An alpha-adaptive approach for stabilized finite element solution of advective-diffusive problems with sharp gradients

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1. INTRODUCTION

The standard Galerkin finite element method is known to fail for solution of advective-diffusive problems for moderate and high values of the advective terms [1,2]. Over the years a number of techniques have been proposed to obtain the so called stable (or oscillation-free) solutions. Original remedies were based on the heuristic addition of the right amount of balancing diffusion to the original problem [1–4].

A more rigorous approach is based in adding to the Galerkin finite element formulation the sum over all elements of the integrals over the element interiors of the residual of the original differential equation times an "ad-hoc" perturbation of the weighting functions and the so called stabilization parameter. By choosing adequately the perturbation function the standard SUPG [5], GLS [6], Taylor-Galerkin [7], Characteristic approximation [1,8] and Subgrid Scale [9] methods can be recovered as shown in [10]. As for the stabilization parameter, this can be interpreted either as a "characteristic length" of the discrete problem, as a proportion of a typical element dimension, or as the "intrinsic time" taken for a particle to travel half the characteristic length at the advective speed.

The precise computation of any of the equivalent forms of the crucial stabilization parameter can only be attempted for simple one dimensional (1D) problems such the sourceless 1D advection-diffusion case [1,2]. Attempts to generalize the computation of this parameter were due to Idelsohn [1] using a pseudo-variational principle. Hughes [9] and later Brezzi and co-workers [12–14] have proposed a numerical expression for the stabilization parameter involving an approximation of the element Green's function using bubble shape functions. None of these procedures has however succeeded so far to present evidence of its usefulness for practical multidimensional problems.

In [15,16] Oñate proposed a different approach for computing the stabilization parameter. The method is based in introducing "a priori" the stabilizing terms within the differential equations governing the balance of fluxes over a finite domain. This kind of *finite increment calculus* (FIC) procedure allows to obtain any stable discretized scheme using finite difference, finite element or finite volume methods in a straight forward manner. For instance it can be shown that the Galerkin finite element form of the new stabilized governing equations is identical to that obtained with the well known SUPG and Characteristic-Galerkin methods, among others [15,16].

The interest of the FIC approach is that it leads naturally to an interative scheme for evaluating the stabilization parameter in terms of the residuals of the numerical solution. The efficiency of the new approach for computing the streamline stabilization parameter in a variety of 1D and 2D advective-diffusive problems was reported in [15–17].

In this paper the FIC method is used as the basis for a new "alpha-adaptive" procedure (where alpha denotes the stabilization parameters) for obtaining stable solution in advective-diffusive problems where arbitrary sharp transverse gradients are present. The new stabilization thechnique can be viewed as an alternative class of adaptive methods where the numerical solution is enhanced by searching "adaptively" the optimal value of the streamline and transverse (crosswing) stabilization parameters while keeping the mesh and the finite element approximation unchanged. Indeed the basic alpha-adaptive process can be enhanced by combining it with standard h, p or hp adaptive schemes.

In the first part of the paper the basis of the FIC stabilized method for advective-diffusive problems are explained. Next the algorithm for computing the streamline and transverse stabilization parameters via the new "alpha-adaptive" procedure is described. Finally, the efficiency and accuracy of the new approach are shown in two examples of application.

2. STABILIZED GOVERNING EQUATIONS FOR ADVECTIVE-DIFFUSIVE TRANSPORT

2.1 One dimensional advective-diffusive problem

Let us consider for simplicity the standard advective-diffusive transport problem to be solved in a one-dimensional domain of length l (Figure 1a). Figure 1b shows a typical segment AB of length $\overline{AB} = h$ where balance (equilibrium) of fluxes must be satisfied. The values of the diffusive flow rate q and the advective transport rate $u\phi$ at a point A with coordinate $x_A = x_B - h$ can be approximated in terms of values at point B using third order Taylor's expansion. A linear variation of the source term Q over the segment is also assumed. Under these assumptions and using Fourier's law the govening balance equation can be obtained as [15,16].

