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JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 158 (2003) 243–276

www.elsevier.com/locate/cam

An iteration-by-subdomain overlapping Dirichlet/Robin domain decomposition method for advection–diffusion problems

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Received 6 February 2002; received in revised form 26 November 2002

Abstract

We present a new overlapping Dirichlet/Robin Domain Decomposition method. The method uses Dirichlet and Robin transmission conditions on the interfaces of an overlapping partitioning of the computational domain. We derive interface equations to study the convergence of the method and show its properties through four numerical examples. The mathematical framework is general and can be applied to derive overlapping versions of all the classical nonoverlapping methods.

Keywords: Overlapping; Mixed domain decomposition method; Finite elements

1. Introduction

In this paper we present a domain decomposition (DD) method to solve scalar advection–diffusion–reaction (ADR) equations which falls into the category of *iteration-by-subdomain* DD methods.

Domain decomposition methods are usually divided into two families, namely overlapping and nonoverlapping methods. The former ones are based on the Schwarz method, first studied by Schwarz in 1869 and more recently returned to focus in [15]. At the differential level, this domain decomposition method uses alternatively the solution on one subdomain to update the Dirichlet data of the other. Although the Schwarz method presents a severe drawback, i.e., the dependence of the rate of convergence upon the overlapping length, as noted in [16]

> [...] the Schwarz algorithm [...] presents some properties (like "robustness", or indifference to the type of equations considered...) which do not seem to be enjoyed by other methods.

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0377-0427/03/$ - see front matter
doi:10.1016/S0377-0427(03)00447-3

Contrary to the Schwarz method, nonoverlapping DD methods use necessarily two different transmission conditions on the interface, in such a way that both the continuity of the unknown and its first derivatives are achieved on the interface (for ADR equations). The transmission conditions can correspond to the essential and natural boundary conditions of the weak form of the problem; however, this is not a requirement. Four types of couplings are possible. The Dirichlet/Neumann method was introduced in [4] and presented in the finite element context in [19] and extensively reviewed in [26]. Alonso et al. [2] developed a coercive Dirichlet/Robin method, called γ-D/R, which uses the Robin transmission condition given by the natural condition of the weak formulation plus a constant to increase the coercivity of the associated preconditioner. The Robin/Robin method was first introduced in [17] for the Poisson equation as a generalization of the Schwarz method to nonoverlapping subdomains. This method was reinterpreted within an augmented Lagrangian framework in [12]. The freedom in choosing the coefficients of the Robin condition, when one does not exactly use the natural boundary condition of the weak form, has led to many formulations: see for example [23,18,24,2] for the coercive γ-Robin/Robin method. The Robin/Neumann coupling is also a possible choice (see, e.g., [26]). Note that the case of the Neumann/Neumann method [5] is different as it introduces an additional system for each subdomain at the differential level. However, it leads at the algebraic level to a preconditioned Richardson method, like the other methods introduced previously (see, e.g., [27,3,1]).

Some of these mixed methods present limitations, related to the fact that the boundary conditions must be imposed in accordance to the direction of the flow when advection is dominant. This requirement is at its turn closely related to the well-posedness of the local variational problems. This was the argument for developing the so-called adaptive methods. Adaptive domain decomposition methods take into account the direction of the flow on the interface. The adaptive Dirichlet/Neumann method imposes a Dirichlet transmission condition at inflows and Neumann transmission condition at outflows, the inflow on one side of the interface being the outflow on the other. See for example [6,7,11,30]. In [10], an iteration-by-subdomain DD method is devised to solve an advection–reaction transport equation, where only Dirichlet data are prescribed on the inflow parts of the interface.

In the literature, all the mixed DD methods mentioned previously have been mainly studied in the context of disjoint partitioning. However, there exists no particular reason for restricting their application only to nonoverlapping subdomains. See for example [21,22,28]. This paper gives a possible line of study for the generalization of the mixed method to overlapping subdomains. We expect that the overlapping mixed DD methods will enjoy some properties of their disjoint brothers as well as some properties of the classical Schwarz method, as for example the dependence on the overlapping length.

