Numerical solution of the Cauchy problem for the Laplace equation: A deterministic and Bayesian approach X International Conference on Adaptive Modeling and Simulation – ADMOS 2021

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ABSTRACT

The problem of determining a harmonic function in a bounded annular region from measurements on part of the boundary (Cauchy data), is called the Cauchy problem for the Laplace equation. We present a numerical study for the solution of the following problem in an annular region Ω: Given a function V defined on the exterior boundary Γ_2, find a function $\varphi = u|_{\Gamma_1}$ defined on the interior boundary Γ_1, which is the trace of a function $u \in H^1(\Omega)$ that satisfies:

$$-\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$u = V \quad \text{on} \quad \Gamma_2, \tag{2}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma_2. \tag{3}$$

This problem is ill-posed and arise in several applications. Many numerical solution techniques has been applied to this problem. In a previous study [1], this problem was solved in annular complex regions with a variational approach based on its reformulation as a boundary control problem, for which the cost function φ incorporates a penalized term with the input Cauchy data. Lagrange linear finite elements are good enough to solve the forward problem.

Here, we go beyond and consider a statistical inversion computational model. We propose a model with Gaussian distributions based on Bayes formula. In particular, the a priori model is built up from Gaussian Markov random fields (GMRF) for spatial statistics [2], and we propose different precision matrices for the Cauchy problem. We take advantage of the relationship between the a priori distribution and traditional Tikhonov regularization to propose different models where smooth and non-smooth regularization is possible.

Concerning the Cauchy data V, we assume white noise on the measurements. So, the solution of the statistical model is given by a Gaussian posterior distribution. This distribution is explored by a MCMC sampling based on a Metropolis–Hasting algorithm known as the t-walk, [3]. We take advantage of the connection between the information of the data and the knowledge of the prior in order to make a low range analysis (see [4]) to estimate the optimal number of unknown parameters (discrete values of φ)with respect to the number of measurements. The numerical results obtained are excellent and consistent with the previous results obtained with the deterministic variational approach.

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