$$\boxed{r - \frac{h}{2}\frac{dr}{dx} = 0} \quad , \quad 0 < x < l \tag{1a}$$

with

$$r = -\nu \frac{d(u\phi)}{dx} + \frac{d}{dx}\left(k \frac{d\phi}{dx}\right) + Q \tag{1b}$$

Figure 1. (a) One-dimensional advection-diffusion problem. (b) Finite balance domain AB

In eq.(1b) ν and k are the advective and diffusive material parameters, respectively.

Note that for $h \to 0$ (i.e. when the length of the balancing domain is infinitesimal) then the standard form of the governing equation for 1D advective-diffusive transport ($r = 0$) is recovered.

The essential (Dirichlet) boundary condition is the standard one given by

$$\phi - \bar{\phi} = 0 \qquad \text{on} \quad x = 0 \tag{2}$$

where $\bar{\phi}$ is the prescribed unknown field at the Dirichlet boundary.

For consistency the stabilized form of the Neumann boundary condition is needed. This can be obtained by invoking again the balance law in a segment AB next to a boundary point. For convenience the length of this segment is taken as half of the characteristic length h for the interior domain points [15,16].

Assuming now second order expansion for the advective and diffusive fluxes and taking the source Q to be constant over AB, the balance equation is obtained as [15,16]

$$\boxed{-\nu u\phi + k\frac{d\phi}{dx} + \bar{q} - \frac{h}{2}r = 0} \qquad \text{on} \quad x = l \tag{3}$$

where r is given by eq.(16). Obviously for $h \to 0$ the standard form of the Neumann boundary condition is recovered.

Equation (1a) can now be solved together with eqs.(2) and (3). These equations are the starting point to derive stabilized numerical schemes using any discretization procedure.

The extension of this stabilization concept to the transient case can be found in [15,16].

2.2 Two dimensional advective-diffusive problem

The concepts of previous section will be extended now to the solution of advection-diffusion problems in a two-dimensional domain Ω with boundary Γ. Let us consider a finite rectangular domain of dimensions h_x and h_y in directions x and y, respectively. Both the advective and diffusive fluxes are assumed to vary linearly along the four sides of the balance domain (Figure 2). The flux balance equation will be obtained using the following Taylor expansions: diffusive term, third order expansion; advective term, third order expansion; source term, second order expansion.

Figure 2. Balance domain for 2D advection-diffusion problem. Advective and diffusive fluxes are assumed to vary linearly along the sides

The balance of fluxes across the four sides of the rectangular domain of Figure 2 gives after some algebra [16]

$$\boxed{r - \frac{1}{2}\mathbf{h}^T \boldsymbol{\nabla} r = 0} \qquad \text{in} \quad \Omega \tag{4}$$

where

$$r = -\nu \boldsymbol{\nabla}^T \mathbf{f} + \boldsymbol{\nabla}^T (\mathbf{D} \boldsymbol{\nabla} \phi) + Q \tag{5}$$

and

$$\mathbf{h} = [h_x, h_y]^T \tag{6}$$

In eq.(5)

$$\mathbf{f} = [u\phi, v\phi]^T, \quad \boldsymbol{\nabla} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]^T, \quad \mathbf{D} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \tag{7}$$

The boundary conditions are written as

$$\phi - \bar{\phi} = 0 \qquad \text{on } \Gamma_\phi \tag{8}$$

where Γ_ϕ is the Dirichlet boundary, where the variable is prescribed, and

$$\boxed{-\nu \mathbf{n}^T \mathbf{u}\phi + \mathbf{n}^T \mathbf{D} \boldsymbol{\nabla} \phi + \bar{q}_n - \frac{1}{2}\mathbf{h}^T \mathbf{n} r = 0} \qquad \text{on } \Gamma_q \tag{9}$$

where $\bar{q}_n$ is the prescribed total flux across the Neumann boundary Γ_q with $\Gamma = \Gamma_\phi \cup \Gamma_q$ and $\mathbf{n} = [n_x, n_y]^T$ is the normal vector. Eq.(9) has been obtained by balance of fluxes in a finite boundary domain [15,16].

