Impulsive Fractional Boundary Value Problems Involving Volterra–Fredholm Integral Operators

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ABSTRACT

This article investigates a class of nonlinear impulsive fractional integro-differential equations involving Riemann–Liouville fractional derivatives and integral boundary conditions. The model incorporates Volterra–Fredholm integral operators to represent both memory effects and non-local interactions in systems experiencing impulsive changes. To address the analytical challenges posed by the nonlocal and impulsive features, we develop a novel hybrid fixed-point approach that combines the Banach contraction principle with Krasnoselskii's theorem in Banach spaces. We establish rigorous existence and uniqueness results under suitable conditions. A detailed example is provided to demonstrate the effectiveness and applicability of the proposed method.

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Symbols

$\mathfrak{D}_0^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r})$	Riemann–Liouville fractional derivative of order $\mathfrak{w} \in (1, 2)$ of $\mathfrak{u}$.
$\mathfrak{u}(\mathfrak{r})$	Unknown function (solution) defined on $\psi = [0, 1]$
$\mathfrak{F}(\mathfrak{r}, \cdot)$	Nonlinear continuous function from $\psi \times \mathbb{R}^4$ to $\mathbb{R}$
$(\mathcal{P}\mathfrak{u})(\mathfrak{r})$	Integral operator defined as $\int_0^{\mathfrak{r}} \mathfrak{p}(\mathfrak{r}, \sigma)\mathfrak{u}(\sigma)\, d\sigma$
$(\mathcal{H}\mathfrak{u})(\mathfrak{r})$	Integral operator defined as $\int_0^{1} \mathfrak{h}(\mathfrak{r}, \sigma)\mathfrak{u}(\sigma)\, d\sigma$
ψ	Interval [0, 1], the domain of $\mathfrak{u}$
$\mathfrak{u}_k$	Impulse magnitude at $\mathfrak{r}_k$, a real constant

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$\mathfrak{r}_k$	Impulse instants: $0 < \mathfrak{r}_0 < \mathfrak{r}_1 < \ldots < \mathfrak{r}_m = 1$
$\mathfrak{u}(\mathfrak{r}_k^+),\ \mathfrak{u}(\mathfrak{r}_k^-)$	Right and left limits of $\mathfrak{u}$ at the impulsive point $\mathfrak{r}_k$
$\Delta\mathfrak{u}\|_{\mathfrak{r}=\mathfrak{r}_k}$	Jump of $\mathfrak{u}$ at the impulse point $\mathfrak{r}_k$
$\mathfrak{a}, \mathfrak{d}$	Real parameters
μ	Real constant in the nonlocal integral boundary condition
κ	Real number in $(0, 1)$
$\mathfrak{e}_1, \mathfrak{e}_2$	Points in $[0, 1]$ at which derivatives

1 Introduction

Fractional calculus has emerged as a powerful tool for modeling complex systems with memory effects [1,2]. Many physical, biological, economic, control-theoretic, stochastic, and engineering phenomena can be effectively described using fractional differential equations (FDEs). In recent years, FDEs have attracted significant attention [3] due to their ability to capture hereditary properties and anomalous diffusion.

Impulsive differential equations, which model systems undergoing sudden changes, have been extensively studied [4]. However, despite recent advancements, the combination of impulsive Riemann-Liouville derivatives with Volterra-Fredholm integro-differential equations (FVFIDEs) and integral boundary conditions remains largely unexplored [5,6]. Addressing this gap is crucial, as such equations arise in real-world applications, including biology, physics, and engineering.

Boundary value problems involving the Riemann-Liouville fractional integral and the Caputo fractional derivative have also gained significant interest. Studies such as [7,8] have explored three-point fractional boundary value problems, while [9–13] extended these investigations to different fractional settings. Despite these efforts, further research is needed to establish the existence and uniqueness of solutions fornonlinear impulsive FDEs under new boundary conditions.

Recent studies [14–16] have demonstrated the effectiveness of fractional calculus in modeling memory-dependent systems. While many works focus oninitial value problems for fractional-order differential equations, boundary value problems for nonlinear FDEs remain an active research area with unresolved questions [17–20].

Recent advances in fractional impulsive systems have explored various extensions, including higher-order Caputo fractional integrodifferential inclusions of Volterra–Fredholm type with impulses and infinite delay [21–25]. While Sobolev-type systems have been examined for fractional delay integrodifferential equations of order [26]. Additionally, existence and controllability for fractional evolution inclusions of Clarke's subdifferential type [27]. Our work builds on these foundations by introducing a novel analytical framework for nonlinear impulsive fractional integro-differential equations with integral boundary conditions, addressing gaps in the existing theory.

While initial value problems (IVPs) are commonly studied, boundary value problems (BVPs) provide a more suitable framework for systems influenced by global or distributed constraints, particularly when dealing with Riemann–Liouville derivatives whose nonlocality reflects the system's entire history. This study focuses on nonlinear impulsive fractional Volterra–Fredholm integro-differential equations (FVFIDEs) in Banach spaces with new nonlocal and integral boundary conditions, which are essential for modeling state discontinuities, delayed feedback, and spatial constraints—phenomena not adequately addressed by IVPs [28]. Thus, the BVP approach is both necessary and more effective for analyzing such systems.

In [29], the authors present results on non-local impulsive implicit Caputo-Hadamard fractional differential equations.

The present study introduces a novel analytical framework for investigating nonlinear impulsive fractional Volterra–Fredholm integro-differential equations (FVFIDEs) with Riemann–Liouville derivatives in Banach spaces under newly formulated boundary conditions. Unlike existing works that typically address either non-impulsive or Caputo-type models with standard initial or Dirichlet-type boundary conditions, our approach uniquely incorporates impulsive effects, integral nonlocal multipoint conditions, and mixed-type fractional operators in a unified system. The boundary condition in Eq. (3) represents a new hybrid formulation involving a linear combination of fractional derivatives at interior points and an integral constraint, which to the best of our knowledge, has not been previously addressed in the literature. Additionally, our analysis provides general existence and uniqueness results by combining fixed point theory with measures of noncompactness, thereby extending current theoretical tools to a more complex and realistic class of fractional systems.

The present study introduces a novel analytical framework for investigating nonlinear impulsive fractional Volterra–Fredholm integro-differential equations (FVFIDEs) with Riemann–Liouville derivatives in Banach spaces under newly formulated boundary conditions. Unlike existing works that typically address either non-impulsive or Caputo-type models with standard initial or Dirichlet-type boundary conditions, our approach uniquely incorporates impulsive effects, integral nonlocal multipoint conditions, and mixed-type fractional operators in a unified system. The boundary condition in Eq. (3) represents a new hybrid formulation involving a linear combination of fractional derivatives at interior points and an integral constraint, which to the best of our knowledge, has not been previously addressed in the literature. Additionally, our analysis provides general existence and uniqueness results by combining fixed point theory with measures of noncompactness, thereby extending current theoretical tools to a more complex and realistic class of fractional systems. This comprehensive treatment offers broader applicability and deeper insights into dynamic processes governed by memory and impulse effects in applied science and engineering.

This study aims to investigate solutions to Riemann-Liouville fractional derivative problems in Banach spaces for systems governed by nonlinear impulsive FVFIDEs with new boundary conditions. Specifically, we analyze the equation:

$$\mathfrak{D}_0^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}) = \mathfrak{F}(\mathfrak{r}, \mathfrak{u}(\mathfrak{r}), \mathfrak{D}^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}), (\mathcal{P}\mathfrak{u})(\mathfrak{r}), (\mathcal{H}\mathfrak{u})(\mathfrak{r})), \quad \mathfrak{r} \in \psi = [0, 1], \tag{1}$$

subject to the conditions:

$$\mathfrak{u}(\mathfrak{r}_k^+) = \mathfrak{u}(\mathfrak{r}_k^-) + \mathfrak{u}_k, \quad \mathfrak{u}_k \in \mathbb{R}, \quad k = 1, \dots, m, \tag{2}$$

$$\mathfrak{u}(0) = 0, \quad \mathfrak{a}\mathfrak{u}'(\mathfrak{e}_1) + \mathfrak{d}\mathfrak{u}'(\mathfrak{e}_2) = \mu \int_0^{\kappa} \mathfrak{u}(\sigma)d\sigma, \quad \mathfrak{a}, \mathfrak{d} \in \mathbb{R}, \quad 0 < \kappa < 1. \tag{3}$$

Here, $\mathfrak{D}^{\mathfrak{w}}$ denotes the Riemann–Liouville fractional derivative of order $1 < \mathfrak{w} < 2$. The nonlinear function $\mathfrak{F} : [0, 1] \times \mathbb{R}^4 \to \mathbb{R}$ is assumed to be continuous. The integral operators $(\mathcal{P}\mathfrak{u})(\mathfrak{r})$ and $(\mathcal{H}\mathfrak{u})(\mathfrak{r})$ are defined as:

$$(\mathcal{P}\mathfrak{u})(\mathfrak{r}) = \int_0^{\mathfrak{r}} \mathfrak{p}(\mathfrak{r}, \sigma)\mathfrak{u}(\sigma)d\sigma, \quad (\mathcal{H}\mathfrak{u})(\mathfrak{r}) = \int_0^1 \mathfrak{h}(\mathfrak{r}, \sigma)\mathfrak{u}(\sigma)d\sigma,$$

where the kernels $\mathfrak{p}, \mathfrak{h} : [0, 1]^2 \to \mathbb{R}$ are continuous functions. The impulse points satisfy:

$$0 < \mathfrak{r}_1 < \mathfrak{r}_2 < \dots < \mathfrak{r}_m = 1, \quad \Delta\mathfrak{u}\big|_{\mathfrak{r}=\mathfrak{r}_k} = \mathfrak{u}(\mathfrak{r}_k^+) - \mathfrak{u}(\mathfrak{r}_k^-), \quad k = 1, \dots, m.$$

To analyze the existence and uniqueness of solutions to the nonlinear impulsive fractional integro-differential Eqs. (1)–(3) involving the Riemann–Liouville fractional derivative in a Banach space, we employ two complementary fixed-point theorems: the Krasnoselskii fixed-point theorem and the Banach contraction principle. The Krasnoselskii theorem is particularly effective due to the composite nature of the nonlinear operator, which can be decomposed into a sum of a compact and a contraction operator. This structure arises naturally from the integral representation of the solution and the properties of the fractional and nonlocal terms. In cases where the nonlinear operator satisfies a global Lipschitz condition with a sufficiently small constant, we use the Banach contraction principle, which ensures not only existence but also uniqueness of the solution. The combination of these tools is well-suited for handling the impulsive effects and nonlocal boundary conditions, making them highly applicable to the present fractional differential system

We thank the reviewer for highlighting the important issue of impulsive discontinuities in the context of Riemann–Liouville fractional derivatives. In our study, we adopt a formulation in which the Riemann–Liouville derivative $\mathfrak{D}_0^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r})$ is considered piecewise on each subinterval $(\mathfrak{r}_{k-1}, \mathfrak{r}_k)$, where $\{\mathfrak{r}_k\}_{k=1}^m$ denote the impulsive points with $\mathfrak{r}_0 = 0$, $\mathfrak{r}_{m+1} = 1$. The function $\mathfrak{u}$ is assumed to be sufficiently regular on each subinterval, and the impulsive conditions are imposed through regulated jumps. This piecewise approach ensures that the Riemann–Liouville derivative is well-defined within each continuous segment, and the influence of impulses is accounted for through the matching conditions at the discontinuities. Such treatment aligns with existing frameworks in the literature dealing with impulsive fractional differential equations.