Our motivation to study these types of methods has been to maintain the implementation advantages of the Schwarz method when used together with a numerical approximation of the problem. The possibility to have some overlapping simplifies enormously the discretization of the subdomains. However, very often this overlapping needs to be very small in practice, and thus the convergence rate of the Schwarz method becomes very small. Contrary to the Schwarz method, the limit case of zero overlapping will be possible using the formulation proposed herein. We have chosen to study an overlapping Dirichlet/Robin method, using the coercive bilinear form presented in [2] in the context of the γ-D/R and γ-R/R methods. This simplifies the analysis of the DD method as no assumption has to be made on the direction of the flow and its amplitude on the interfaces of the overlapping subdomains.

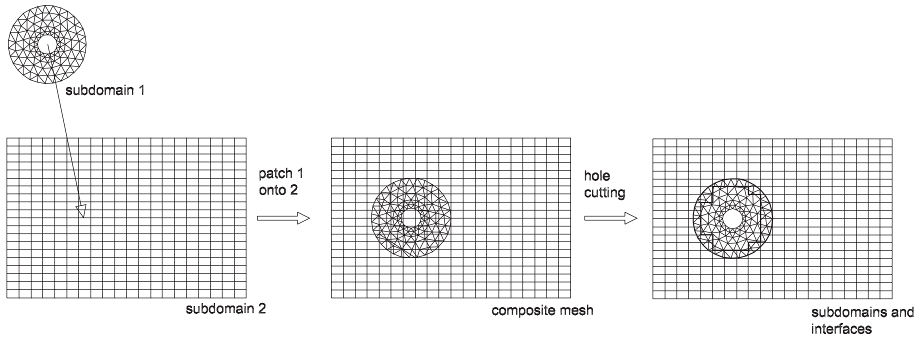


Fig. 1. Chimera method.

We would like to stress that our approach *is not* to view domain decomposition as a preconditioner for solving the linear systems of equations arising after the space discretization of the differential equations. In our case, the domain is decomposed at *the continuous level*. We are not concerned with the scaling properties with respect to the number of subdomains of the iteration-by-subdomain strategy we propose. For our purposes, it is enough to analyze *two* subdomains, in the same spirit as [19,2,3,10]. More precisely, our final goal is to devise a Chimera-type strategy, and this paper must be understood as a theoretical basis for such a formulation. We recall briefly the Chimera method, of which we give an example in Fig. 1. Firstly, independent meshes are generated for the background mesh and the mesh around the cylinder. Secondly, the mesh around the cylinder is placed on the background mesh. Then, according to some criteria (order of interpolation, geometrical overlap prescribed, etc.), we can impose in a simple way a Dirichlet condition on some nodes of the background located inside the cylinder subdomain (this task is called hole cutting, as some elements do not participate any longer to the solution process). Doing so, we form an apparent interface on the background subdomain to set up an iteration-by-subdomain method. Note that a natural condition of Neumann or Robin type is in general not possible as the apparent interface is irregular. Finally, by imposing a Dirichlet, Neumann or Robin condition on the outer boundary of the cylinder subdomain we can define completely an iteration-by-subdomain method to couple both subdomains. The Chimera method was first thought as a tool to simplify the meshing of complicated geometry. It is also a powerful tool to treat subdomains in relative motion.

The paper is organized as follows. In the following section, we introduce the continuous problem and derive the corresponding variational formulation. Then we present the new domain decomposition method. For simplicity, we shall restrict ourselves to a two-domain variational formulation of the problem, originating from a geometrical decomposition of the original domain of study; we follow the strategy presented in [20] for the classical Dirichlet/Neumann method and extensively studied in [26]. We show how the formulation can be reformulated into an overlapping domain decomposition method based on a Dirichlet/Robin coupling and how this formulation can be simply derived from the continuous problem. We then present the corresponding interface equations in terms of Steklov–Poincaré operators and show the convergence of the relaxed algorithm. Finally, we present four numerical examples and compare the results of this new domain decomposition method to the classical Dirichlet/Robin and Schwarz methods.

2. Problem statement

Let us consider the advection–diffusion–reaction problem of finding u such that

$$\begin{aligned} Lu &:= -\varepsilon\Delta u + \nabla\cdot(\boldsymbol{a}u) + \sigma u = f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a d-dimensional domain ($d=1,2,3$) with boundary $\partial\Omega$, ε is the diffusion constant of the medium, f is the force term, $\boldsymbol{a}$ is the advection field (not necessarily solenoidal) and σ is a source (reaction) term.