The standard differential equations are simply obtained by neglecting the stabilizing terms in eqs.(4) and (9) (i.e making $\mathbf{h} = 0$). The extension to three-dimensional problems is straightforward and identical stabilized expressions are obtained.

REMARK 1

It is interesting to note that the finite element Galerkin form of the new stabilized governing equations leads to a set of discretized equations identical to those obtained with the standard SUPG formulation [15]. Alternatively, the stabilized transient form leads to the well known Characteristic-Galerkin procedure [15]. This indicates that the new governing equations can be considered as the *intrinsic stabilized equations* of the problem.

2.3 The concept of intrinsic time

It is usual to accept that $\mathbf{h}$ and $\mathbf{u}$ are parallel, so that $\mathbf{h} = \frac{h}{|\mathbf{u}|}\mathbf{u}$. The distance $h = (h_x^2 + h_y^2)^{1/2}$ is then called the characteristic length of the 2D advective-diffusive problem. The *intrinsic time parameter* is now defined as [6]

$$\tau = \frac{h}{2|\mathbf{u}|} \tag{10}$$

Note that this coincides with the time taken for a particle to travel the distance $h/2$ at the speed $|\mathbf{u}|$.

The assumption of the characteristic length vector $\mathbf{h}$ being parallel to the velocity vector $\mathbf{u}$ is a *simplification* which elliminates any transverse diffusion effect. This assumption is the basis of the standard SUPG approach. However it is well known that when arbitrary sharp transverse layers are present, additional transverse (or crosswind) diffusion is required to capture these discontinuities. Different "ad hoc" expressions for the transverse diffusion terms, typically of non linear nature, have been proposed [18–20]. Indeed the introduction of this additional stabilizing effect can be simply reproduced in the FIC approach here proposed by abandoning the assumption of $\mathbf{h}$ being parallel to $\mathbf{u}$ and keeping the two characteristic lengths h_x and h_y as "free" stabilization parameters. The computation of these two parameters is described in the following section.

3. COMPUTATION OF THE STABILIZATION PARAMETERS

Let us consider the finite element solution of an advective-diffusive problem. The standard interpolation within an element e with n nodes can be written as

$$\phi \simeq \hat{\phi} = \sum_{i=1}^{n} N_i \phi_i \tag{11}$$

where N_i are the element shape functions and ϕ_i are nodal values of the approximate function $\hat{\phi}$. Substitution of eq.(11) into eq.(4) gives

$$\hat{r} - \frac{1}{2}\mathbf{h}^T \boldsymbol{\nabla} \hat{r} = r_\Omega \qquad \text{in} \quad \Omega \tag{12}$$

where $\hat{r} = r(\hat{\phi})$.

Let us now define the average residual of a particular numerical solution over an element as

$$r^{(e)} = \frac{1}{\Omega^{(e)}} \int_{\Omega^{(e)}} r_\Omega \, d\Omega \tag{13}$$

Substituting eq.(13) into (12) gives

$$r^{(e)} = \hat{r}^{(e)} - \left(\frac{1}{2}\mathbf{h}^T \boldsymbol{\nabla} \hat{r}\right)^{(e)} \tag{14}$$

where

$$a^{(e)} \equiv \frac{1}{\Omega^{(e)}} \int_{\Omega^{(e)}} a \, d\Omega \tag{15}$$

For simplicity the characteristic length vector will be assumed to be constant over each element, i.e. $\mathbf{h} = \mathbf{h}^{(e)}$. With this assumption eq.(14) can be simplified to

$$r^{(e)} = \hat{r}^{(e)} - \frac{1}{2}\left[\mathbf{h}^{(e)}\right]^T (\boldsymbol{\nabla} \hat{r})^{(e)} \tag{16}$$