The remaining content is structured as follows: Section 2 deals with the fundamental ideas and Theorems that will underpin the results. The uniqueness of solutions (1)–(3) and the existence of the system under adequate assumptions are demonstrated in Section 3. In Section 4, we illustrate our results with a relevant example.

2 Auxiliary Results

Before presenting our primary results, we introduce essential Definitions, preliminary concepts, and assumptions that will be used in our subsequent discussion, see [30–32].

Let $\psi = [0, 1]$ be the domain of interest. We define the Banach space $\mathbb{C}(\psi, \mathbb{R})$ as the set of all real-valued continuous functions on $[0, 1]$, equipped with the supremum norm:

$$\|\mathfrak{u}\| = \sup_{\mathfrak{r}\in[0,1]} |\mathfrak{u}(\mathfrak{r})|.$$

Define the subset $\mathfrak{z} \subset \mathbb{C}(\psi, \mathbb{R})$ as: $\mathfrak{z} = \{\mathfrak{u} \in \mathbb{C}(\psi, \mathbb{R}) : \mathfrak{u}(\mathfrak{r}) \geq 0 \text{ for all } \mathfrak{r} \in \psi\}$. This subset forms a closed convex cone of nonnegative functions.

We consider functions $\mathfrak{u}$ that belong to the space $\mathbb{PC}^1([0, 1])$, consisting of functions that are continuously differentiable except at a finite number of impulse points $\{\mathfrak{r}_k\}_{k=1}^m$, and have finite left- and right-hand limits at each impulse point:

$$\mathfrak{u}(\mathfrak{r}_k^+) = \lim_{\eta\to 0^+} \mathfrak{u}(\mathfrak{r}_k + \eta), \quad \mathfrak{u}(\mathfrak{r}_k^-) = \lim_{\eta\to 0^-} \mathfrak{u}(\mathfrak{r}_k + \eta).$$

Definition 1: *([1,2]) For* $\mathfrak{w} > 0$*, the integral*

$$\mathfrak{I}_{0^+}^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}) = \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1}\mathfrak{u}(\sigma)\, d\sigma, \tag{4}$$

is called the Riemann–Liouville (R–L) fractional integral of order $\mathfrak{w}$.

Definition 2: *([1,2]) The Riemann–Liouville fractional derivative of order* $\mathfrak{w}$ *(where* $n-1<\mathfrak{w}<n$, $n\in\mathbb{N}$*) for a function* $\mathfrak{u}(\mathfrak{r})$ *is defined by*

$$\mathfrak{D}^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r})=\frac{d^n}{d\mathfrak{r}^n}\left(\mathfrak{I}^{n-\mathfrak{w}}\mathfrak{u}(\mathfrak{r})\right).$$

For $0<\mathfrak{w}<1$, *this reduces to*

$$\mathfrak{D}^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r})=\frac{1}{\Gamma(1-\mathfrak{w})}\frac{d}{d\mathfrak{r}}\int_0^{\mathfrak{r}}(\mathfrak{r}-\sigma)^{-\mathfrak{w}}\mathfrak{u}(\sigma)\,d\sigma.$$

Definition 3: *([1,12]) The Caputo fractional derivative of order* $\mathfrak{w}$ *(where* $n-1<\mathfrak{w}<n$, $n\in\mathbb{N}$*) for a function* $\mathfrak{u}(\mathfrak{r})$ *is defined by*

$${}^{C}\mathfrak{D}^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r})=\mathfrak{I}^{n-\mathfrak{w}}\left(\frac{d^n}{d\mathfrak{r}^n}\mathfrak{u}(\mathfrak{r})\right).$$

For $0<\mathfrak{w}<1$, *this simplifies to*

$${}^{C}\mathfrak{D}^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r})=\frac{1}{\Gamma(1-\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\sigma)^{-\mathfrak{w}}\mathfrak{u}'(\sigma)\,d\sigma.$$

Lemma 1: *([1,2])(Semigroup Property of Fractional Integrals) For* $\mathfrak{w}_1,\mathfrak{w}_2>0$, *the following identities hold:*

1) $\mathfrak{I}^{\mathfrak{w}_1}\left(\mathfrak{I}^{\mathfrak{w}_2}\mathfrak{u}(\mathfrak{r})\right)=\mathfrak{I}^{\mathfrak{w}_1+\mathfrak{w}_2}\mathfrak{u}(\mathfrak{r})$, 2) $\mathfrak{D}^{\mathfrak{w}_1}\mathfrak{I}^{\mathfrak{w}_2}\mathfrak{u}(\mathfrak{r})=\mathfrak{I}^{\mathfrak{w}_1-\mathfrak{w}_2}\mathfrak{u}(\mathfrak{r})$, $\forall\mathfrak{r}\in[\mathfrak{a},\mathfrak{d}]$.

Lemma 2: *([1]) Let* $\mathfrak{u}\in\mathcal{C}^{\nu}[0,\xi]$, $\mathfrak{w}\in(\nu-1,\nu)$, $\nu\in\mathbb{N}$. *Then, for all* $\mathfrak{r}\in\psi$,

$$\mathfrak{I}^{\mathfrak{w}}_{0+}\mathfrak{D}^{\mathfrak{w}}_{0+}\mathfrak{u}(\mathfrak{r})=\mathfrak{u}(\mathfrak{r})-\sum_{k=0}^{\nu-1}\frac{\mathfrak{r}^k}{k!}\mathfrak{u}^{(k)}(0).$$

Theorem 1: *([33]) (Banach's Fixed Point Theorem) Let* Ω *be a nonempty complete metric space, and let* $\varphi:\Omega\to\Omega$ *be a contraction mapping. Then, there exists a unique point* $\mathfrak{e}\in\Omega$ *such that* $\varphi(\mathfrak{e})=\mathfrak{e}$.

Theorem 2: *[33] ("Krasnoselskii's fixed point theorem")*

Consider the Banach space $(\Phi;\|.\|)$ *and* Ω *as a non-empty closed bounded convex subset. If* $\mathfrak{Z}_1$ *and* $\mathfrak{Z}_2$ *transform* Ω *into* Φ, *then for every* $\mathfrak{s}_1,\mathfrak{s}_2\in\Omega$, *we have*

(a) $\mathfrak{Z}_1\mathfrak{s}_1+\mathfrak{Z}_2\mathfrak{s}_2\in\Omega$. *for all* $\mathfrak{s}_1,\mathfrak{s}_2\in\Omega$,

(b) $\mathfrak{Z}_1$ *is continuous and compact,*

(c) $\mathfrak{Z}_2$ *is a contraction with constant* $\ell<1$. *Then* $\mathfrak{Z}_1\mathfrak{s}+\mathfrak{Z}_2\mathfrak{s}=\mathfrak{s}$.

Lemma 3: *Assuming* $\mathfrak{u}$ *is an impulsive IBC solution,*

$$\xi:\psi\times\mathbb{R}^4\longrightarrow\mathbb{R}$$

where

$$\xi(\mathfrak{r})=\mathfrak{F}(\mathfrak{r},\mathfrak{u}(\mathfrak{r}),\mathfrak{D}^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}),(P\mathfrak{u})(\mathfrak{r}),(\delta\mathfrak{u})(\mathfrak{r})),\mathfrak{r}\in\psi=[0,1],$$

is given by

$$\mathfrak{D}^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}) = \xi(\mathfrak{r}), \quad 1 < \mathfrak{w} \leq 2 \tag{5}$$

$$\mathfrak{u}(\mathfrak{r}_k^+) = \mathfrak{u}(\mathfrak{r}_k^-) + \mathfrak{u}_k, \ \ \mathfrak{u}_k \in \mathbb{R}, \ \ k = 1, ..., m, \tag{6}$$

$$\mathfrak{u}(0) = 0, \ \ \mathfrak{a}\mathfrak{u}'(\mathfrak{e}_1) + \mathfrak{d}\mathfrak{u}'(\mathfrak{e}_2) = \mu \int_0^{\kappa} \mathfrak{u}(\sigma)d\sigma, \tag{7}$$