We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$, and by $V := H_0^1(\Omega)$ the space where u will be sought. Likewise, we use the notation

$$\langle\cdot,\cdot\rangle_\omega := \langle\cdot,\cdot\rangle_{H^{-1/2}(\omega)\times H_{00}^{1/2}(\omega)} \quad \text{for } \omega \ d-1\text{-dimensional},$$

$$\langle\cdot,\cdot\rangle_\omega := \langle\cdot,\cdot\rangle_{(H^1(\omega))'\times H^1(\omega)} \quad \text{for } \omega \ d\text{-dimensional}$$

for the duality pairings to be used. We endow $H^1(\Omega)$ with the usual scalar product $(w,v)_{1,\Omega} = (w,v) + (\nabla w, \nabla v)$ and the associated norm $\|\cdot\|_{1,\Omega}$.

Let us consider our differential problem (1). We restrict ourselves to solutions in V. To guarantee existence, we take $f\in(H^1(\Omega))'$ and $\boldsymbol{a},\sigma,\nabla\cdot\boldsymbol{a}\in L^\infty(\Omega)$. By noting that

$$\int_\Omega v\boldsymbol{a}\cdot\nabla u \,\mathrm{d}\Omega = -\int_\Omega u\boldsymbol{a}\cdot\nabla v\,\mathrm{d}\Omega - \int_\Omega uv\nabla\cdot\boldsymbol{a}\,\mathrm{d}\Omega \quad \forall u,v\in V,$$

we transform the convective term into a skew symmetric operator, and we can enunciate our problem as follows: find $u\in V$ such that

$$a(u,v) = \langle f,v\rangle_\Omega \quad \forall v\in V, \tag{2}$$

where the bilinear form is

$$a(w,v) := \varepsilon(\nabla w,\nabla v) + \tfrac{1}{2}(\boldsymbol{a}\cdot\nabla w, v) - \tfrac{1}{2}(w,\boldsymbol{a}\cdot\nabla v) + (\sigma_0 w, v)$$

with

$$\sigma_0 = \sigma + \tfrac{1}{2}\nabla\cdot\boldsymbol{a}.$$

By applying Lax–Milgram lemma, it can be easily shown that if $\sigma_0 \geqslant 0$ almost everywhere, Problem (2) has a unique solution.

3. Overlapping Dirichlet/Robin method

3.1. Domain partitioning and definitions

We perform a geometrical decomposition of the original domain Ω into three disjoint and connected subdomains Ω_3, Ω_4 and Ω_5 such that

$$\Omega = \mathrm{int}(\overline{\Omega_3\cup\Omega_4\cup\Omega_5}).$$

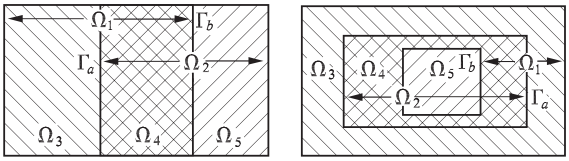


Fig. 2. Examples of decomposition of a domain Ω into two overlapping subdomains Ω_1 and Ω_2.

From this partition, we define Ω_1 and Ω_2, as two overlapping subdomains:

$$\Omega_1 := \mathrm{int}(\overline{\Omega_3 \cup \Omega_4}), \quad \Omega_2 := \mathrm{int}(\overline{\Omega_5 \cup \Omega_4}).$$

Finally, we define Γ_a as the part of $\partial\Omega_2$ lying in Ω_1, and Γ_b as the part of $\partial\Omega_1$ lying in Ω_2, formally given by

$$\Gamma_a := \overline{\partial\Omega_2 \cap \Omega_1}, \quad \Gamma_b := \overline{\partial\Omega_1 \cap \Omega_2}.$$

The geometrical nomenclature is shown in Fig. 2. Γ_b and Γ_a are the *interfaces* of the domain decomposition method we now present. Ω_4 is the overlap zone. In the following, index i or j refer to a subdomain or an interface.