Let us express the characteristic length vector in terms of the components along the velocity vector $\mathbf{u}$ and the normal velocity direction $\mathbf{u}_n$ (Figure 3) as

$$\mathbf{h} = \frac{1}{|\mathbf{u}|}[h_s \mathbf{u} + h_n \mathbf{u}_n] \tag{17}$$

where $\mathbf{u}_n = [-v, u]^T$ and h_s and h_n are streamline and transverse (crosswind) characteristic lengths, respectively.

Figure 3. Characteristic length in global and velocity axes.

Substituting (17) into (16) gives

$$r^{(e)} = \hat{r}^{(e)} - \frac{1}{2|\mathbf{u}|}\left[h_s \mathbf{u}^T + h_n \mathbf{u}_n^T\right]^{(e)} (\boldsymbol{\nabla}\hat{r})^{(e)} \tag{18}$$

The characteristic lengths h_s and h_n can be expressed now as a proportion of a typical element dimension $l^{(e)}$

$$h_s^{(e)} = \alpha_s^{(e)} l^{(e)} \quad , \quad h_n = \alpha_n^{(e)} l^{(e)} \tag{19}$$

where $\alpha_s^{(e)}$ and $\alpha_n^{(e)}$ are the streamline and transverse stabilization parameters, respectively. In the examples shown next $l^{(e)}$ has been taken equal to the length of the longest side of each triangular element.

Clearly for $\alpha_n^{(e)} = 0$ just the streamline diffusive effect, typical of the SUPG approach, is reproduced.

Let us consider now that an enhanced numerical solution has been found for a given finite element mesh. This can be simply achieved by projecting into the original mesh an improved solution obtained via global/local smoothing or superconvergent recovery of derivatives [21,22]. If $r_1^{(e)}$ and $r_2^{(e)}$ respectively denote the element residuals of the original and the enhanced numerical solutions for a given mesh it is obvious that

$$r_1^{(e)} - r_2^{(e)} \geq 0 \tag{20}$$

Eq. (11) assumes that r_1 is positive. Clearly for the negative case the inequality should be appropiately reversed.

Combining eqs.(18),(19) and (20) gives

$$[\alpha_s \mathbf{u}^T + \alpha_n \mathbf{u}_n^T]^{(e)} (\boldsymbol{\nabla}\hat{r}_2^{(e)} - \boldsymbol{\nabla}\hat{r}_1^{(e)}) \geq \frac{2|\mathbf{u}|}{l^{(e)}} (\hat{r}_2^{(e)} - \hat{r}_1^{(e)}) \tag{21}$$

3.1 Computation of α for elements at the boundaries

The stabilized balance equation at a boundary can be written after discretization as

$$-\nu \mathbf{n}^T \mathbf{u}\hat{\phi} + \mathbf{n}^T \mathbf{D}\boldsymbol{\nabla}\hat{\phi} + q - \frac{1}{2}\mathbf{h}^T \mathbf{n}\hat{r} = r_\Gamma \tag{22}$$

where q represents the prescribed normal flux at a Neumann boundary, or alternatively the unknown normal flux at the Dirichlet boundary where ϕ is prescribed.

Following the arguments used previously the equation defining the stabilization parameters at a boundary element can be obtained as

$$[\alpha_s \mathbf{u}^T + \alpha_n \mathbf{u}_n^T]^{(e)} \mathbf{n}(\hat{r}_2^{(e)} - \hat{r}_1^{(e)}) \geq -\nu \mathbf{n}^T \mathbf{u}(\hat{\phi}_2^{(e)} - \hat{\phi}_1^{(e)}) + \mathbf{n}^T \mathbf{D}(\boldsymbol{\nabla}\hat{\phi}_2^{(e)} - \boldsymbol{\nabla}\hat{\phi}_1^{(e)}) + q_2^{(e)} - q_1^{(e)} \tag{23}$$

where $(\cdot)^{(e)}$ denotes average values over a boundary domain and indexes 1 and 2 refer to the original and enhanced solutions, respectively. The enhanced nodal values $\hat{\phi}_2^{(e)}$ can be obtained by superconvergent nodal recovery of primary variables [22].