if and only if

$$\mathfrak{u}(\mathfrak{r}) = \begin{cases} \dfrac{1}{\Gamma(\mathfrak{w})}\displaystyle\int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\xi(\sigma)d\sigma + \dfrac{\mathfrak{r}}{\Im}\Big\{\dfrac{1}{\Gamma(\mathfrak{w}-1)}\displaystyle\int_0^{\mathfrak{e}_1} (\mathfrak{e}_1-\sigma)^{\mathfrak{w}-2}\xi(\sigma)d\sigma \\ +\dfrac{1}{\Gamma(\mathfrak{w}-1)}\displaystyle\int_0^{\mathfrak{e}_2} (\mathfrak{e}_2-\sigma)^{\mathfrak{w}-2}\xi(\sigma)d\sigma \\ -\dfrac{\mu}{\Gamma(\mathfrak{w})}\displaystyle\int_0^{\kappa}\Big(\int_0^{\sigma}(\sigma-\varphi)^{\mathfrak{w}-1}\xi(\varphi)d\varphi\Big)d\sigma\Big\}, \textit{for } \mathfrak{r}\in[0,\mathfrak{r}_1), \\ \mathfrak{u}_1 + \dfrac{1}{\Gamma(\mathfrak{w})}\displaystyle\int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\ \xi(\sigma)d\sigma + \dfrac{\mathfrak{r}}{\Im}\Big\{\dfrac{1}{\Gamma(\mathfrak{w}-1)}\displaystyle\int_0^{\mathfrak{e}_1} (\mathfrak{e}_1-\sigma)^{\mathfrak{w}-2}\xi(\sigma)d\sigma \\ +\dfrac{1}{\Gamma(\mathfrak{w}-1)}\displaystyle\int_0^{\mathfrak{e}_2} (\mathfrak{e}_2-\sigma)^{\mathfrak{w}-2}\xi(\sigma)d\sigma \\ -\dfrac{\mu}{\Gamma(\mathfrak{w})}\displaystyle\int_0^{\kappa}\Big(\int_0^{\sigma}(\sigma-\varphi)^{\mathfrak{w}-1}\xi(\varphi)d\varphi\Big)d\sigma\Big\}, \textit{ for } \mathfrak{r}\in(\mathfrak{r}_1,\mathfrak{r}_2), \\ \mathfrak{u}_1+\mathfrak{u}_2 + \dfrac{1}{\Gamma(\mathfrak{w})}\displaystyle\int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\ \xi(\sigma)d\sigma + \dfrac{\mathfrak{r}}{\Im}\Big\{\dfrac{1}{\Gamma(\mathfrak{w}-1)}\displaystyle\int_0^{\mathfrak{e}_1} (\mathfrak{e}_1-\sigma)^{\mathfrak{w}-2}\ \xi(\sigma)d\sigma \\ +\dfrac{1}{\Gamma(\mathfrak{w}-1)}\displaystyle\int_0^{\mathfrak{e}_2} (\mathfrak{e}_2-\sigma)^{\mathfrak{w}-2}\xi(\sigma)d\sigma \\ -\dfrac{\mu}{\Gamma(\mathfrak{w})}\displaystyle\int_0^{\kappa}\Big(\int_0^{\sigma}(\sigma-\varphi)^{\mathfrak{w}-1}\xi(\varphi)d\varphi\Big)d\sigma\Big\}, \text{ for } \mathfrak{r}\in(\mathfrak{r}_2,\mathfrak{r}_3), \\ \dots \\ \displaystyle\sum_{j=1}^{i}\mathfrak{u}_j + \dfrac{1}{\Gamma(\mathfrak{w})}\displaystyle\int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\ \xi(\sigma)d\sigma + \dfrac{\mathfrak{r}}{\Im}\Big\{\dfrac{1}{\Gamma(\mathfrak{w}-1)}\displaystyle\int_0^{\mathfrak{e}_1} (\mathfrak{e}_1-\sigma)^{\mathfrak{w}-2}\xi(\sigma)d\sigma \\ +\dfrac{1}{\Gamma(\mathfrak{w}-1)}\displaystyle\int_0^{\mathfrak{e}_2} (\mathfrak{e}_2-\sigma)^{\mathfrak{w}-2}\xi(\sigma)d\sigma \\ -\dfrac{\mu}{\Gamma(\mathfrak{w})}\displaystyle\int_0^{\kappa}\Big(\int_0^{\sigma}(\sigma-\varphi)^{\mathfrak{w}-1}\xi(\varphi)d\varphi\Big)d\sigma\Big\}, \textit{ for } \mathfrak{r}\in(\mathfrak{r}_j,\mathfrak{r}_{j+1}). \end{cases}$$

where

$$\Im = \Big(\frac{\mu\kappa^2}{2} - \mathfrak{a}\mathfrak{e}_1 - \mathfrak{d}\mathfrak{e}_2\Big) \neq 0$$

Proof: Assume $\mathfrak{u}$ satisfies (5) and (6). If $\mathfrak{r} \in [0, \mathfrak{r}_1)$, then

$$\mathfrak{D}^{\mathfrak{w}}\mathfrak{u}(\mathfrak{r}) = \xi(\mathfrak{r}),\ \mathfrak{r} \in [0, \mathfrak{r}_1),$$

$$\mathfrak{u}(0) = 0,\ \mathfrak{a}\mathfrak{u}'(\mathfrak{e}_1) + \mathfrak{d}\mathfrak{u}'(\mathfrak{e}_2) = \mu \int_0^{\kappa} \mathfrak{u}(\sigma)d\sigma. \tag{8}$$

From Lemma 1, we have

$$\begin{aligned}\mathfrak{u}(\mathfrak{r}) = &\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\ \xi(\sigma)d\sigma + \frac{\mathfrak{r}}{\mathfrak{I}}\Bigg\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1} (\mathfrak{e}_1-\sigma)^{\mathfrak{w}-2}\ \xi(\sigma)d\sigma \\ &+ \frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2} (\mathfrak{e}_2-\sigma)^{\mathfrak{w}-2}\ \xi(\sigma)d\sigma \\ &- \frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma} (\sigma-\varphi)^{\mathfrak{w}-1}\xi(\varphi)d\varphi\right)d\sigma\Bigg\}\end{aligned} \tag{9}$$

If $\mathfrak{r} \in (\mathfrak{r}_1, \mathfrak{r}_2)$, then we have

$$\begin{aligned}\mathfrak{u}(\mathfrak{r}) = &\ \mathfrak{u}(\mathfrak{r}_1^+) - \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}_1} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\ \xi(\sigma)d\sigma + \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\ \xi(\sigma)d\sigma \\ &+ \frac{\mathfrak{r}}{\mathfrak{I}}\Bigg\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1} (\mathfrak{e}_1-\sigma)^{\mathfrak{w}-2}\ \xi(\sigma)d\sigma \\ &+ \frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2} (\mathfrak{e}_2-\sigma)^{\mathfrak{w}-2}\ \xi(\sigma)d\sigma \\ &- \frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma} (\sigma-\varphi)^{\mathfrak{w}-1}\xi(\varphi)d\varphi\right)d\sigma\Bigg\} \\ = &\ \mathfrak{u}(\mathfrak{r}_1^+) + \mathfrak{u}_1 - \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}_1} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\ \xi(\sigma)d\sigma \\ &+ \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\ \xi(\sigma)d\sigma \\ &+ \frac{\mathfrak{r}}{\mathfrak{I}}\Bigg\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1} (\mathfrak{e}_1-\sigma)^{\mathfrak{w}-2}\ \xi(\sigma)d\sigma \\ &+ \frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2} (\mathfrak{e}_2-\sigma)^{\mathfrak{w}-2}\ \xi(\sigma)d\sigma \\ &- \frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma} (\sigma-\varphi)^{\mathfrak{w}-1}\xi(\varphi)d\varphi\right)d\sigma\Bigg\}, \\ &\ldots\ldots\ldots\ldots\ldots \\ = &\ \mathfrak{u}_1 + \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} (\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\ \xi(\sigma)d\sigma \\ &+ \frac{\mathfrak{r}}{\mathfrak{I}}\Bigg\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1} (\mathfrak{e}_1-\sigma)^{\mathfrak{w}-2}\ \xi(\sigma)d\sigma + \frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2} (\mathfrak{e}_2-\sigma)^{\mathfrak{w}-2}\ \xi(\sigma)d\sigma\end{aligned}$$

$$- \frac{\mu}{\Gamma(\mathfrak{w})} \int_0^{\kappa} \left(\int_0^{\sigma} (\sigma - \varphi)^{\mathfrak{w}-1} \xi(\varphi) d\varphi \right) d\sigma \Bigg\}$$

If $\mathfrak{r} \in (\mathfrak{r}_2, \mathfrak{r}_3)$, then we have

$$\begin{aligned}
\mathfrak{u}(\mathfrak{r}) = \ & \mathfrak{u}(\mathfrak{r}_2^+) - \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}_2} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \ \xi(\sigma) d\sigma + \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \ \xi(\sigma) d\sigma \\
& + \frac{\mathfrak{r}}{\mathfrak{I}} \Bigg\{ \frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_1} (\mathfrak{e}_1 - \sigma)^{\mathfrak{w}-2} \ \xi(\sigma) d\sigma \\
& + \frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_2} (\mathfrak{e}_2 - \sigma)^{\mathfrak{w}-2} \ \xi(\sigma) d\sigma \\
& - \frac{\mu}{\Gamma(\mathfrak{w})} \int_0^{\kappa} \left(\int_0^{\sigma} (\sigma - \varphi)^{\mathfrak{w}-1} \xi(\varphi) d\varphi \right) d\sigma \Bigg\} \\
= \ & \mathfrak{u}(\mathfrak{r}_2^+) + \mathfrak{u}_2 - \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}_2} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \ \xi(\sigma) d\sigma \\
& + \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \ \xi(\sigma) d\sigma \\
& + \frac{\mathfrak{r}}{\mathfrak{I}} \Bigg\{ \frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_1} (\mathfrak{e}_1 - \sigma)^{\mathfrak{w}-2} \ \xi(\sigma) d\sigma \\
& + \frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_2} (\mathfrak{e}_2 - \sigma)^{\mathfrak{w}-2} \ \xi(\sigma) d\sigma \\
& - \frac{\mu}{\Gamma(\mathfrak{w})} \int_0^{\kappa} \left(\int_0^{\sigma} (\sigma - \varphi)^{\mathfrak{w}-1} \xi(\varphi) d\varphi \right) d\sigma \Bigg\}, \\
& \ldots\ldots\ldots\ldots\ldots \\
= \ & \mathfrak{u}_1 + \mathfrak{u}_2 + \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \ \xi(\sigma) d\sigma \\
& + \frac{\mathfrak{r}}{\mathfrak{I}} \Bigg\{ \frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_1} (\mathfrak{e}_1 - \sigma)^{\mathfrak{w}-2} \ \xi(\sigma) d\sigma \\
& + \frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_2} (\mathfrak{e}_2 - \sigma)^{\mathfrak{w}-2} \ \xi(\sigma) d\sigma \\
& - \frac{\mu}{\Gamma(\mathfrak{w})} \int_0^{\kappa} \left(\int_0^{\sigma} (\sigma - \varphi)^{\mathfrak{w}-1} \xi(\varphi) d\varphi \right) d\sigma \Bigg\}.
\end{aligned}$$

If $\mathfrak{r} \in (\mathfrak{r}_j, \mathfrak{r}_{j+1})$ and we again try to apply Lemma 1, we get the other equation of (5). □

3 Main Results

To demonstrate the main conclusions, we need the following presumptions:

(H_1) $\mathfrak{F} : \psi = [0, 1] \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ are continuous.

