ON A DIRECT PROCEDURE TO CONSTRUCT A BASIS FOR THE DIVERGENCE FREE POLYNOMIAL STRESS FIELD SPACE IN 3D

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Summary. A procedure is proposed for the direct construction of a basis of a space of symmetric divergence free polynomial stress fields in 3D. Such a basis may be used in the formulation of equilibrium finite elements.

1 INTRODUCTION

It appears to be common practice to form a divergence free basis for polynomial stress fields in a 3D domain, either by exploiting the Maxwell[1] or Morera[2] stress functions, or by forming a basis for the complete stress space and then applying a reduction process. Such a basis is referred to here as "hyperstatic" since the domain may be without limit and boundary traction conditions are not relevant.

The use of the stress functions is complicated by the fact that the dimension of a basis of stress functions is greater than the dimension of the hyperstatic space, so that zero and dependent stress fields need to be discarded[3]. On the other hand, reduction from a complete basis of the stress space requires a transformation that zeroes the body forces, and this could become quite complicated[4].

In this paper an alternative simpler procedure is proposed which directly constructs a hyperstatic basis ***S*** of polynomial stress fields in 3D. The stresses are considered in terms of their six contravariant components related to skew references axes *x*, *y*, *z*. The construction of the basis relies on considering two subspaces comprised of two types of stress field: type (1) where each of the three direct stress components (σ^{xx}, σ^{yy}, or σ^{zz}) is defined by the monomial $x^i y^j z^k$, and the condition for zero body force is enforced by including components of shear stress defined by appropriate monomials; and type (2) where each of the three shear stress components $\left(\sigma^{xy},\ \sigma^{yz},\ \sigma^{zx}\right)$ is defined by a monomial, e.g. $x^i y^j$, and is balanced with zero body forces by one of the other components of shear stress defined by another monomial.

2 EQUILIBRIUM CONDITIONS WITHIN A 3D DOMAIN

Infinitesimal quasi-static displacements are assumed so we use Cauchy stresses $\boldsymbol{\sigma}$ assuming an undeformed body. The equilibrium condition can be succinctly expressed as

$$\nabla\boldsymbol{\sigma}+\boldsymbol{b}=\mathbf{0} \tag{1}$$

where ∇ is the divergence operator, and $\boldsymbol{b}$ represents the vector of body force intensities which do not change with time.

3 DIVERGENCE-FREE STRESS FIELDS

Rotational equilibrium of the infinitesimal parallelepiped element in Figure 1 implies that the shear stresses are symmetric, and so

$$\begin{Bmatrix}\sigma^{yx}\\ \sigma^{zy}\\ \sigma^{xz}\end{Bmatrix}=\begin{Bmatrix}\sigma^{xy}\\ \sigma^{yz}\\ \sigma^{zx}\end{Bmatrix} \tag{2}$$

and the general equation of equilibrium with zero body forces has the form:

$$\begin{bmatrix}\partial/\partial x & 0 & 0\\ 0 & \partial/\partial y & 0\\ 0 & 0 & \partial/\partial z\end{bmatrix}\begin{Bmatrix}\sigma^{xx}\\ \sigma^{yy}\\ \sigma^{zz}\end{Bmatrix}+\begin{bmatrix}\partial/\partial y & 0 & \partial/\partial z\\ \partial/\partial x & \partial/\partial z & 0\\ 0 & \partial/\partial y & \partial/\partial x\end{bmatrix}\begin{Bmatrix}\sigma^{xy}\\ \sigma^{yz}\\ \sigma^{zx}\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ 0\end{Bmatrix}. \tag{3}$$

Note that the form of these equations applies equally when the reference axes (x,y,z) are orthogonal Cartesian or more generally oblique.

Two types of stress field are generated as bases of complementary subspaces

$$\boldsymbol{S}=\boldsymbol{S}_1\oplus\boldsymbol{S}_2\,. \tag{4}$$

Stress components are defined by monomials in x, y, and z.