To state the variational formulation of the two-domain problem, we introduce the following definitions:

$$\begin{aligned}
&(w, v)_{\Omega_i} := \int_{\Omega_i} wv \,\mathrm{d}\Omega,\\
&a_i(w, v) := \varepsilon(\nabla w, \nabla v)_{\Omega_i} + \tfrac{1}{2}(\boldsymbol{a} \cdot \nabla w, v)_{\Omega_i} - \tfrac{1}{2}(w, \boldsymbol{a} \cdot \nabla v)_{\Omega_i} + (\sigma_0 w, v)_{\Omega_i},\\
&V_i := \{v \in H^1(\Omega_i) \,|\, v_{|\partial\Omega\cap\partial\Omega_i} = 0\},\\
&V_i^0 := H_0^1(\Omega_i),
\end{aligned} \tag{3}$$

where i can be any of the five subdomains introduced previously, i.e., $i = 1, 2, 3, 4$ or 5.

Let $\gamma_{0,i}$ be the trace operators

$$\gamma_{0,i} : V_i \to H^{1/2}(\partial\Omega_i), \quad i = 1, 2, 3, 4, 5,$$

which are linear and continuous, like the trace operators T_a and T_b defined by

$$T_a : V \to H_{00}^{1/2}(\Gamma_a), \quad T_a v = v_{|\Gamma_a},$$

$$T_b : V \to H_{00}^{1/2}(\Gamma_b), \quad T_b v = v_{|\Gamma_b}.$$

We explicitly define the trace spaces on Γ_a and Γ_b as $\Lambda_a := \{\mu_a \in H_{00}^{1/2}(\Gamma_a)\}$ and $\Lambda_b := \{\mu_b \in H_{00}^{1/2}(\Gamma_b)\}$, respectively.

We also need to introduce some basic properties of the space we are working with; as many constants are going to be introduced, we adopt a general nomenclature. We enunciate three inequalities (Poincaré–Friedrichs, trace inequalities and an a priori estimate) that characterize the functions belonging to our working spaces, i.e., $H^1(\Omega)$ and $H_0^1(\Omega)$. The domains of study are the original domain Ω and its five partitions Ω_i, with $i = 1, 2, 3, 4, 5$. The Poincaré–Friedrichs inequality reads

$$\|v\|_{0,\Omega_i}^2 \leqslant C_{\Omega_i} \|\nabla v\|_{0,\Omega_i}^2 \quad \forall v \in H_0^1(\Omega_i), \tag{4}$$

where C_{Ω_i} is a positive constant depending on the size of the domain Ω_i. The space of application $H^1_0(\Omega_i)$ can be actually extended to any subspace of $H^1(\Omega_i)$ for which the trace is specified somewhere on $\partial\Omega_i$.

The trace inequality is a direct consequence of the trace theorem; it states that there exists a positive constant C_i^* such that

$$\|v_{|\partial\Omega_i}\|_{1/2,\partial\Omega_i} \leqslant C_i^* \|v\|_{1,\Omega_i} \quad \forall v \in H^1(\Omega_i). \tag{5}$$

Finally, the following a priori estimate for the solution v of homogeneous elliptic problems with Dirichlet data holds (see, e.g., [9,25]):

$$\|v\|_{1,\Omega_i} \leqslant C_i \|v_{|\partial\Omega_i}\|_{1/2,\partial\Omega_i} \quad \forall v \in H^1(\Omega_i). \tag{6}$$

This establishes the continuous dependence of the solution on the boundary data and closes the list of properties we need.

3.2. Variational formulation

We propose to solve the following problem: find $u_1 \in V_1$ and $u_2 \in V_2$ such that

$$\begin{aligned}
&a_1(u_1, v_1) = \langle f, v_1\rangle_{\Omega_1} \quad \forall v_1 \in V_1^0,\\
&u_1 = u_2 \quad \text{on } \Gamma_b,\\
&a_2(u_2, v_2) = \langle f, v_2\rangle_{\Omega_2} \quad \forall v_2 \in V_2^0,\\
&a_3(u_1, E_3\mu_a) + a_2(u_2, E_2\mu_a) = \langle f, E_3\mu_a\rangle_{\Omega_3} + \langle f, E_2\mu_a\rangle_{\Omega_2} \quad \forall \mu_a \in \Lambda_a,
\end{aligned} \tag{7}$$

where E_i denotes any possible extension operator from Λ_a to $H^1(\Omega_i)$, that is to say,

$$E_i : \Lambda_a \to H^1(\Omega_i), \quad T_a E_i \mu_a = \mu_a \quad \forall \mu_a \in \Lambda_a.$$

Eqs. $(7)_1$ and $(7)_3$ are the equations for the unknown in subdomains Ω_1 and Ω_2 respectively. Eq. $(7)_2$ is the condition that ensures continuity of the primary variable across Γ_b, and levels the solution in both subdomains. Finally, Eq. $(7)_4$ is the equation for the primary variable on the interface Γ_a.