(H_2) There exist non-negative constants $\lambda_1 > 0, 0 < \lambda_2 < 1, \lambda_3, \lambda_4 > 0$, $\forall \mathfrak{r} \in \psi$ such that the function $\mathfrak{F}$ satisfies

$$\|\mathfrak{F}(\mathfrak{r}, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4) - \mathfrak{F}(\mathfrak{r}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4)\| \le \lambda_1\|\mathfrak{u}_1 - \mathfrak{v}_1\| + \lambda_2\|\mathfrak{u}_2 - \mathfrak{v}_2\| + \lambda_3\|\mathfrak{u}_3 - \mathfrak{v}_3\| + \lambda_4\|\mathfrak{u}_4 - \mathfrak{v}_4\|$$

(H_3) There exist constants $\mathcal{P} > 0$ and $\mathcal{H}^* > 0$ such that

$$\mathcal{P}^* = \left|\sup_{\mathfrak{r}\in\psi} \int_0^{\mathfrak{r}} \left|\mathfrak{p}(\mathfrak{r}, \sigma)d\sigma\right| < \infty, \quad \mathcal{H}^* = \left|\sup_{\mathfrak{r}\in\psi} \int_0^1 \left|\mathfrak{h}(\mathfrak{r}, \sigma)d\sigma\right| < \infty$$

(H_4) There exists $\delta, \nu, \zeta, \eta, \tau \in \mathfrak{C}(\psi, \mathbb{R})$, with

$$\delta^* = \sup_{\mathfrak{r}\in\psi} \delta(\mathfrak{r}) < 1, \quad \nu^* = \sup_{\mathfrak{r}\in\psi} \nu(\mathfrak{r}) < 1, \quad \zeta^* = \sup_{\mathfrak{r}\in\psi} \zeta(\mathfrak{r}) < 1,$$
$$\eta^* = \sup_{\mathfrak{r}\in\psi} \eta(\mathfrak{r}) < 1, \quad \tau^* = \sup_{\mathfrak{r}\in\psi} \tau(\mathfrak{r}) < 1$$

then

$$\delta(\mathfrak{r}, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4) + \delta(\mathfrak{r}) + \nu(\mathfrak{r})|\mathfrak{u}_1| + \zeta(\mathfrak{r})|\mathfrak{u}_2| + \eta(\mathfrak{r})|\mathfrak{u}_3| + \tau(\mathfrak{r})|\mathfrak{u}_4|$$

(H_5) There exists a constant $\mathcal{M} > 0$ then $\mathfrak{F}(\mathfrak{r}, \mathfrak{u}, \mathfrak{v}, \mathfrak{z}, \mathfrak{r}) \le \mathcal{M}$, for $a.e \quad \mathfrak{r} \in \psi$

Theorem 3: *Assume that (H_1), (H_2), and (H_3) hold. Then, the following inequality holds:*

$$\Xi := \left[\frac{\lambda_1 + \left[\lambda_3\mathcal{P}^* + \lambda_4\mathcal{H}^*\right]}{1 - \lambda_2}\right]\left(\frac{1}{\Gamma(\mathfrak{w} + 1)} + \frac{1}{\Im}\left[\frac{\mathfrak{e}_1^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} + \frac{\mathfrak{e}_2^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} - \frac{\mu\kappa^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}\right]\right) < 1 \tag{10}$$

If (10) is satisfied, then the system (1)–(3) has a unique solution in $\mathfrak{r} \in \mathfrak{C}(\psi, \mathbb{R})$.

Proof: Consider the operator $\Omega : \mathfrak{C}(\psi, \mathbb{R}) \to \mathfrak{C}(\psi, \mathbb{R})$ defined by

$$\begin{aligned}\Omega(\mathfrak{u}(\mathfrak{r})) = {} & \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \quad \xi_{\mathfrak{u}}(\sigma)d\sigma + \frac{\mathfrak{r}}{\Im}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_1} (\mathfrak{e}_1 - \sigma)^{\mathfrak{w}-2} \quad \xi_{\mathfrak{u}}(\sigma)d\sigma \right. \\ & \left. + \frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_2} (\mathfrak{e}_2 - \sigma)^{\mathfrak{w}-2} \quad \xi_{\mathfrak{u}}(\sigma)d\sigma - \frac{\mu}{\Gamma(\mathfrak{w})} \int_0^{\kappa} \left(\int_0^{\sigma} (\sigma - \varphi)^{\mathfrak{w}-1} \xi_{\mathfrak{u}}(\varphi)d\varphi\right)d\sigma\right\} \\ & + \sum_{j=1}^{i} \mathfrak{u}_j\end{aligned} \tag{11}$$

for any $\mathfrak{u}, \mathfrak{u}_0 \in C(\psi, \mathbb{R})$, we have

$$\begin{aligned}& \|(\Omega\mathfrak{u})(\mathfrak{r}) - (\Omega\mathfrak{u}_0)(\mathfrak{r})\| \\ & \le \frac{1}{\Gamma(\mathfrak{w})} \int_0^{\mathfrak{r}} (\mathfrak{r} - \sigma)^{\mathfrak{w}-1} \quad \left|\xi_{\mathfrak{u}}(\sigma) - \xi_{\mathfrak{u}_0}(\sigma)\right| d\sigma \\ & + \frac{\mathfrak{r}}{\Im}\left\{\left[\frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_1} (\mathfrak{e}_1 - \sigma)^{\mathfrak{w}-2} + \frac{1}{\Gamma(\mathfrak{w}-1)} \int_0^{\mathfrak{e}_2} (\mathfrak{e}_2 - \sigma)^{\mathfrak{w}-2}\right]\left|\xi_{\mathfrak{u}}(\sigma) - \xi_{\mathfrak{u}_0}(\sigma)\right| d\sigma \right. \\ & \left. - \frac{\mu}{\Gamma(\mathfrak{w})} \int_0^{\kappa} \left(\int_0^{\sigma} (\sigma - \varphi)^{\mathfrak{w}-1} \left|\xi_{\mathfrak{u}}(\varphi) - \xi_{\mathfrak{u}_0}(\varphi)\right| d\varphi\right) d\sigma\right\}\end{aligned} \tag{12}$$

with

$$\xi_{\mathfrak{u}}(\mathfrak{r})=\mathfrak{F}\left(\mathfrak{r},\mathfrak{u}(\mathfrak{r}),\xi_{\mathfrak{u}}(\mathfrak{r}),\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma)d\sigma,\int_0^1\mathfrak{h}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma)d\sigma\right),$$

and

$$\xi_{\mathfrak{u}_0}(\mathfrak{r})=\mathfrak{F}\left(\mathfrak{r},\mathfrak{u}_0(\mathfrak{r}),\xi_{\mathfrak{u}_0}(\mathfrak{r}),\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}_0(\sigma)d\sigma,\int_0^1\mathfrak{h}(\mathfrak{r},\sigma)\mathfrak{u}_0(\sigma)d\sigma\right)$$

By using (H_3), we find

$$\left|\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma)d\sigma-\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}_0(\sigma)d\sigma\right|\leq\left|\sup_{\mathfrak{r}\in\psi}\int_0^{\mathfrak{r}}\left|\mathfrak{p}(\mathfrak{r},\sigma)d\sigma\right|\left|\mathfrak{u}(\sigma)-\mathfrak{u}_0(\sigma)\right|\leq P^*\|\mathfrak{u}-\mathfrak{u}_0\|$$

and

$$\left|\int_0^1\mathfrak{h}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma)d\sigma-\int_0^1\mathfrak{h}(\mathfrak{r},\sigma)\mathfrak{u}_0(\sigma)d\sigma\right|\leq\left|\sup_{\mathfrak{r}\in\psi}\int_0^1\left|\mathfrak{h}(\mathfrak{r},\sigma)d\sigma\right|\left|\mathfrak{u}(\sigma)-\mathfrak{u}_0(\sigma)\right|\leq\delta^*\|\mathfrak{u}-\mathfrak{u}_0\|$$

From (H_2), we get

$$\begin{aligned}\left|\xi_{\mathfrak{u}}(\mathfrak{r})-\xi_{\mathfrak{u}_0}(\mathfrak{r})\right|=&\left\|\mathfrak{F}\left((\mathfrak{r},\mathfrak{u}(\mathfrak{r}),\xi_{\mathfrak{u}}((\mathfrak{r}),\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma)d\sigma,\int_0^1\mathfrak{h}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma)d\sigma\right)\right.\\&\left.-\mathfrak{F}\left((\mathfrak{r},\mathfrak{u}_0(\mathfrak{r}),\xi_{\mathfrak{u}_0}(\mathfrak{r}),\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}_0(\sigma)d\sigma,\int_0^1\mathfrak{h}(\mathfrak{r},\sigma)\mathfrak{u}_0(\sigma)d\sigma\right)\right\|\\&\leq\lambda_1\left|\mathfrak{u}(\mathfrak{r})-\mathfrak{u}_0(\mathfrak{r})\right|+\lambda_2\left|\xi_{\mathfrak{u}}(\mathfrak{r})-\xi_{\mathfrak{u}_0}(\mathfrak{r})\right|+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]\|\mathfrak{u}-\mathfrak{u}_0\|\end{aligned}$$

Thus

$$\left|\xi_{\mathfrak{u}}(\mathfrak{r})-\xi_{\mathfrak{u}_0}(\mathfrak{r})\right|\leq\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\|\mathfrak{u}-\mathfrak{u}_0\|\tag{13}$$

