon}$ denotes the strain-rate tensor field,

$$\boldsymbol{\varepsilon}(\boldsymbol{v}) = \tfrac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top), \tag{13}$$

and $L$ is the load functional defined by

$$L(\boldsymbol{v}) = \int_{\Omega} \boldsymbol{F} \cdot \boldsymbol{v}\, dx + \int_{\partial\Omega_f} \boldsymbol{f} \cdot \boldsymbol{v}\, ds. \tag{14}$$

Let $\Lambda$ denote the set of stresses $\boldsymbol{\sigma} \in \Sigma$ satisfying (12) and let

$$P_{q(\lambda)} := \{\boldsymbol{\sigma} \in \Sigma \mid \ \Phi\big(q(\lambda); \boldsymbol{\sigma}\big) \leq 0 \ \text{ in } \Omega\}. \tag{15}$$

We see that the set $P_{q(\lambda)}$ represents the constraint $(8)_2$ and thus we can write

$$\begin{aligned}\omega^* &= \sup\{\lambda \geq 0 \mid \ P_{q(\lambda)} \cap \Lambda \neq \emptyset\} \\ &= \sup_{\lambda \geq 0} \sup_{\boldsymbol{\sigma} \in P_{q(\lambda)} \cap \Lambda} \{\lambda\}.\end{aligned} \tag{16}$$

Releasing the constraint set $\Lambda$, we arrive at the following dual problem in terms of velocity fields:

$$\omega^* = \sup_{\lambda \geq 0} \inf_{\boldsymbol{v} \in V} \left[\lambda + \int_{\Omega} D(q(\lambda); \boldsymbol{\varepsilon}(\boldsymbol{v}))\, dx - L(\boldsymbol{v})\right], \tag{17}$$

where

$$D(q(\lambda); \boldsymbol{\varepsilon}) = \sup_{\substack{\boldsymbol{\sigma} \in \mathbb{R}^{3\times 3}_{sym} \\ \Phi(q(\lambda); \boldsymbol{\sigma}) \leq 0}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon} \tag{18}$$

denotes the local dissipation function depending on $\lambda \geq 0$. The function $D(q(\lambda); \boldsymbol{\varepsilon})$ is finite-valued only for some $\boldsymbol{\varepsilon} \in \mathbb{R}^{3\times 3}_{sym}$ belonging to a convex cone. Therefore, the inner problem in (17) can be classified as cone programming. (17) can be interpreted as the *kinematic principle* of the OPT-SSR method. We expect that this duality holds without any gap, which is partially justified by the results presented in [4, 9].

## 5 REGULARIZED PROBLEM AND ITS SOLUTION

For numerical solution of the SSR-OPT problem, we suggest to use the regularization method introduced in [18]. This method has also been considered for the solution of similar problems, see [17, 12, 9, 13, 8, 7, 16].

We arise from (16) and regularize this problem with respect to a parameter $\alpha > 0$ as follows:

$$\omega^*_\alpha = \sup_{\lambda \geq 0} \sup_{\boldsymbol{\sigma} \in P_{q(\lambda)} \cap \Lambda} \left[\lambda - \frac{1}{2\alpha} \int_{\Omega} \mathbb{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma}\, dx\right], \tag{19}$$

where $\mathbb{C}$ is a positive definite fourth order tensor, for example, the elastic tensor. One can also write

$$\omega^*_\alpha = \max_{\lambda \geq 0} [\lambda - G_\alpha(\lambda)] = \lambda^*_\alpha - G_\alpha(\lambda^*_\alpha), \tag{20}$$

where $\lambda^*_\alpha$ maximizes the middle term in (20) and

$$G_\alpha(\lambda) = \inf_{\boldsymbol{\sigma} \in P_{q(\lambda)} \cap \Lambda} \frac{1}{2\alpha} \int_{\Omega} \mathbb{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma}\, dx \tag{21}$$

The sequence $\{\lambda^*_\alpha\}_{\alpha>0}$ defined by (21) satisfies [18]:

$$\omega^*_\alpha \leq \lambda^*_\alpha \leq \omega^*, \quad \lim_{\alpha\to+\infty} \lambda^*_\alpha = \omega^*, \tag{22}$$

Therefore, the value $\lambda^*_\alpha$ is convenient for the approximation of $\omega^*$.