Theorem 1. *Problems* (7) *and* (2) *are equivalent.*

Proof. We first show that the solution is the same in both subdomains inside the overlap zone Ω_4, i.e., that the two transmission conditions on the interfaces are sufficient to uniquely define the solution. For any $v_4 \in V_4^0$, construct

$$v_1 = \begin{cases} 0 & \text{in } \Omega_3,\\ v_4 & \text{in } \Omega_4 \end{cases} \quad \text{and} \quad v_2 = \begin{cases} v_4 & \text{in } \Omega_4,\\ 0 & \text{in } \Omega_5. \end{cases}$$

Clearly, $v_1 \in V_1^0$ and $v_2 \in V_2^0$ and therefore subtracting $(7)_1$ and $(7)_3$, we obtain

$$a_4(u_1 - u_2, v_4) = 0 \quad \forall v_4 \in V_4^0$$

together with the condition $u_1 - u_2 = 0$ on Γ_b, derived from $(7)_2$. Now, we need to derive a boundary condition on Γ_a in order to close the problem for the unknown $u_1 - u_2$. For any $\mu_a \in \Lambda_a$

define

$$v_1 = \begin{cases} E_3\mu_a & \text{in } \Omega_3, \\ E_4\mu_a & \text{in } \Omega_4. \end{cases}$$

Since $v_1 \in V_1^0$, Eqs. $(7)_1$ and $(7)_4$ give

$$a_2(u_2, E_2\mu_a) - a_4(u_1, E_4\mu_a) = \langle f, E_2\mu_a\rangle_{\Omega_2} - \langle f, E_4\mu_a\rangle_{\Omega_4} \quad \forall \mu_a \in \Lambda_a. \tag{8}$$

Now we define for all $\mu_a \in \Lambda_a$

$$v_2' = \begin{cases} E_4\mu_a & \text{in } \Omega_4, \\ 0 & \text{in } \Omega_5. \end{cases}$$

Eq. (8) can be rewritten as

$$\begin{aligned} & a_2(u_2, E_2\mu_a - v_2') + a_2(u_2, v_2') - a_4(u_1, E_4\mu_a) \\ & \quad = \langle f, E_2\mu - v_2'\rangle_{\Omega_2} + \langle f, v_2'\rangle_{\Omega_2} - \langle f, E_4\mu_a\rangle_{\Omega_4} \quad \forall \mu_a \in \Lambda_a. \end{aligned} \tag{9}$$

According to the definition of v_2', $(E_2\mu_a - v_2') \in V_2^0$ and consequently, applying $(7)_3$, we obtain

$$a_2(u_2, E_2\mu_a - v_2') = \langle f, E_2\mu - v_2'\rangle_{\Omega_2}.$$

Eq. (9) gives therefore

$$a_4(u_1 - u_2, E_4\mu_a) = 0 \quad \forall \mu_a \in \Lambda_a.$$

As a result, the complete system of equations for $w = u_1 - u_2$ is

$$a_4(w, v_4) = 0 \quad \forall v_4 \in V_4^0,$$

$$w = 0 \quad \text{on } \Gamma_b,$$

$$a_4(w, E_4\mu_a) = 0 \quad \forall \mu_a \in \Lambda_a.$$

From the Lax–Milgram lemma, this problem has a unique solution $w = 0$; this implies that $u_1 = u_2$ in Ω_4.

We now show that the solution of the original problem is also solution of the domain decomposition problem. Let u be solution of Eq. (2), and define $u_i = u_{|\Omega_i}$ for $i = 1, 2$. Clearly, $u_i \in V_i$ and therefore Eqs. $(7)_1$, $(7)_2$ and $(7)_3$ are trivially satisfied. Now for all $\mu_a \in \Lambda_a$ define γ as

$$\gamma = \begin{cases} E_3\mu_a & \text{in } \Omega_3, \\ E_2\mu_a & \text{in } \Omega_2. \end{cases}$$

We have that $\gamma \in V$, which implies that

$$a(u, \gamma) = \langle f, \gamma\rangle_\Omega,$$

and substituting the definition of γ we recover Eq. $(7)_4$.