Replacing (13) in (12), we obtain

$$\begin{aligned}&\|(\Omega\mathfrak{u})(\mathfrak{r})-(\Omega\mathfrak{u}_0)(\mathfrak{r})\|\\&\leq\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}(\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\left|\mathfrak{u}(\sigma)-\mathfrak{u}_0(\sigma)\right|d\sigma\\&+\frac{\mathfrak{r}}{\mathfrak{J}}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1}(\mathfrak{e}_1-\sigma)^{\mathfrak{w}-2}\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\left|\mathfrak{u}(\sigma)-\mathfrak{u}_0(\sigma)\right|d\sigma\right.\\&+\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2}(\mathfrak{e}_2-\sigma)^{\mathfrak{w}-2}\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\left|\mathfrak{u}(\sigma)-\mathfrak{u}_0(\sigma)\right|d\sigma\\&\left.-\frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma}(\sigma-\varphi)^{\mathfrak{w}-1}\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\left|\mathfrak{u}(\varphi)-\mathfrak{u}_0(\varphi)\right|d\varphi\right)d\sigma\right\}\\&\leq\frac{1}{\Gamma(\mathfrak{w})}\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\int_0^{\mathfrak{r}}(\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\left|\mathfrak{u}(\sigma)-\mathfrak{u}_0(\sigma)\right|d\sigma\end{aligned}$$

$$+\frac{\mathfrak{r}}{\mathfrak{I}}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\int_0^{\mathfrak{e}_1}\left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2}\left|\mathfrak{u}(\sigma)-\mathfrak{u}_0(\sigma)\right|d\sigma\right.$$

$$+\frac{1}{\Gamma(\mathfrak{w}-1)}\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\int_0^{\mathfrak{e}_2}\left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2}\left|\mathfrak{u}(\sigma)-\mathfrak{u}_0(\sigma)\right|d\sigma$$

$$\left.-\frac{\mu}{\Gamma(\mathfrak{w})}\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\int_0^{\kappa}\left(\int_0^{\sigma}\left(\sigma-\varphi\right)^{\mathfrak{w}-1}\left|\mathfrak{u}(\varphi)-\mathfrak{u}_0(\varphi)\right|d\varphi\right)d\sigma\right\}$$

$$\leq\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\left(\frac{1}{\Gamma(\mathfrak{w}+1)}+\frac{1}{\mathfrak{I}}\left[\frac{\mathfrak{e}_1^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})}+\frac{\mathfrak{e}_2^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})}-\frac{\mu\kappa^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}\right]\right)|\mathfrak{u}(\sigma)-\mathfrak{u}_0(\sigma)|$$

Hence,

$$\|(\Omega\mathfrak{u})(\mathfrak{r})-(\Omega\mathfrak{u}_0)(\mathfrak{r})\|_\infty\leq\Xi\|\mathfrak{u}-\mathfrak{u}_0\|_\infty,$$

where

$$\Xi=\left[\frac{\lambda_1+\left[\lambda_3\mathcal{P}^*+\lambda_4\mathcal{H}^*\right]}{1-\lambda_2}\right]\left(\frac{1}{\Gamma(\mathfrak{w}+1)}+\frac{1}{\mathfrak{I}}\left[\frac{\mathfrak{e}_1^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})}+\frac{\mathfrak{e}_2^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})}-\frac{\mu\kappa^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}\right]\right)<1$$

By inequality (10), the operator Ω satisfies the conditions of a contraction mapping. Applying Banach's contraction principle, we conclude that Ω possesses a unique fixed point. Consequently, the system (1)–(3) admits a unique solution. □

Next, we study the second result, by using the fixed point Theorem of Schauder.

Theorem 4: *Assume that* (H_2), (H_3), *and* (H_4) *are true. Then, the problems (1)–(3) admit at least one solution.*

Proof: Consider

$$\mathfrak{e}_{\mathfrak{z}}=\{\mathfrak{u}\in\varphi(\psi,\mathbb{R}):|\mathfrak{u}|\leq\mathfrak{z}\}.$$

Let $\mathfrak{K}$ and $\mathfrak{D}$ be the two operators defined on $\mathfrak{e}_{\mathfrak{z}}$. The integral operator $\mathfrak{K}\colon\mathfrak{e}_{\mathfrak{z}}\to\varphi(\psi,\mathbb{R})$ is given by:

$$(\mathfrak{K}\mathfrak{u})(\mathfrak{r})=\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}\left(\mathfrak{r}-\sigma\right)^{\mathfrak{w}-1}\xi_{\mathfrak{u}}(\sigma)d\sigma,$$

and the source term operator $\mathfrak{D}\colon\mathfrak{e}_{\mathfrak{z}}\to\mathbb{R}$ (a constant function depending on $\mathfrak{u}_0$) is defined by:

$$(\mathfrak{D}\mathfrak{u}_0)(\mathfrak{r})=\frac{\mathfrak{r}}{\mathfrak{I}}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1}\left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2}\xi_{\mathfrak{u}_0}(\sigma)\,d\sigma+\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2}\left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2}\xi_{\mathfrak{u}_0}(\sigma)\,d\sigma\right.$$

$$\left.-\frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma}\left(\sigma-\varphi\right)^{\mathfrak{w}-1}\xi_{\mathfrak{u}_0}(\varphi)\,d\varphi\right)d\sigma\right\}+\sum_{j=1}^{i}(\mathfrak{u}_0)_j.$$

The operator $\mathfrak{D}$ depends only on the fixed function $\mathfrak{u}_0$ and acts as a constant source term for the fractional integral equation.

Step 1: Show That $\mathfrak{K}$ Is a Contraction

From assumption (H_6), the function $\xi_{\mathfrak{u}}(\mathfrak{r})$ is Lipschitz continuous in $\mathfrak{u}$, i.e.,

$$|\xi_{\mathfrak{u}_1}(\sigma)-\xi_{\mathfrak{u}_2}(\sigma)|\leq\delta_{\xi}\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\infty}$$

Now for any $\mathfrak{u}_1,\mathfrak{u}_2\in\mathfrak{e}_{\mathfrak{z}}$, we estimate:

$$\begin{aligned}\|\mathfrak{K}\mathfrak{u}_1-\mathfrak{K}\mathfrak{u}_2\|_{\infty}&\leq\frac{1}{\Gamma(\mathfrak{w})}\sup_{\mathfrak{r}\in\psi}\int_0^{\mathfrak{r}}(\mathfrak{r}-\sigma)^{\mathfrak{w}-1}|\xi_{\mathfrak{u}_1}(\sigma)-\xi_{\mathfrak{u}_2}(\sigma)|d\sigma\\&\leq\frac{\delta_{\xi}}{\Gamma(\mathfrak{w})}\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\infty}\sup_{\mathfrak{r}\in\psi}\int_0^{\mathfrak{r}}(\mathfrak{r}-\sigma)^{\mathfrak{w}-1}\,d\sigma\\&\leq\Upsilon\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\infty}\\&=\frac{\delta_{\xi}}{\Gamma(\mathfrak{w}+1)}\|\mathfrak{u}_1-\mathfrak{u}_2\|_{\infty},\end{aligned}$$

where

$$\Upsilon:=\frac{\delta_{\xi}}{\Gamma(\mathfrak{w}+1)}.$$

Under (H_3), we assume $\Upsilon<1$, so $\mathfrak{K}$ is a contraction.

Step 2: Show That $\mathfrak{D}\mathfrak{u}_0\in\mathfrak{e}_{\mathfrak{z}}$.

Under assumption (H_4), the fixed function $\mathfrak{u}_0$ is such that the output of $\mathfrak{D}\mathfrak{u}_0$ remains in the ball of radius $\mathfrak{z}$. That is,

$$\|(\mathfrak{D}\mathfrak{u}_0)\|_{\infty}\leq C<\mathfrak{z},\quad\forall\mathfrak{r}\in\psi$$

This ensures $\mathfrak{D}\mathfrak{u}_0\in\mathfrak{e}_{\mathfrak{z}}$.

Step 3:

Note that $\mathfrak{u},\mathfrak{u}_0\in\mathfrak{e}_{\mathfrak{z}}$. Then

$$\mathfrak{K}\mathfrak{u}+\mathfrak{D}\mathfrak{u}_0\in\mathfrak{e}_{\mathfrak{z}}$$

thus verifying the inequality in the equation above.

$$\begin{aligned}&|\mathfrak{K}\mathfrak{u}+\mathfrak{D}\mathfrak{u}_0|\\&=\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}\left(\mathfrak{r}-\sigma\right)^{\mathfrak{w}-1}\quad|\xi_{\mathfrak{u}}(\sigma)|d\sigma+\frac{1}{\mathfrak{I}}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1}\left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2}|\xi_{\mathfrak{u}}(\sigma)|d\sigma\right.\\&\left.+\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2}\left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2}\quad|\xi_{\mathfrak{u}}(\sigma)|d\sigma-\frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma}\left(\sigma-\varphi\right)^{\mathfrak{w}-1}|\xi_{\mathfrak{u}}(\varphi)|d\varphi\right)d\sigma+\sum_{j=1}^{i}\mathfrak{u}_j\right\}\end{aligned}\tag{14}$$

where

$$\xi(\mathfrak{r})=\mathfrak{F}\left(\mathfrak{r},\mathfrak{u}_0(\mathfrak{r}),\xi_{\mathfrak{u}0}(\mathfrak{r}),\textstyle\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}_0(\sigma)d\sigma,\int_0^1\mathfrak{h}(\mathfrak{r},\delta,\sigma)\mathfrak{u}_0(\sigma))d\sigma\right)$$

From (H_3) and (H_4), we get

$$|\xi(\mathfrak{r})|=\left|\mathfrak{F}\left(\mathfrak{r},\mathfrak{u}(\mathfrak{r}),\xi(\mathfrak{r}),\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma)d\sigma,\int_0^1\mathfrak{h}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma))d\sigma\right)\right|$$

$$\leq \delta(\mathfrak{r}) + \nu(\mathfrak{r})|\mathfrak{u}(\mathfrak{r})| + \zeta(\mathfrak{r})|\xi(\mathfrak{r})| + \mathcal{P}^*\|\mathfrak{u}\|_\infty + \mathcal{H}^*\|\mathfrak{u}\|_\infty$$
$$\leq \delta^* + (\nu^* + \mathcal{P}^*\eta^* + \mathcal{H}^*\tau^*)\|\mathfrak{u}\|_\infty + \zeta^*|\xi(\mathfrak{r})|$$

