ADAPTIVE MESH REFINEMENT TECHNIQUES FOR STRUCTURAL PROBLEMS

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SUMMARY

In this paper some adaptive mesh refinement (AMR) strategies for finite element analysis of structural problems are discussed. Two mesh optimality criteria based on the equal distribution of: (a) the global error, and (b) the specific error over the elements are studied. It is shown that the correct evaluation of the rate of convergence of the different error norms involved in the AMR procedures is essential to avoid oscillations in the refinement process. The behaviour of the different AMR strategies proposed is compared in the analysis of some structural problems.

1. INTRODUCTION

The evaluation of discretization errors and the design of suitable meshes via adaptive mesh refinement (AMR) are nowadays two of the challenging issues in the analysis of structures using the finite element method (FEM).

The topic of error estimation and mesh adaptivity in the FEM is by no means new. For a comprehensive review of the topic see the reference list of Chapter 14 of Volume I of [26]. Zienkiewicz and Zhu [1-5] have introduced a successful AMR strategy for elasticity problems using a simple error estimate based on the difference of the discontinuous finite element stress (or strain) field with an "improved" smooth solution. These authors and others have extended this AMR procedure for plate bending and shell analysis[9-10],[19]. This AMR strategy has also been successfully used for fluid flow problems [2],[19].

In this paper we present an overview of different AMR strategies based on the Zienkiewicz and Zhu error estimator [1-6] for structural analysis. We will show first how the standard AMR algorithm based on equal distribution of the error over all the elements requires a careful identification of the rate of convergence of the different error terms involved in the design of the new element size to avoid oscillations in the refinement process.

An alternative AMR strategy based on the equal-distribution of the "specific error", i.e. the error per unit area (or volume) in all the elements in the mesh will be presented next. We will see in the examples shown that this
strategy allows to concentrate more and smaller elements in zones where stress concentrations occur, as it should be expected from the engineering point of view.

In next section some basic concepts of the error estimation process are given.

2. BASIC CONCEPTS OF ERROR ESTIMATION

In dealing with adaptive mesh refinement the following two concepts should be clearly defined:

(a) **Error estimator.** Since the "exact" solution is not known, a method to approximately evaluate the error of the finite element solution should be defined.

(b) **Acceptable solution.** A finite element solution is "acceptable" if the estimated error satisfies some prescribed global and local conditions.

Both concepts (a) and (b) are further explained in next sections.

2.1 Error estimator

One of the most popular error estimators for structural problems is based on the error energy norm expressed as [1-6]

$$
\|e\| = \left( \int_\Omega (\sigma - \hat{\sigma})^T D^{-1} (\sigma - \hat{\sigma}) d\Omega \right)^{1/2}
$$

(1)

where \( \sigma \) are the exact stresses, \( \hat{\sigma} \) are the stress values obtained from the finite element solution, \( D \) is the constitutive matrix and \( \Omega \) is the domain structure. For plate and shell problems \( \sigma \) should here be interpreted as the "resultant stresses" [10], [16].

Since the exact stresses are usually not known they are approximated by

$$
\sigma \approx \sigma^* = N\hat{\sigma}^*
$$

(2)

where \( \hat{\sigma}^* \) are nodal values obtained by simple nodal averaging of the finite element values, least squares local and global smoothing, or other adequate projection methods [1-5]. A simple approach is to use a global nodal smoothing with a lumped "mass" matrix giving the nodal smoothed values, \( \hat{\sigma}^* \), as

$$
\hat{\sigma}^* = M_D^{-1} \int_\Omega N_\sigma \hat{\sigma} d\Omega
$$

(3)

where \( N_\sigma \) are the chosen stress interpolating functions giving a smooth stress field in terms of the nodal stress values obtained [1-6] and \( M_D = \int_\Omega N_\sigma N_\sigma d\Omega \). Eq. (3) can be obviously applied to solve independently for each individual stress component. It can be verified that eq.(3) yields an accurate smoothed stress field for elements with linear interpolation and is adequate for quadratic elements providing correction factors are used [1]. Recent much improved methods have been devised by Zienkiewicz and Zhu [23] to obtain accurately recovered \( \sigma^* \) and these can be used as alternatives.

The strain energy of the exact solution is estimated as

$$
\|e\|^2 = \left( \int_\Omega (\sigma - \hat{\sigma})^T D^{-1} (\sigma - \hat{\sigma}) d\Omega \right)
$$

(4)

Both \( \|e\|^2 \) and \( \|U\|^2 \) can be evaluated as sum of their respective element contributions so that

$$
\|e\|^2 = \sum_{i=1}^n \|e_i\|^2 \quad \|U\|^2 = \sum_{i=1}^n \|U_i\|^2
$$

(5)

where \( n \) is the total number of elements in the mesh.

2.2 Definition of acceptable solution

There is unfortunately no general consensus of what accuracy is acceptable. However, it is usually accepted that a solution is "correct" if the two following conditions are satisfied:

(a) The global error in energy norm is less than a specified percentage value of the total strain energy.

$$
\|e\| \leq \eta \|U\|
$$

(6)

where \( \eta \) is the user's specified limit value of the permissible relative global error.