Note that the equality sign in eqs.(21) and (23) provides the value of the stabilization parameters ensuring no growth of the numerical error. In reference [15] it is proved that this yields the standard critical value of α in the simplest sourceless one dimensional problem solved with linear elements.

Eqs.(21) and (23) are the basis for the alpha-adaptive scheme to be described in next section.

4. ALPHA-ADAPTIVE STABILIZATION SCHEME

The following scheme can be devised to obtain an stable numerical solution in an adaptive manner.

(1) Solve the stabilized problem defined by eqs.(4), (8) and (9) using the FEM with an initial guess of the stabilization parameters, i.e.

$$\alpha_s^{(e)} = {}^o\alpha_s^{(e)} \quad , \quad \alpha_n^{(e)} = {}^o\alpha_n^{(e)} \tag{24}$$

(2) Recover an enhanced derivatives field. Evaluate $\hat{r}^{(1)}, \hat{r}^{(2)}, \boldsymbol{\nabla}\hat{r}_1^{(e)}$ and $\boldsymbol{\nabla}\hat{r}_2^{(e)}$.

(3) Compute an enhanced valued of the streamline stabilization parameter $\alpha_s^{(e)}$ by

$${}^1\alpha_s^{(e)} = \frac{2|\mathbf{u}|}{l^{(e)}\mathbf{u}^T(\boldsymbol{\nabla}\hat{r}_2^{(e)} - \boldsymbol{\nabla}\hat{r}_1^{(e)})}[\hat{r}_2^{(e)} - \hat{r}_1^{(e)} - \alpha_n^{(e)}\mathbf{u}_n^T(\boldsymbol{\nabla}\hat{r}_2^{(e)} - \boldsymbol{\nabla}\hat{r}_1^{(e)})] \tag{25}$$

If the element lays in one of the boundaries the expression for $\alpha_s^{(e)}$ as deduced from eq.(23) should be used.

(4) Repeat steps (1)–(3) until convergence is found for the value of $\alpha_s^{(e)}$ while keeping $\alpha_n^{(e)}$ constant.

(5) Repeat steps (1)–(4) for computing $\alpha_n^{(e)}$ while keeping $\alpha_s^{(e)}$ constant and equal to the previously converged value. In the first iteration $\alpha_n = {}^o\alpha^{(e)} + \varepsilon$ where ε is a small value should be used. The updated value of $\alpha_n^{(e)}$ is computed as

$${}^i\alpha_n^{(e)} = \frac{2|\mathbf{u}|}{l^{(e)}\mathbf{u}_n^T(\boldsymbol{\nabla}\hat{r}_2^{(e)} - \boldsymbol{\nabla}\hat{r}_1^{(e)})}[\hat{r}_2^{(e)} - \hat{r}_1^{(e)} - \alpha_s^{(e)}\mathbf{u}^T(\boldsymbol{\nabla}\hat{r}_2^{(e)} - \boldsymbol{\nabla}\hat{r}_1^{(e)})] \tag{26}$$

Again for a boundary element the expression for $\alpha_n^{(e)}$ deduced from eq.(23) should be used.

(6) Once $\alpha_n^{(e)}$ has been found steps (1)–(5) can be repeated to obtain yet more improved values of both $\alpha_s^{(e)}$ and $\alpha_n^{(e)}$.