Hence,

$$|\xi(\mathfrak{r})| \leq \frac{\delta^* + (\nu^* + \mathcal{P}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{z}}{1-\zeta^*} \tag{15}$$

In the inequality (14), we obtain by substituting (15)

$$\begin{aligned}
|\mathfrak{K}\mathfrak{u} + \mathfrak{D}\mathfrak{u}_0| \leq{}& \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} \left(\mathfrak{r}-\sigma\right)^{\mathfrak{w}-1} \left[\frac{\delta^* + (\nu^* + \mathcal{P}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{z}}{1-\zeta^*}\right] d\sigma \\
&+ \frac{\mathfrak{r}}{\mathfrak{I}}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1} \left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2} \left[\frac{\delta^* + (\nu^* + \mathcal{P}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{z}}{1-\zeta^*}\right] d\sigma\right. \\
&+ \frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2} \left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2} \left[\frac{\delta^* + (\nu^* + \mathcal{P}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{z}}{1-\zeta^*}\right] d\sigma \\
&\left.- \frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa} \left(\int_0^{\sigma} \left(\sigma-\varphi\right)^{\mathfrak{w}-1}\left[\frac{\delta^* + (\nu^* + \mathcal{P}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{z}}{1-\zeta^*}\right] d\sigma + \sum_{j=1}^{i}\mathfrak{u}_j\right\}\right. \\
&\leq \left[\frac{\delta^* + (\nu^* + \mathcal{P}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{z}}{1-\zeta^*}\right]\left(\frac{1}{\Gamma(\mathfrak{w}+1)} + \frac{1}{\mathfrak{I}}\left[\frac{\mathfrak{e}_1^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} + \frac{\mathfrak{e}_2^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} - \frac{\mu\kappa^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}\right]\right) \\
&+ \sum_{j=1}^{i}\mathfrak{u}_j \\
&= \Xi\left[\frac{\delta^* + (\nu^* + \mathcal{P}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{z}}{1-\zeta^*}\right] + \sum_{j=1}^{i}\mathfrak{u}_j \\
&\leq \mathfrak{z}
\end{aligned}$$

Thus,

$$\mathfrak{K}\mathfrak{u} + \mathfrak{D}\mathfrak{u}_0 \in \mathfrak{e}_{\mathfrak{z}}$$

Furthermore unambiguously is $\mathfrak{D}$ a contraction mapping. From continuity of $\mathfrak{K}$, the operator $(\mathfrak{K}\mathfrak{u})(\mathfrak{r})$ is continuous in line with ξ. Also we observe that

$$(\mathfrak{K}\mathfrak{u})(\mathfrak{r})) = \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} \left(\mathfrak{r}-\sigma\right)^{\mathfrak{w}-1} \xi_{\mathfrak{u}}(\sigma)d\sigma \leq \frac{1}{\Gamma(\mathfrak{w}+1)}\left[\frac{\delta^* + (\nu^* + \mathcal{P}^*\eta^* + \mathcal{H}^*\tau^*)\mathfrak{z}}{1-\zeta^*}\right]$$

Hence, $\mathfrak{K}$ is uniformly bounded on $\mathfrak{e}_{\mathfrak{z}}$.

Now, we show that $(\mathfrak{K}\mathfrak{u})(\mathfrak{r}))$ is equicontinuous.

Let $\mathfrak{r}_1, \mathfrak{r}_2 \in \psi = (0, 1], \quad \mathfrak{r}_1 < \mathfrak{r}_2$ and let $\mathfrak{u} \in \mathfrak{e}_{\mathfrak{z}}$

$$|(\mathfrak{K}\mathfrak{u})(\mathfrak{r}_2)) - (\mathfrak{K}\mathfrak{u})(\mathfrak{r}_1))|$$

$$= \left|\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}_1} \left(\mathfrak{r}_2-\sigma\right)^{\mathfrak{w}-1} \xi_{\mathfrak{u}}(\sigma)d\sigma - \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}_1} \left(\mathfrak{r}_1-\sigma\right)^{\mathfrak{w}-1} \xi_{\mathfrak{u}}(\sigma)d\sigma\right|$$

$$\leq \frac{1}{\Gamma(\mathfrak{w})}\left|\int_0^{\mathfrak{r}_2}\left(\left(\mathfrak{r}_2-\sigma\right)^{\mathfrak{w}-1}-\left(\mathfrak{r}_1-\sigma\right)^{\mathfrak{w}-1}\right)\ \xi_{\mathfrak{u}}(\sigma)d\sigma\right|$$
$$+\frac{1}{\Gamma(\mathfrak{w})}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left(\mathfrak{r}_2-\sigma\right)^{\mathfrak{w}-1}\ \xi_{\mathfrak{u}}(\sigma)d\sigma$$
$$\leq\left[\frac{\delta^*+(\nu^*+\mathcal{P}^*\eta^*+\mathcal{H}^*\tau^*)\mathfrak{z}}{(1-\zeta^*)\Gamma(\mathfrak{w}+1)}\right]\left|\left(\mathfrak{r}_2^{\mathfrak{w}}-\mathfrak{r}_1^{\mathfrak{w}}\right)+2(\mathfrak{r}_2^{\mathfrak{w}}-\mathfrak{r}_1^{\mathfrak{w}})\right|$$
$$\longrightarrow 0\quad\text{as } \mathfrak{r}_1\longrightarrow\mathfrak{r}_2$$

The Arzela-Ascoli Theorem shows that $\mathfrak{K}(\mathfrak{e}_{\mathfrak{z}})$ is relatively $\mathfrak{K}$ is compact. The problem (1)–(3) has a fixed point. Then, the problem (1)–(3) have at least one solution. □

Theorem 5: *Assume condition* (H_5) *holds. Then, problems (1)–(3) admit at least one solution.*

Proof: We will demonstrate that Ω, as defined by (11), has a fixed point by applying Schaefer's fixed point Theorem. The proof is going to be presented in stages.

Step 1. Ω is continuous. Let $\{\mathfrak{u}_m\}$ be a sequence as follows $\mathfrak{u}_m\longrightarrow\mathfrak{u}$ in $\mathfrak{e}\mathfrak{C}(\psi,\mathbb{R})$ Then for any $\mathfrak{r}\in\psi$

$$\begin{aligned}\left|(\Omega(\mathfrak{u}_m)(\mathfrak{r})-(\Omega(\mathfrak{u})(\mathfrak{r})\right|\leq&\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}\left(\mathfrak{r}-\sigma\right)^{\mathfrak{w}-1}\ |\xi_{\mathfrak{u}_n}(\sigma)-\xi_{\mathfrak{u}}(\sigma)|d\sigma\\&+\frac{\mathfrak{r}}{\mathfrak{I}}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1}\left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2}\ |\xi_{\mathfrak{u}_n}(\sigma)-\xi_{\mathfrak{u}}(\sigma)|d\sigma\right.\\&+\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2}\left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2}\ |\xi_{\mathfrak{u}_n}(\sigma)-\xi_{\mathfrak{u}}(\sigma)|d\sigma\\&\left.-\frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma}\left(\sigma-\varphi\right)^{\mathfrak{w}-1}|\xi_{\mathfrak{u}_n}(\varphi)-\xi_{\mathfrak{u}}(\varphi)|d\varphi\right)d\sigma\right\}\\&+\sum_{j=1}^{i}\mathfrak{u}_j|\xi_{\mathfrak{u}_n}(\varphi)-\xi_{\mathfrak{u}}(\varphi)|\end{aligned}$$

where $\xi_{\mathfrak{u}},\xi_{\mathfrak{u}_m}\in\mathfrak{C}(\psi,\mathbb{R})$ are

$$\xi_{\mathfrak{u}}(\mathfrak{r})=\mathfrak{F}\left(\mathfrak{r},\mathfrak{u}(\mathfrak{r}),\xi_{\mathfrak{u}}(\mathfrak{r}),\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma)d\sigma,\int_0^1\mathfrak{h}(\mathfrak{r},\sigma)\mathfrak{u}(\sigma)d\sigma\right)$$

and

$$\xi_{\mathfrak{u}_m}(\mathfrak{r})=\mathfrak{F}\left(\mathfrak{r},\mathfrak{u}_m(\mathfrak{r}),\xi_{\mathfrak{u}_m}(\mathfrak{r}),\int_0^{\mathfrak{r}}\mathfrak{p}(\mathfrak{r},\sigma)\mathfrak{u}_m(\delta)d\sigma,\int_0^1\mathfrak{h}(\mathfrak{r},\sigma)\mathfrak{u}_m(\delta)d\sigma\right)$$

Since Ω is continuous functions (i.e., $\mathfrak{F}$ is continuous), we then from the Lebesgue Theorem of dominated convergence, we get

$$\left\|(\Omega(\mathfrak{u}_m)(\mathfrak{r})-(\Omega(\mathfrak{u})(\mathfrak{r})\right\|_\infty\longrightarrow 0\text{ as }\ m\longrightarrow+\infty$$

Hence, $\Omega(\mathfrak{u}_m)(\mathfrak{r})\longrightarrow(\Omega(\mathfrak{u})(\mathfrak{r})$ as $m\longrightarrow+\infty$ which implies that Ω is continuous