3.1 Type 1

Consider the typical stress field:

$$\boldsymbol{\sigma}^T = \lfloor \sigma^{xx} \quad 0 \quad 0 \,|\, \sigma^{xy} \quad \sigma^{yz} \quad \sigma^{zx} \rfloor \quad \text{with } \sigma^{xx} = x^i y^j z^k \tag{5}$$

and

$$\lfloor \sigma^{xy} \quad \sigma^{yz} \quad \sigma^{zx} \rfloor = \frac{i}{2(j+1)(k+1)} \lfloor x^{(i-1)} y^{(j+1)} z^k \quad x^{(i-2)} y^{(j+1)} z^{(k+1)} \quad x^{(i-1)} y^j z^{(k+1)} \rfloor \begin{bmatrix} -(k+1) & 0 & 0 \\ 0 & (i-1) & 0 \\ 0 & 0 & -(j+1) \end{bmatrix} \tag{6}$$

These expressions for shear stresses apply for all $i > 1$, so to complete the definitions when $i < 2$, consider:

When

$$i = 1,\ \sigma^{xx} = xy^j z^k \text{, and the body force } b_x = -y^j z^k . \tag{7}$$

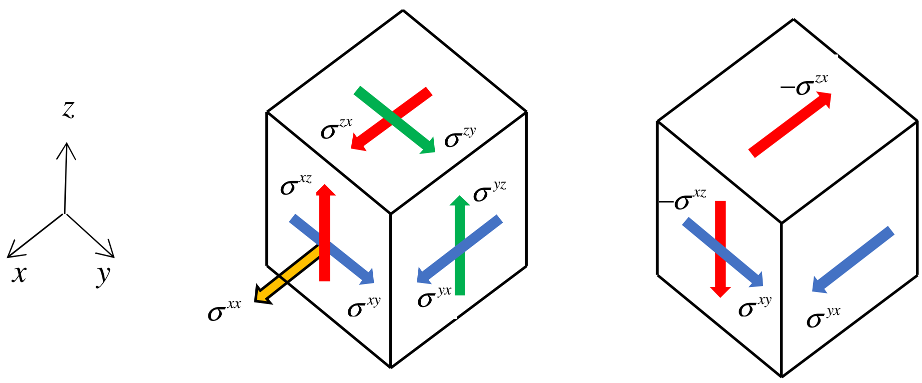
To counterbalance this force we only require σ^{xy} to be $-\dfrac{1}{(j+1)} y^{(j+1)} z^k$;

When

$$i = 0,\ \sigma^{xx} \text{ is not a function of } x, \tag{8}$$

and so is self-equilibrated without the need for shear stress fields.

Hence when the Type 1 stress fields are driven by σ^{xx}, $(n+1)(n+2)(n+3)/6$ admissible stress fields are defined for **all** values of *i, j,* and *k*. By appealing to cyclic symmetry when stress fields are driven by σ^{yy} or σ^{zz}, the total number of independent stress fields of type 1 is $3(n+1)(n+2)(n+3)/6$.



Components of a Type 1 stress field　　Components of a Type 2 stress field

Figure 1: Stress components on an infinitesimal parallelepiped.

3.2 Type 2

Consider the typical stress field:

$$\boldsymbol{\sigma}^T = \lfloor 0 \quad 0 \quad 0 \mid \sigma^{xy} \quad 0 \quad \sigma^{zx} \rfloor \tag{9}$$

with

$$\lfloor \sigma^{xy} \quad 0 \quad \sigma^{zx} \rfloor = \frac{1}{(k+1)} \lfloor x^0 y^j z^k \quad 0 \quad x^0 y^{(j-1)} z^{(k+1)} \rfloor \begin{bmatrix} (k+1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -j \end{bmatrix} \tag{10}$$

The first field holds for all values of indices j and k since, when

$$j = 1,\ \sigma^{xy} = yz^k \text{ and } \sigma^{zx} = -\frac{z^{(k+1))}}{(k+1)}, \tag{11}$$

and when

$$j = 0,\ \ \sigma^{xy} = z^k \text{ and } \sigma^{zx} = 0\,. \tag{12}$$

Hence, the number of combinations of j and k with j,k in the range 0 to n and $(j+k) \le n$ is $(n+1)(n+2)/2$. Thus, by again appealing to cyclic symmetry with alternative pairs of shear stress fields, the total number of independent stress fields of type 2 is $3(n+1)(n+2)/2$.

The total number of independent stress fields of types 1 and 2 is thus:

$$3[\,(n+1)(n+2)(n+3)/6 \ + \ (n+1)(n+2)/2\,] = \ n_s = (n+1)(n+2)(n+6)/2\,, \tag{13}$$

and this number agrees with that derived in reference[3].

4 CONCLUDING REMARKS

- The proposed procedure defines directly a basis for the space of polynomial stress fields with zero divergence up to any desired degree;
- Each component of stress is defined as a monomial;
- The directness of the definitions should simplify computations and hence is expected to improve computational efficiency;
- The procedure should be of particular benefit in the implementation of hybrid equilibrium solid finite elements in computational models that complement displacement based conforming models in the context of adaptive procedures.

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