Let us note that the scalar optimization problem in (20) can be solved, for example, by sequential enlarging $\lambda$ for fixed $\alpha$.

Next, for the solution of (20), it is crucial to evaluate the function $G_\alpha$. Using the duality approach, we arrive at the following kinematic definition of $G_\alpha$:

$$G_\alpha(\lambda) = -\inf_{\boldsymbol{v}\in V} \left[\int_\Omega D_\alpha(q(\lambda); \boldsymbol{\varepsilon}(\boldsymbol{v}))\, dx - L(\boldsymbol{v})\right], \tag{23}$$

where

$$D_\alpha(q(\lambda); \boldsymbol{\varepsilon}) = \sup_{\substack{\boldsymbol{\sigma}\in\mathbb{R}^{3\times3}_{sym} \\ \Phi(q(\lambda);\boldsymbol{\sigma})\leq 0}} \left[\boldsymbol{\sigma} : \boldsymbol{\varepsilon} - \frac{1}{2\alpha}\mathbb{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\sigma}\right]. \tag{24}$$

The function $D_\alpha$ is finite-valued and differentiable with respect to $\boldsymbol{\varepsilon}$ unlike the original dissipation $D$. Moreover, the second derivative of $D_\alpha$ exists almost everywhere. Let $T_\alpha(q(\lambda); \boldsymbol{\varepsilon}) \in \mathbb{R}^{3\times3}_{sym}$ denote the derivative of $D_\alpha(q(\lambda); \boldsymbol{\varepsilon})$ with respect to $\boldsymbol{\varepsilon}$. Then the problem (23) is equivalent to the following nonlinear variational equation:

$$\text{find } \boldsymbol{v}_{q(\lambda)} \in V: \quad \int_\Omega T_\alpha(q(\lambda); \boldsymbol{\varepsilon}(\boldsymbol{v}_{q(\lambda)})) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, dx = L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V. \tag{25}$$

This equation is practically the same as time-discretized elasto-plastic problem with the Mohr-Coulomb yield criterion. For its solution, one can use the standard finite element method and a non-smooth variant of the Newton method as in [15]. In [18], we have also completed the solution with continuation over $\alpha$ and simple mesh adaptivity in order to receive more reliable and accurate results.

The suggested numerical solution has been implemented within in-house Matlab codes which have been systematically developed and described in [15, 3]. For particular numerical examples we refer to [18]. In [18], the computed FoS have been compared with results from the commercial softwares Plaxis and Comsol Multiphysics.

## 6 CONCLUSIONS

We have proposed to approximate the standard SSR method for the non-associated model by the associated model and by a modified reduction of the strength parameters. Such an approach is inspired by recently developed Davis approaches to the SSR method [19, 20, 21]. The approximation (modification) of the SSR method has enabled the introduction of a rigorous optimization framework and the derivation of the duality between the static and kinematic principles as in limit analysis. For numerical solution, we have suggested a regularization method and its combination with the finite element, continuation and Newton-like methods. For more details, we refer to [18].

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## REFERENCES

[1] Brinkgreve, R.B.J. and Bakker, H.L. Non-linear finite element analysis of safety factors. *Proceedings of the international conference on computer methods and advances in geomechanics.* Rotterdam, the Netherlands: Balkema. (1991) 1117–1122.

[2] Brinkgreve, R.B.J., Swolfs, W.M. and Engin, E. *Plaxis 2D 2011 – user manual.* Delft, the Netherlands: Plaxis bv. (2011).

[3] Čermák, M., Sysala, S. and Valdman, J. Efficient and flexible MATLAB implementation of 2D and 3D elastoplastic problems. *Applied Mathematics and Computation* **355** (2019) 595–614.