Step 2. Ω maps bounded sets into bounded sets in $\varphi\mathfrak{C}(\psi,\mathbb{R})$. Now, for any $\mathfrak{z} > 0$, we take Consider $\mathfrak{e}_{\mathfrak{z}} = \{\mathfrak{u} \in \varphi(\psi,\mathbb{R}) : \|\mathfrak{u}\|_\infty \le \mathfrak{z}\}$

we have

$$\|(\Omega\mathfrak{u})(\mathfrak{r}))\| = \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}} \left(\mathfrak{r}-\sigma\right)^{\mathfrak{w}-1} \quad \xi_{\mathfrak{u}}(\sigma)d\sigma + \frac{\mathfrak{r}}{\mathfrak{I}}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1} \left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2} \quad \xi_{\mathfrak{u}_0}(\sigma)d\sigma\right.$$
$$+ \frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2} \left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2} \quad \xi_{\mathfrak{u}_0}(\sigma)d\sigma$$
$$\left. - \frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma}\left(\sigma-\varphi\right)^{\mathfrak{w}-1}\xi_{\mathfrak{u}_0}(\varphi)d\varphi\right)d\sigma + \sum_{j=1}^{i}\mathfrak{u}_{0j}\right\}$$

where

$$|\xi_{\mathfrak{u}}(\sigma)| = \left|\mathfrak{F}\left(\sigma,\mathfrak{u}(\sigma),\xi_{\mathfrak{u}}(\sigma),\int_0^{\sigma}\mathfrak{p}(\sigma,\delta)\mathfrak{u}(\sigma)d\sigma,\int_0^1\mathfrak{h}(\sigma,\delta)\mathfrak{u}(\sigma)d\sigma\right)\right|$$

by using (H_5) we have

$$|(\Omega\mathfrak{u})(\mathfrak{r}))| \le \mathcal{M}\left(\frac{1}{\Gamma(\mathfrak{w}+1)} + \frac{1}{\mathfrak{I}}\left[\frac{\mathfrak{e}_1^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} + \frac{\mathfrak{e}_2^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} - \frac{\mu\kappa^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}\right]\right)$$
$$+ \sum_{j=1}^{i}\mathfrak{u}_j = \mathcal{M}T + \sum_{j=1}^{i}\mathfrak{u}_j \le \mathfrak{z}$$

Step 3. Ω maps bounded sets into equicontinuous sets of $\varphi\mathfrak{C}(\psi,\mathbb{R})$.

let $\mathfrak{r}_1,\mathfrak{r}_2 \in \psi = (0,1]$, $\quad \mathfrak{r}_1 < \mathfrak{r}_2$ and let $\mathfrak{u} \in \mathfrak{e}_{\mathfrak{z}}$

$$|(\Omega\mathfrak{u})(\mathfrak{r}_2)) - (\Omega\mathfrak{u})(\mathfrak{r}_1))|$$
$$= \left|\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}_1}\left(\mathfrak{r}_2-\sigma\right)^{\mathfrak{w}-1} \quad \xi_{\mathfrak{u}}(\sigma)d\sigma - \frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}_1}\left(\mathfrak{r}_1-\sigma\right)^{\mathfrak{w}-1} \quad \xi_{\mathfrak{u}}(\sigma)d\sigma\right|$$
$$+ \frac{|\mathfrak{r}_2-\mathfrak{r}_1|}{\mathfrak{I}}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1}\left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2} \quad \xi_{\mathfrak{u}_0}(\sigma)d\sigma\right.$$
$$+ \frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2}\left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2} \quad \xi_{\mathfrak{u}_0}(\sigma)d\sigma$$
$$\left. - \frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma}\left(\sigma-\varphi\right)^{\mathfrak{w}-1}\xi_{\mathfrak{u}_0}(\varphi)d\varphi\right)d\sigma + \sum_{j=1}^{i}\mathfrak{u}_{0j}\right\}$$

$$\le \frac{1}{\Gamma(\mathfrak{w})}\left|\int_0^{\mathfrak{r}_1}\left(\left(\mathfrak{r}_2-\sigma\right)^{\mathfrak{w}-1} - \left(\mathfrak{r}_1-\sigma\right)^{\mathfrak{w}-1}\right) \quad \xi_{\mathfrak{u}}(\sigma)\right|d\sigma$$
$$+ \frac{1}{\Gamma(\mathfrak{w})}\int_{\mathfrak{r}_1}^{\mathfrak{r}_2}\left|\left(\mathfrak{r}_2-\sigma\right)^{\mathfrak{w}-1} \quad \xi_{\mathfrak{u}}(\sigma)\right|d\sigma$$

$$+\frac{|\mathfrak{r}_2-\mathfrak{r}_1|}{\Im}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1}\left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2}\ \xi_{\mathfrak{u}_0}(\sigma)d\sigma\right.$$
$$+\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2}\left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2}\ \xi_{\mathfrak{u}_0}(\sigma)d\sigma$$
$$\left.-\frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma}(\sigma-\varphi)^{\mathfrak{w}-1}\xi_{\mathfrak{u}_0}(\varphi)d\varphi\right)d\sigma+\sum_{j=1}^{i}\mathfrak{u}_{0j}\right\}$$
$$\leq\mathcal{M}\left[\frac{1}{\Gamma(\mathfrak{w}+1)}\left|\left(\mathfrak{r}_2^{\mathfrak{w}}-\mathfrak{r}_1^{\mathfrak{w}}\right)+2(\mathfrak{r}_2^{\mathfrak{w}}-\mathfrak{r}_1^{\mathfrak{w}})\right|+\frac{|\mathfrak{r}_1-\mathfrak{r}_2|}{\Im}\left(\frac{\mathfrak{e}_1^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})}+\frac{\mathfrak{e}_2^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})}-\frac{\mu\kappa^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}\right)\right]$$
$$\longrightarrow 0\quad \text{as}\mathfrak{r}_1\longrightarrow\mathfrak{r}_2$$

Step 4. boundaries a priori. The set

$$\Psi=\{\mathfrak{u}\in\varphi\mathfrak{C}(\psi,\mathbb{R}):\mathfrak{u}=\Theta\Omega(\mathfrak{u})\ \text{for some}\ \ 1<\Theta<T\}$$

is bounded, as must be demonstrated now.

$$\mathfrak{u}(\mathfrak{r})\leq\Theta\left\{\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}\left(\mathfrak{r}-\sigma\right)^{\mathfrak{w}-1}\ \xi_{\mathfrak{u}}(\sigma)d\sigma+\frac{\mathfrak{r}}{\Im}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1}\left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2}\ \xi_{\mathfrak{u}_0}(\sigma)d\sigma\right.\right.$$
$$+\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2}\left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2}\ \xi_{\mathfrak{u}_0}(\sigma)d\sigma$$
$$\left.\left.-\frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma}(\sigma-\varphi)^{\mathfrak{w}-1}\xi_{\mathfrak{u}_0}(\varphi)d\varphi\right)d\sigma\right\}+\sum_{j=1}^{i}\mathfrak{u}_{0j}\right\}$$

For $\Theta\in[0,\ 1]$, let $\mathfrak{u}\in\varphi\mathfrak{C}(\psi,\mathbb{R})$ for each $\mathfrak{r}\in\psi$

$$\|(\Omega\mathfrak{u})(\mathfrak{r}))\|\leq\frac{1}{\Gamma(\mathfrak{w})}\int_0^{\mathfrak{r}}\left(\mathfrak{r}-\sigma\right)^{\mathfrak{w}-1}\ \xi_{\mathfrak{u}}(\sigma)d\sigma+\frac{\mathfrak{r}}{\Im}\left\{\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_1}\left(\mathfrak{e}_1-\sigma\right)^{\mathfrak{w}-2}\ \xi_{\mathfrak{u}_0}(\sigma)d\sigma\right.$$
$$+\frac{1}{\Gamma(\mathfrak{w}-1)}\int_0^{\mathfrak{e}_2}\left(\mathfrak{e}_2-\sigma\right)^{\mathfrak{w}-2}\ \xi_{\mathfrak{u}_0}(\sigma)d\sigma$$
$$\left.-\frac{\mu}{\Gamma(\mathfrak{w})}\int_0^{\kappa}\left(\int_0^{\sigma}(\sigma-\varphi)^{\mathfrak{w}-1}\xi_{\mathfrak{u}_0}(\varphi)d\varphi\right)d\sigma\right\}+\sum_{j=1}^{i}\mathfrak{u}_{0j}$$
$$\leq\mathcal{M}\left(\frac{1}{\Gamma(\mathfrak{w}+1)}+\frac{1}{\Im}\left[\frac{\mathfrak{e}_1^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})}+\frac{\mathfrak{e}_2^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})}-\frac{\mu\kappa^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}\right]\right)+\sum_{j=1}^{i}\mathfrak{u}_j=\mathcal{M}\Xi+\sum_{j=1}^{i}\mathfrak{u}_j$$

Thus,

$$\|(\Omega\mathfrak{u})(\mathfrak{r}))\|\leq\infty$$

Since $\|(\Omega\mathfrak{u})(\mathfrak{r})\|<\infty$. The Arzela-Ascoli theorem shows that Ω is relatively compact in both scenarios, and Schauder's fixed point theorem states that Ω has a fixed point. Then, Ω is a solution of system (1)–(3) on ψ. □

4 Application

In this section, we present a concrete example to demonstrate the applicability of our main theoretical results to a nonlinear fractional integro differential equation. The system involves a fractional derivative of order $\frac{5}{2}$ and incorporates both integral and nonlocal conditions. Such models arise in various applied fields where memory effects and nonlocal interactions are present, including anomalous diffusion, viscoelasticity, and signal processing. The following equations represent a fully fractional variational integro-differential equation (FVFIDE) accompanied by interface and nonlocal boundary conditions.