We now prove the reciprocal. Let

$$u = \begin{cases} u_{1|\Omega_3} & \text{in } \Omega_3, \\ u_2 & \text{in } \Omega_2. \end{cases}$$

We have shown that $u_1 = u_2$ in Ω_4 and in particular that $u_1 = u_2$ on Γ_a. This implies that $u \in V$ and, as a result, we have

$$a(u, v) = a_3(u_1, v) + a_2(u_2, v) \quad \forall v \in V. \tag{10}$$

For each $v \in V$, set $\mu_a = T_a v \in \Lambda_a$. Let us define

$$\gamma_3 = v_{|\Omega_3} - E_3\mu_a \quad \text{and} \quad \gamma_2 = v_{|\Omega_2} - E_2\mu_a,$$

and rewrite Eq. (10) as

$$a(u, v) = a_3(u_1, \gamma_3) + a_3(u_1, E_3\mu_a) + a_2(u_2, \gamma_2) + a_2(u_2, E_2\mu_a). \tag{11}$$

By definition, $\gamma_3 \in V_3^0$. Let us now define γ_1 as

$$\gamma_1 = \begin{cases} \gamma_3 & \text{in } \Omega_3, \\ 0 & \text{in } \Omega_4. \end{cases}$$

$\gamma_1 \in V_1^0$ and therefore $(7)_1$ implies that

$$a_3(u_1, \gamma_3) = \langle f, \gamma_3 \rangle_{\Omega_3}.$$

Similarly, knowing also that $\gamma_2 \in V_2^0$, we can show that

$$a_2(u_2, \gamma_2) = \langle f, \gamma_2 \rangle_{\Omega_2},$$

and from the latter two equations, Eq. (11) becomes

$$a(u, v) = \langle f, \gamma_3 \rangle_{\Omega_3} + a_3(u_1, E_3\mu_a) + \langle f, \gamma_2 \rangle_{\Omega_2} + a_2(u_2, E_2\mu_a) \quad \forall \mu_a \in \Lambda_a.$$

From Eq. $(7)_4$, the last equation reads

$$a(u, v) = \langle f, \gamma_3 \rangle_{\Omega_3} + \langle f, E_3\mu_a \rangle_{\Omega_3} + \langle f, \gamma_2 \rangle_{\Omega_2} + \langle f, E_2\mu_a \rangle_{\Omega_2} \quad \forall \mu_a \in \Lambda_a,$$

which gives from the definitions of γ_3 and γ_2 yields,

$$\begin{aligned} a(u, v) &= \langle f, v_{|\Omega_3} \rangle_{\Omega_3} + \langle f, v_{|\Omega_2} \rangle_{\Omega_2}, \\ &= \langle f, v \rangle \quad \forall v \in V, \end{aligned}$$

and hence the theorem follows. □

Remark 2. The variational formulation given by Eqs. $(7)_{1–4}$ provides a general setting for an overlapping domain decomposition method. On the one hand, we have a Dirichlet condition on Γ_b; on the other hand, the transmission condition $(7)_4$ on Γ_a depends on the bilinear chosen to represent the original differential operator in the weak formulation. For the particular case of the ADR problem, this condition can be written in the more familiar form presented next.

3.3. Alternative formulation

We develop an alternative formulation for the domain decomposition method given by Eqs. $(7)_{1–4}$.

Lemma 3. *The solution of the domain decomposition problem satisfies*

$$\varepsilon\frac{\partial u_1}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1=\varepsilon\frac{\partial u_2}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2 \quad \text{on } \Gamma_a,$$

where $\partial(\cdot)/\partial n_2=\boldsymbol{n}_2\cdot\nabla(\cdot)$, $\boldsymbol{n}_2$ *being the exterior normal to* Ω_2 *on* Γ_a.