Eq.(6) allows to define a global error parameter, \( \xi_G \), as

$$
\xi_G = \frac{\|e\|}{\|U\|}
$$

(7)

Clearly the values \( \xi_G \geq 1 \) denote satisfaction of the global error criterion, whereas \( \xi_G > 1 \) indicates that further refinement is necessary.

(b) The distribution of the elements in the refined mesh satisfies a "mesh optimality criterion". The local condition can be expressed as

$$
\|e_i\| \leq \|e_i\|
$$

(8)

where \( \|e_i\| \) is the actual error norm in each element \( i \) and \( \|e_i\| \) is the "required" error norm in the element.

From eq.(8) we can define a local error parameter \( \xi_i \) for each element \( i \) as

$$
\xi_i = \frac{\|e_i\|}{\|e_i\|}
$$

(9)
Note that a value of $\xi_i = 1$ defines an "optimal" element size, whereas $\xi_i > 1$ and $\xi_i < 1$ indicate that the size of element $i$ needs refinement and refinement, respectively.

The definition of the required error in each element $\|e_i\|_H$ is a key issue and it only affects the distribution of element sizes in the mesh. This definition can be based on different mesh optimality criteria and some of these are presented in a later section.

3 Element refinement parameter

We can define now a single element refinement parameter, combining the definition of the global and local conditions (a) and (b) of previous section as:

$$\xi_i = \xi_i\xi_{Ei} = \frac{\|e_i\|_{\text{local}}}{\eta(U)^{\frac{1}{p+1}}} \xi(E_i)$$

(10)

The element refinement parameter $\xi_i$ was first introduced in ref[1] and since on it has been used by many authors as the basis for deciding the new element size in a general AMR strategy [1-9]. However, $\xi_i$ can also be interpreted from (3) as the result of trying to satisfy both the global and local error conditions in a successive manner. Eq.(10) provides all the terms involved in this combined access and these could play individually a very different role as explained in the section.

MESH OPTIMALITY CRITERIA AND AMR PROCEDURES

1 Mesh optimality criterion based on the equal-distribution of the global error

A very popular mesh optimality criterion for structural analysis is based on the so-called equal-distribution of the error, i.e., a mesh is defined as optimal if the global error is equally distributed over all the elements [1-10]. On the basis of this assumption we can define the required error for each element as the ratio between the global error and the total number of elements in the mesh. This means that only the square of the error norm is additive (see eq.(5)) we have:

$$\|e_i\|_H = \frac{\|e_i\|}{\sqrt{m}}$$

(11)

Combining (9) and (11) yields the expression of the local error parameter as:

$$\xi_i = \frac{\|e_i\|_H}{\|e_i\|_{\text{local}}} \xi(E_i)$$

(12)

The element refinement parameter is now obtained vis. eq.(19) as:

$$\xi_i = \xi_{Ei} = \frac{\|e_i\|}{\eta(U)^{\frac{1}{p+1}}} \xi(E_i)$$

(13)

The parameter $\xi_i$ can now be readily interpreted as the ratio between the element error and the distributed value of the permissible error over the mesh.

Expression (13) is is identical to that used in ref[1]. However, the multiplicative form $\xi_i = \xi_{Ei}$ allows to derive the correct AMR strategy. Thus by noting that the convergence rates of the element and global error norms are:

$$\|e_i\|_H \Omega(H_i^{m+\frac{1}{2}}) \simeq O(H_i^{m-\frac{1}{2}})$$

(14a)

$$\|e_i\|_H \Omega(H_i^{m+\frac{1}{2}}) \simeq O(H_i^{m})$$

(14b)

where $h_i$ and $h$ are the existing element size and the average size of all the elements in the mesh, respectively, $m$ is the degree of the shape function polynomials (m = 1 for linear elements, m = 2 for quadratic elements, etc.), and d is the number of dimensions of the problem (d = 1, 2, 3 for 1D, 2D and 3D problems, respectively). It can be deduced that the new element size $h_i$ can be obtained in terms of the existing size using the expression:

$$\frac{h_i}{h} = \frac{\xi_i}{\xi_{Ei}}$$

(15)

$$\xi_i = \xi_{Ei} = \frac{\|e_i\|}{\eta(U)^{\frac{1}{p+1}}} \xi(E_i)$$

(16)

The expression of the element size parameter $\xi$ as given by (16) takes into account the different convergence rates of the element and global error norms.

Zienkiewicz and Zhu, followed by others [1-10], use a simpler expression for $\xi$ based directly on the element refinement parameter $\xi_i$ as:

$$\xi_i = \frac{\|e_i\|^2}{\eta(U)^{\frac{1}{p+1}}} \xi(E_i)$$

(17)

where $c$ is a relaxation factor and the new exponent $m'$ is taken as $m$ except for elements adjacent to singularities where $m' = \lambda$ is used ($\lambda$ being the singularity strength).