$$\mathfrak{D}^{\frac{5}{2}}\mathfrak{u}(\mathfrak{r}) = \frac{\mathfrak{r}^2}{\mathfrak{r}+1} + \frac{e^{\mathfrak{r}^2}}{\sqrt{\mathfrak{r}+36}}\sin(\mathfrak{u}(\mathfrak{r})) + \frac{1}{36(1+\mathfrak{r}^4)}\mathfrak{D}^{\frac{5}{2}}\mathfrak{u}(\mathfrak{r}) + \frac{1}{18}\int_0^{\mathfrak{r}} e^{(\mathfrak{r}-\delta)}\cos(\mathfrak{u}(\sigma))d\sigma$$
$$+ \int_0^1 e^{-(\mathfrak{r}-\delta)^2}\frac{\sin(\mathfrak{u}(\sigma))}{\sqrt{81+|\mathfrak{r}|}}d\sigma \tag{16}$$

$$\mathfrak{u}(\mathfrak{r}_k^+) = \mathfrak{u}(\mathfrak{r}_k^-) + \frac{1}{8},, \quad k = 1, ..., m, \quad \mathfrak{r} \in \psi = [0,\ 1] \tag{17}$$

$$\mathfrak{u}(0) = 0, \quad 2\mathfrak{u}'\left(\frac{2}{9}\right) + \mathfrak{u}'\left(\frac{1}{3}\right) = 5\int_0^{\frac{2}{3}} \mathfrak{u}(\sigma)d\sigma, \quad \mathfrak{a}, \mathfrak{d} \in \mathbb{R}, \quad 0 < \kappa < 1 \tag{18}$$

we see that

$$\mathfrak{w} = \frac{5}{2}, \quad \mathfrak{a} = 2, \quad \mathfrak{d} = 1, \quad \mathfrak{e}_1 = \frac{2}{9}, \quad \mathfrak{e}_2 = \frac{1}{3}, \quad \mu = 5, \quad \kappa = \frac{2}{3}$$

and

Derivation of Lipschitz Constants for $\mathfrak{F}$

Given the function:

$$\mathfrak{F}(\mathfrak{r}, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4) = \frac{t^2}{t+1} + \frac{e^{\mathfrak{r}^2}}{\sqrt{\mathfrak{r}+36}}\sin(\mathfrak{u}_1(\mathfrak{r})) + \frac{1}{36(1+\mathfrak{r}^4)}\mathfrak{u}_2 + \frac{1}{18}\cos\mathfrak{u}_3$$
$$+ \frac{\sin(\mathfrak{u}_4)}{\sqrt{81+|\mathfrak{r}|}}$$

Clearly, $\mathfrak{F}$ is a continuous, and $\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4 \in \mathbb{R}$ for $\mathfrak{r} \in [0,\ 1]$ we get

$$\|\mathfrak{F}(\mathfrak{r}, \mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4) - \mathfrak{F}(\mathfrak{r}, \mathfrak{v}_1, \mathfrak{v}_2, \mathfrak{v}_3, \mathfrak{v}_4)\| \leq \frac{1}{6}\|\mathfrak{u}_1 - \mathfrak{v}_1\| + \frac{1}{36}\|\mathfrak{u}_2 - \mathfrak{v}_2\|$$
$$+ \frac{1}{18}\|\mathfrak{u}_3 - \mathfrak{v}_3\| + \frac{1}{9}\|\mathfrak{u}_4 - \mathfrak{v}_4\|$$

1. For λ_1 (Term with $\sin\mathfrak{u}_1$):

$$\left|\frac{e^{\mathfrak{r}^2}}{\sqrt{\mathfrak{r}+36}}(\sin\mathfrak{u}_1 - \sin\mathfrak{v}_1)\right| \leq \frac{e^{\mathfrak{r}^2}}{\sqrt{\mathfrak{r}+36}}|\mathfrak{u}_1 - \mathfrak{v}_1|$$
$$\leq \frac{1}{6}|\mathfrak{u}_1 - \mathfrak{v}_1|$$

$$\Rightarrow \lambda_1 = \frac{1}{6}$$

2. For λ_2 (Linear term with $\mathfrak{u}_2$):

$$\left|\frac{1}{36(1+\mathfrak{r}^4)}(\mathfrak{u}_2 - \mathfrak{v}_2)\right| = \frac{1}{36}|\mathfrak{u}_2 - \mathfrak{v}_2|$$

$$\Rightarrow \lambda_2 = \frac{1}{36}$$

3. For λ_3 (Term with $\cos \mathfrak{u}_3$):

$$\left|\frac{1}{18}(\cos \mathfrak{u}_3 - \cos \mathfrak{v}_3)\right| \leq \frac{1}{18}|\mathfrak{u}_3 - \mathfrak{v}_3|$$

$$\Rightarrow \lambda_3 = \frac{1}{18}$$

4. For λ_4 (Term with $\sin \mathfrak{u}_4$):

$$\left|\frac{1}{\sqrt{81+|\mathfrak{r}|}}(\sin \mathfrak{u}_4 - \sin \mathfrak{v}_4)\right| \leq \frac{1}{9}|\mathfrak{u}_4 - \mathfrak{v}_4|$$

$$\Rightarrow \lambda_4 = \frac{1}{9}$$

Hence, condition (H_2) is satisfied with

$$\lambda_1 = \frac{1}{6}, \quad \lambda_2 = \frac{1}{36}, \quad \lambda_3 = \frac{1}{18}, \quad \lambda_4 = \frac{1}{9}$$

and

$$\mathcal{P}^* = \sup_{\mathfrak{r}\in[0,1]}\left|\int_0^{\mathfrak{r}} e^{(\mathfrak{r}-\sigma)}\cos(\mathfrak{u}(\sigma))\,d\sigma\right| \leq \sup_{\mathfrak{r}\in[0,1]}\int_0^{\mathfrak{r}} e^{(\mathfrak{r}-\sigma)}\,d\sigma \quad (\text{since } |\cos(\mathfrak{u}(\sigma))| \leq 1)$$

$$= \sup_{\mathfrak{r}\in[0,1]}\left[e^{\mathfrak{r}}\int_0^{\mathfrak{r}} e^{-\sigma}\,d\sigma\right] = \sup_{\mathfrak{r}\in[0,1]}[e^{\mathfrak{r}}(1-e^{-\mathfrak{r}})] = \sup_{\mathfrak{r}\in[0,1]}(e^{\mathfrak{r}}-1) = e^1 - 1 \approx 1.7183.$$

$$\mathcal{H}^* = \sup_{\mathfrak{r}\in[0,1]}\left|\int_0^1 e^{-(\mathfrak{r}-\sigma)}\sin(\mathfrak{u}(\sigma))d\sigma\right| \leq \sup_{\mathfrak{r}\in[0,1]}\left[e^{-\mathfrak{r}}\int_0^1 e^{\sigma}\,d\sigma\right] (\text{since } |\sin(\mathfrak{u}(\sigma))| \leq 1)$$

$$= \sup_{\mathfrak{r}\in[0,1]}[e^{-\mathfrak{r}}(e-1)] = (e-1)\sup_{\mathfrak{r}\in[0,1]} e^{-\mathfrak{r}} = (e-1)\cdot e^0 = \approx 1.7183.$$

Step 1: Compute the Numerator

$$\lambda_1 + (\lambda_3\mathcal{P}^* + \lambda_4\mathcal{H}^*) = \frac{1}{6} + \frac{e-1}{6} = \frac{e}{6}$$

Step 2: Compute the Denominator:

$$1 - \lambda_2 = 1 - \frac{1}{36} = \frac{35}{36}$$

Step 3: Fractional Term

$$\frac{\frac{e}{6}}{\frac{35}{36}} = \frac{e}{6} \times \frac{36}{35} = \frac{6e}{35}$$

Step 4: Gamma Function Values

$$\Gamma(\mathfrak{w}+1) = \Gamma\left(\tfrac{7}{2}\right) = \tfrac{15\sqrt{\pi}}{8}, \quad \Gamma(\mathfrak{w}) = \Gamma\left(\tfrac{5}{2}\right) = \tfrac{3\sqrt{\pi}}{4}, \quad \Gamma(\mathfrak{w}+2) = \Gamma\left(\tfrac{9}{2}\right) = \tfrac{105\sqrt{\pi}}{16}$$

Step 5: First Parenthetical Term

$$\frac{1}{\Gamma(\mathfrak{w}+1)} = \frac{8}{15\sqrt{\pi}}$$

Step 6: Second Parenthetical Term

Compute each component:

$$\frac{\mathfrak{e}_1^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} = \frac{(2/9)^{3/2}}{3\sqrt{\pi}/4} = \frac{4(2/9)^{3/2}}{3\sqrt{\pi}}, \quad \frac{\mathfrak{e}_2^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} = \frac{(1/3)^{3/2}}{3\sqrt{\pi}/4} = \frac{4(1/3)^{3/2}}{3\sqrt{\pi}}$$

$$\frac{\mu\kappa^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)} = \frac{5(2/3)^{7/2}}{105\sqrt{\pi}/16} = \frac{80(2/3)^{7/2}}{105\sqrt{\pi}}$$

Combine terms:

$$\frac{9}{13}\left[\frac{4(2/9)^{3/2}}{3\sqrt{\pi}} + \frac{4(1/3)^{3/2}}{3\sqrt{\pi}} - \frac{80(2/3)^{7/2}}{105\sqrt{\pi}}\right]$$

Step 7: Numerical Evaluation

Approximate values:

$$\frac{8}{15\sqrt{\pi}} \approx 0.300, \quad \text{and the second term} \approx 0.0839$$

Total parenthetical term $\approx 0.300 + 0.0839 = 0.3839$

Final Calculation of $\Xi = \frac{6e}{35} \times 0.3839 \approx 0.1824 < 1$

$$\Xi = \left[\frac{\lambda_1 + [\lambda_3\mathcal{P}^* + \lambda_4\mathcal{H}^*]}{1-\lambda_2}\right]\left(\frac{1}{\Gamma(\mathfrak{w}+1)} + \frac{1}{\Im}\left[\frac{\mathfrak{e}_1^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} + \frac{\mathfrak{e}_2^{\mathfrak{w}-1}}{\Gamma(\mathfrak{w})} - \frac{\mu\kappa^{\mathfrak{w}+1}}{\Gamma(\mathfrak{w}+2)}\right]\right) = 0.1824 < 1$$

so it follows from Theorem 3 that the problem (16)–(18) has a unique solution.

5 Conclusion

This work presents an analytical framework for solving a class of nonlinear impulsive fractional integro-differential equations with Riemann–Liouville derivatives and integral boundary conditions. By combining fixed-point theorems and Volterra–Fredholm operators, we established existence, uniqueness, and solution representation for these systems, validated through an illustrative example.

The proposed approach opens several research directions, particularly in control and stability theory:

1. Controllability: Extending the framework to study exact and approximate controllability in impulsive fractional systems with memory effects.

2. Optimal Control: Investigating necessary optimality conditions for fractional impulsive systems with integral constraints.

3. Stability Analysis: Exploring Mittag-Leffler, finite-time, and input-to-state stability under impulsive perturbations.

Further extensions include variable-order derivatives, numerical approximations, and applications in viscoelasticity, biological systems, and hybrid control processes. These advancements would strengthen the theoretical foundations while enhancing practical utility in engineering and applied sciences.

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