Proof. According to Green's formula, for all $\mu_a\in\Lambda_a$ we have

$$a_3(u_1,E_3\mu_a)=-\left\langle\varepsilon\frac{\partial u_1}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1,\mu_a\right\rangle_{\Gamma_a}+\langle Lu_1,E_3\mu_a\rangle_{\Omega_3}, \tag{12}$$

$$a_2(u_2,E_2\mu_a)=\left\langle\varepsilon\frac{\partial u_2}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2,\mu_a\right\rangle_{\Gamma_a}+\langle Lu_2,E_2\mu_a\rangle_{\Omega_2}. \tag{13}$$

In addition, from Eqs. $(7)_1$ and $(7)_3$, we have that

$$Lu_1=f \text{ in } \Omega_1 \quad \text{and} \quad Lu_2=f \text{ in } \Omega_2,$$

in the sense of distributions. As a result, Eqs. (12) and (13) become

$$\begin{aligned}a_3(u_1,E_3\mu_a)&=-\left\langle\varepsilon\frac{\partial u_1}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1,\mu_a\right\rangle_{\Gamma_a}+\langle f,E_3\mu_a\rangle_{\Omega_3},\\ a_2(u_2,E_2\mu_a)&=\left\langle\varepsilon\frac{\partial u_2}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2,\mu_a\right\rangle_{\Gamma_a}+\langle f,E_2\mu_a\rangle_{\Omega_2}.\end{aligned} \tag{14}$$

Adding up these two equations, and substituting the result into Eq. $(7)_4$, we find

$$\left\langle-\varepsilon\frac{\partial u_1}{\partial n_2}+\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1+\varepsilon\frac{\partial u_2}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2,\mu_a\right\rangle_{\Gamma_a}=0 \quad \forall\mu_a\in\Lambda_a,$$

and thus the lemma holds. □

Theorem 4. *System of Eqs.* $(7)_{1\text{–}4}$ *can be reformulated as follows*: *find* $u_1\in V_1$ *and* $u_2\in V_2$ *such that*

$$\begin{aligned}&a_1(u_1,v_1)=\langle f,v_1\rangle_{\Omega_1} \quad \forall v_1\in V_1^0,\\ &u_1=u_2 \quad \text{on } \Gamma_b,\\ &a_2(u_2,v_2')=\langle f,v_2'\rangle_{\Omega_2}+\left\langle\varepsilon\frac{\partial u_1}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1,v_2'\right\rangle_{\Gamma_a} \quad \forall v_2'\in V_2.\end{aligned} \tag{15}$$

Proof. We first substitute Eq. (14) into Eq. $(7)_4$, and add the result to Eq. $(7)_3$:

$$a_2(u_2,v_2+E_2\mu_a)=\left\langle\varepsilon\frac{\partial u_1}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1,\mu_a\right\rangle_{\Gamma_a}+\langle f,v_2+E_2\mu_a\rangle_{\Omega_2} \quad \forall v_2\in V_2^0,\ \mu_a\in\Lambda_a.$$

Let us define $v_2'=v_2+E_2\mu_a$. Clearly, $v_2'\in V_2$ and $\mu_a=T_av_2'$; consequently, the last equation is equivalent to

$$a_2(u_2,v_2')=\left\langle\varepsilon\frac{\partial u_1}{\partial n_2}-\frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1,v_2'\right\rangle_{\Gamma_a}+\langle f,v_2'\rangle_{\Omega_2} \quad \forall v_2'\in V_2.$$

The proof is completed by substituting Eqs. $(7)_3$ and $(7)_4$ of the system of equations $(7)_{1\text{–}4}$ by the latter equation. □

The interpretation of the domain decomposition method now appears clearly. A Dirichlet problem is solved in Ω_1 using as Dirichlet data on the interface Γ_b the solution in Ω_2, whereas a mixed Dirichlet/Robin problem is solved in Ω_2 using as Robin data on Γ_a the solution in Ω_1. This formulation justifies the name *overlapping Dirichlet/Robin method* to designate this domain decomposition method.

Remark 5. The system of equations $(15)_{1\text{–}3}$ could have been derived directly from the following DD problem applied at the differential level:

$$
\begin{aligned}
&Lu_1 = f \quad \text{in } \Omega_1,\\
&u_1 = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega,\\
&u_1 = u_2 \quad \text{on } \Gamma_b,\\
&Lu_2 = f \quad \text{in } \Omega_2,\\
&u_2 = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega,\\
&\varepsilon\frac{\partial u_2}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2 = \varepsilon\frac{\partial u_1}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1 \quad \text{on } \Gamma_a.
\end{aligned}
\tag{16}
$$

The interface conditions on Γ_a and Γ_b are usually referred to as *matching conditions* or *transmission conditions*. The first one is of Dirichlet type while the second one is of Robin type. At the variational level, we have just shown they correspond to essential and natural boundary conditions.