The authors have found that the computation of $\xi$ as given by (17) with the very common choice of $c = 1$ and $m' = m = 1$ [1-10], leads to a non consistent mesh refinement [16]. This is shown by an oscillatory re- and de-refinement of the same mesh zones in the AMR process. This problem disappears if the correct expression (16) for $\xi$ used in the first example and (16).
2 A mesh optimality criterion based on the equal-distribution of the specific error

An alternative criterion is to assume that a mesh is optimal if the square of the error per unit area (or volume) is the same over the whole mesh. It is clear then that in the optimal mesh

$$\frac{\|e\|_{1,\Omega}}{\Omega^{1/2}} = \frac{\|e\|_{1,\Omega'}}{\Omega'^{1/2}}$$

(18) 

- obviously in (18) \(\Omega\) and \(\Omega'\) denote the element and total area (or volume) respectively.

Comparing (8) and (18) yields the expression of the required error norm for each element as

$$\frac{\|e\|_{1,\Omega}}{\Omega^{1/2}} = \frac{\|e\|_{1,\Omega'}^{1/2}}{\Omega'^{1/2}}$$

(19) 

The element error parameter \(\xi\) is obtained now using (9) and (19) as

$$\xi = \frac{\|e\|_{1,\Omega}}{\Omega^{1/2}} = \frac{\|e\|_{1,\Omega'}^{1/2}}{\Omega'^{1/2}}$$

(20) 

The element refinement parameter is obtained from (9), (10) and (30) as

$$\xi = \xi e \eta = \frac{\|e\|_{1,\Omega'}}{\eta^{1/2}}$$

(21) 

Note that the equations (18) and (21) coincide if \(\xi e = \eta\) (i.e. all elements are equal size). This is however not the case for unstructured meshes which results in the different mesh distributions for each mesh optimality criteria as shown in the examples.

Moreover, the way we have defined now the element error eliminates its dependence on the element area. Therefore, the new convergence rate of the element error can be deduced from (14a) as

$$\frac{\|e\|_{1,\Omega}}{\Omega^{1/2}} \approx O(h_p)$$

(22) 

Note the coincidence of the convergence rates of the element and global error norms (eqs. (14b) and (22)).

The new element size is obtained from (15) with \(\xi\) given now by

$$\xi = (\xi e \eta)^{1/m} = (\xi)_{1/m}$$

(23)
Results labelled as strategy B in Figure 2 have been obtained with the same mesh optimality criterion, but using now the correct expression for $\xi$ as given by eq. (16). Note that the refined regions are oscillated and the AMR process converges in a consistent manner.

Results for strategy C have been obtained with the mesh optimality criterion based on the equal distribution of the specific error, with the element size parameter $\xi$ as given by eq. (23). It can be seen that (a) The AMR process converges without oscillations, and (b) This AMR strategy concentrates more and smaller elements in the vicinity of the free edge (where the error is greater due to the higher membrane stress gradients), whereas in the rest of the mesh bigger elements than in the previous cases are allowed. The prior to be paid is the increase in the total number of elements with respect to strategies A and B for the same global accuracy as shown in Figure 2.

Further details of this example and of the general AMR strategy for plates and shells can be found in [16].

4.2 Analysis of a cylindrical shell with a circular perforation under uniform traction

Figure 3 shows the geometrical and material properties of the shell and the initial mesh used. The analysis has been performed using the same facet shell element as in the previous example. A value of the permissible global error $\eta = 10\%$ has been taken.

Figure 3 also shows the sequence of refined meshes obtained with the criterion of equal-distribution of the global error and the correct value for the element size parameter $\xi$ as given by eq. (16) (Strategy B), and also with the criterion of equal distribution of the specific error with $\xi$ given by eq. (23) (Strategy C).

Table 4 shows some characteristic results for each solution like the number of elements, the global error parameter $\xi$, the average of the local error parameter $E_1$ and its mean deviation $E_2$ over each mesh for the two AMR strategies used. From the numbers shown in this table we deduce:

- Both AMR strategies converge to the global permissible error chosen.
- Both AMR processes converge to an "optimal mesh", characterised by the apriori values $(E_1)_0 = 1.0$ and $(E_2)_0 = 0.0$. However, the number of elements and its distribution in each mesh is very different for the two AMR strategies.
- Again, the distribution of specific error (strategy C) tends to concentrate more and smaller elements in the vicinity of the central hole where stress gradients are higher (see Figure 3).

6. CONCLUDING REMARKS

We have shown that the correct evaluation of the rate of convergence of the different error norms involved in the AMR strategy is essential to avoid oscillations in the refinement process. Also, a mesh optimality criteria based on global and specific error distributions are conceptually very different. Thus, whereas the former leads to meshes with a larger number of elements, the second captures
better the effect of high stress gradients. Further research should allow to balance the possibilities of these two criteria for use in practical structural applications.

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REFERENCES