Note that for $\alpha_n^{(e)} = 0$ above adaptive scheme provides the value of the critical streamline stabilization parameter $\alpha_s^{(e)}$ corresponding to the well known SUPG procedure. It can be shown that for the simplest 1D sourceless advective-diffusive case solved with linear elements the well known critical value $\alpha_s^{(e)} = 1 - \frac{1}{\gamma^{(e)}}$, where $\gamma^{(e)} = \frac{ul^{(e)}}{2k}$ is the element Peclet number is obtained. Indeed accounting for the cross-wind stabilization parameter α_n has proved to be essential for obtaining stable solution in presence of arbitrary transverse sharp layers.

In the examples shown next the enhanced derivative field has been obtained by the simplest nodal averaging procedure. It has also been found useful to smooth the distribution of the $\alpha_s^{(e)}$ and $\alpha_n^{(e)}$ values and this has been done again using nodal averaging. Note also that the number of iterations in the above adaptive process is substantially reduced if the initital guess for $\alpha_s^{(e)}$ and $\alpha_n^{(e)}$ are not far from the final converged values. This can be ensured by using as initial value for $\alpha_s^{(e)}$ the standard expression derived from the straight forward extension of the simple 1D case, whereas the initial guess ${}^o\alpha_n^{(e)} = 0$ provides a good approximation in zones far from sharp layers non orthogonal to the velocity vector.

5. EXAMPLES

5.1 Example 1. Two dimensional advective-diffusive problem with no source, diagonal velocity and uniform Dirichlet boundary conditions

The first 2D example chosen is the solution of the standard advection-diffusion equation in a square domain of unit size with

$$k_x = k_y = 1 \quad , \quad \mathbf{u} = [1, 1]^T \quad , \quad \nu = 1 \times 10^{10} \quad , \quad Q = 0$$

The following Dirichlet boundary conditions are assumed

$\phi = 0$ along the boundary lines $x = 0$ and $y = 0$
$\phi = 100$ along the boundary line $x = 1$
$q_n = 0$ along the boundary line $y = 1$

The expected solution in this case is a uniform distribution of $\phi = 0$ over the whole domain except in the vecinity of the boundary $y = 1$ where a boundary layer is formed.

The domain has been discretized with a uniform mesh of 800 three node triangles as shown in Figure 4. The initial values ${}^o\alpha_s^{(e)} = {}^o\alpha_n^{(e)} = 0$ have been taken in all elements.

Figure 5 shows the initial distributions of ϕ for $\alpha_s^{(e)} = \alpha_n^{(e)} = 0$ (standard Galerkin solution). Note the strong oscillations obtained as expected.

The final converged solution for ϕ after 7 iterations is displayed in Figure 6. Note that the boundary layer originated in the vecinity of the boundary at $y = 1$ is well reproduced with minimum oscillations. These oscillations grows considerably higher if the value of the transverse stabilization parameter $\alpha_n^{(e)}$ is kept equal to zero during the adaptive process, thus yielding the standard SUPG solution, as shown in Figure 7.

Figure 8 shows finally the smoothed distribution of the stabilization vector $\boldsymbol{\alpha} = \alpha_s \mathbf{u} + \alpha_n \mathbf{u}_n$. Note that in the central part of the domain the $\boldsymbol{\alpha}$ vectors are aligned with the velocity direction (i.e. $\alpha_n = 0$), whereas in the vecinity of the boundaries the effect of the transverse stabilization parameter α_n leads to a noticeable change of the direction of $\boldsymbol{\alpha}$.

Figure 4. Example 1. Sourceless advective-diffusive problem with diagonal velocity. Finite element mesh of 800 linear triangles.

Figure 5. Example 1. Initial oscillatory distribution of ϕ for $\alpha_s^{(e)} = \alpha_n^{(e)} = 0$.

Figure 6. Example 1. Final distribution of ϕ after 7 iterations.