[4] Christiansen, E. *Limit analysis of colapse states.* In P. G. Ciarlet and J. L. Lions, editors, *Handbook of Numerical Analysis*, Vol IV, Part 2, North-Holland, (1996) 195–312.

[5] Dawson, E.M., Roth, W.H. and Drescher, A.A. (1999). Slope stability analysis by strength reduction. *Géotechnique* **49** (1999) 835–840

[6] Griffiths, D.V. & Lane, P.A. . Slope stability analysis by finite elements. *Géotechnique* **49** (1999) 387–403.

[7] Haslinger, J., Repin and S., Sysala, S. A reliable incremental method of computing the limit load in deformation plasticity based on compliance: Continuous and discrete setting. *Journal of Computational and Applied Mathematics* **303** (2016) 156–170.

[8] Haslinger, J., Repin, S. and Sysala, S. Guaranteed and computable bounds of the limit load for variational problems with linear growth energy functionals. *Applications of Mathematics* **61** (2016) 527–564.

[9] Haslinger, J., Repin, S. and Sysala, S. Inf-sup conditions on convex cones and applications to limit load analysis. *Mathematics and Mechanics of Solids* **24** (2019) 3331–3353.

[10] Oberhollenzer, S., Tschuchnigg, F. and Schweiger, H.F. Finite element analyses of slope stability problems using non-associated plasticity. *Journal of Rock Mechanics and Geotechnical Engineering*, **10** (2018) 1091–1101.

[11] Krabbenhoft, K., Lyamin, A. and Krabbenhoft, J. *OptumG2: theory.* Newcastle, Australia: Optum Computational Engineering, (2016).

[12] Reddy, B.D. and Sysala, S. Bounds on the elastic threshold for problems of dissipative strain-gradient plasticity. *Journal of the Mechanics and Physics of Solids* **143** (2020) 104089.

[13] Repin, S., Sysala, S. and Haslinger, J. Computable majorants of the limit load in Hencky's plasticity problems. *Comp. & Math. with Appl.* **75** (2018) 199–217.

[14] Sloan, S.W. Geotechnical stability analysis, *Géotechnique*, **63** (2013) 531–572.

[15] Sysala, S., Čermák, M. and Ligurský, T. Subdifferential-based implicit return-mapping operators in Mohr-Coulomb plasticity. *ZAMM* **97** (2017) 1502–1523.

[16] Sysala, S., Haslinger, J., Hlaváček, I. and Cermak, M. Discretization and numerical realization of contact problems for elastic-perfectly plastic bodies. PART I–discretization, limit analysis. *ZAMM* **95** (2015) 333–353.

[17] Sysala, S., Haslinger, J., Reddy, B.D. and Repin, S. An abstract inf-sup problem inspired by limit analysis in perfect plasticity and related applications. *Mathematical Models and Methods in Applied Sciences.* To appear in 2021, http://arxiv.org/abs/2009.03535.

[18] Sysala, S., Hrubesova, E., Michalec, Z. and Tschuchnigg, F. Optimization and variational principles for the shear strength reduction method. Submitted (2021), https://arxiv.org/pdf/2101.01005.pdf.

[19] Tschuchnigg, F., Schweiger, H.F., Sloan, S.W., Lyamin, A.V. and Raissakis, I. Comparison of finite-element limit analysis and strength reduction techniques. *Géotechnique* **65** (2015) 249–257.

[20] Tschuchnigg, F., Schweiger, H.F. and Sloan, S.W. Slope stability analysis by means of finite element limit analysis and finite element strength reduction techniques. Part I: Numerical studies considering non-associated plasticity. *Computers and Geotechnics* **70** (2015) 169–177.

[21] Tschuchnigg, F., Schweiger, H.F. and Sloan, S.W. Slope stability analysis by means of finite element limit analysis and finite element strength reduction techniques. Part II: Back analyses of a case history. *Computers and Geotechnics* **70** (2015), 178–189.

[22] Zienkiewicz, O.C., Humpheson, C., and Lewis, R.W. Associated and non-associated visco-plasticity and plasticity in soil mechanics. *Géotechnique* **25** (1975) 671–689.