5.2 Example 2. Two dimensional advective-diffusive problem with no source and non uniform Dirichlet boundary conditions

The advection-diffusion equations are now solved with

$$\Omega =]-\frac{1}{2},\frac{1}{2}[\times]-\frac{1}{2},-\frac{1}{2}[\quad , \quad \mathbf{u} = [\cos\theta, -\sin\theta]^T$$

$$k_x = k_y = 10^{-6}, \quad Q(x,y) = 0 \quad , \quad \bar{\phi}(x,y) = \begin{cases} 1 & \text{if} \quad (x,y) \in \Gamma_{\phi_1} \\ 0 & \text{if} \quad (x,y) \in \Gamma_{\phi_2} \end{cases}$$

with $\Gamma_{\phi_1} = \{-1/2\} \times [1/4, 1/2] \cup]-1/2, 1/2[\times\{1/2\}$, $\Gamma_{\phi_2} = \Gamma_\phi - \Gamma_{\phi_1}$ and $\Gamma_q = 0$.

Figure 7. Distribution of ϕ along a central line obtained with the present discontinuity capturing method (DC) and the SUPG formulation ($\alpha_n = 0$).

Figure 8. Example 1. Final distribution of the stabilization vector $\boldsymbol{\alpha} = \alpha_s \mathbf{u} + \alpha_n \mathbf{u}_n$.

A unstructured mesh of 902 linear triangles has been chosen (Figure 9.). The problem has been chosen for an angle of $\mathbf{u}$ given by $\tan\theta = 2$. Once again the initial values ${}^o\alpha_s^{(e)} = {}^o\alpha_n^{(e)} = 0$ have been taken.

Figure 10 shows the oscillatory distribution of ϕ obtained for the first solution, as expected. The final distribution of ϕ after 7 iterations is displayed in Figures 11 and

Figure 9. Example 2. Two dimensional advective-diffusive problem with zero source and non uniform boundary conditions. Geometry and unstructured finite element mesh of 902 linear triangles.

Figure 10. Example 2. Initial oscillatory solution for ϕ obtained for $\alpha^{(e)} = \alpha_n^{(e)} = 0$.

Figure 11. Final solution of ϕ after 7 iterations.

12. Note that both the boundary layers at the edges and the internal sharp layer are captured with minor oscillations. These oscillations are more pronounced near the right hand side edge (Figure 13 and 15) when $\alpha_n^{(e)} = 0$ is taken through out the adaptive process (SUPG solution).

Figure 14 shows finally the distribution of the stabilization vector $\boldsymbol{\alpha} = \alpha_s \mathbf{u} + \alpha_n \mathbf{u}_n$. Again note that the direction of $\boldsymbol{\alpha}$ in the smooth part of the solution is aligned with that of the velocity vector, whereas the effect of the transverse stabilization term is very pronounced near the sharp gradient boundary regions. This leads to a change in the direction of $\boldsymbol{\alpha}$ in these zones.

Figure 12. Example 2. Final distribution of ϕ after 7 iterations.

Figure 13. Example 2. Final distribution of ϕ obtained with $\alpha_n = 0$ (SUPG method).

Figure 14. Example 2. Final distribution of the stabilization vector $\boldsymbol{\alpha} = \alpha_s \mathbf{u} + \alpha_n \mathbf{u}_n$.

CONCLUSIONS

The new stabilized form of the governing differential equations derived via a “finite increment calculus” approach seems to be the natural root for obtaining stable finite element methods for advective-diffusive problems. The stabilized governing

Figure 15. Distribution of ϕ along a central line obtained with the present discontinuity capturing method (DC) and the SUPG formulation ($\alpha_n = 0$).

equations are also the basis for computing line the streamline and crosswind stabilization parameters necessary to capture arbitrary sharp transverse layers. The new stabilization approach can be interpreted as a class of adaptive methods where the numerical solution is enhanced by progressively improving the value of the stabilization parameter, while keeping the mesh and the finite element approximation unchanged. The efficiency of this alpha-adaptive procedure has been shown for two problems with sharp gradients where accounting for the crosswind stabilization parameter has proved to be essential to obtain accurate solutions.

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