3.4. Interface equations

A convenient way to study DD methods is to derive equations for the interface unknown(s). To do so, the problem is first rewritten into two purely Dirichlet problems for which the Dirichlet data are the unknowns on the interfaces. For the sake of clarity, the derivation of the interface equations is carried out at the differential level, starting from Eqs. $(16)_{1\text{–}6}$. The problems to consider are:

$$
\begin{aligned}
&Lw_1 = f \quad \text{in } \Omega_1, && Lw_2 = f \quad \text{in } \Omega_2,\\
&w_1 = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, && w_2 = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega,\\
&w_1 = \lambda_b \quad \text{on } \Gamma_b, && w_2 = \lambda_a \quad \text{on } \Gamma_a.
\end{aligned}
\tag{17}
$$

Now let us decompose w_1 and w_2 into L-homogeneous and Dirichlet-homogeneous parts,

$$w_1 = u_1^0 + u_1^*, \quad w_2 = u_2^0 + u_2^*,$$

where the L-homogeneous parts u_1^0 and u_2^0 are the solutions of the following systems:

$$
\begin{aligned}
&Lu_1^0 = 0 \quad \text{in } \Omega_1, && Lu_2^0 = 0 \quad \text{in } \Omega_2,\\
&u_1^0 = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, && u_2^0 = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega,\\
&u_1^0 = \lambda_b \quad \text{on } \Gamma_b, && u_2^0 = \lambda_a \quad \text{on } \Gamma_a
\end{aligned}
\tag{18}
$$

and the Dirichlet-homogeneous parts u_1^* and u_2^* are the solutions of the following systems:

$$\begin{aligned} &Lu_i^* = f \quad \text{in } \Omega_i, \\ &u_i^* = 0 \quad \text{on } \partial\Omega_i \end{aligned} \tag{19}$$

for $i = 1,2$. We refer to u_1^0 as the L-homogeneous extension of λ_b into Ω_1, and we denote it by $\mathscr{L}_1\lambda_b$. Similarly, we call u_2^0 the L-homogeneous extension of λ_a into Ω_2, and we denote it by $\mathscr{L}_2\lambda_a$. In the case when $L = -\Delta$, $\mathscr{L}$ is called the harmonic extension and is usually denoted by H. The Dirichlet-homogeneous parts u_1^* and u_2^* are rewritten as $\mathscr{G}_1 f$ and $\mathscr{G}_2 f$, respectively.

Comparing systems (17) with system (16), we have that $w_i = u_i$ for $i = 1,2$ if and only if the following two conditions are satisfied:

$$\begin{aligned} &\varepsilon \frac{\partial w_2}{\partial n_2} - \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2) w_2 = \varepsilon \frac{\partial w_1}{\partial n_2} - \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2) w_1 \quad \text{on } \Gamma_a, \\ &w_1 = w_2 \quad \text{on } \Gamma_b. \end{aligned} \tag{20}$$

Using the previous definitions, conditions (20) can be rewritten as

$$\begin{aligned} &\varepsilon \frac{\partial \mathscr{L}_2\lambda_a}{\partial n_2} - \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2)\mathscr{L}_2\lambda_a \\ &\quad = \varepsilon \frac{\partial \mathscr{L}_1\lambda_b}{\partial n_2} - \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2)\mathscr{L}_1\lambda_b + \varepsilon \frac{\partial \mathscr{G}_1 f}{\partial n_2} - \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2)\mathscr{G}_1 f - \varepsilon \frac{\partial \mathscr{G}_2 f}{\partial n_2} + \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2)\mathscr{G}_2 f \quad \text{on } \Gamma_a, \\ &\lambda_b = T_b\mathscr{L}_2\lambda_a + T_b\mathscr{G}_2 f \quad \text{on } \Gamma_b. \end{aligned}$$

Let us clean up this system by introducing some definitions. In the first equation, we recognize the Steklov–Poincaré operator S_2 associated to subdomain Ω_2, and defined as

$$\begin{aligned} &S_2 : \Lambda_a \to H^{-1/2}(\Gamma_a), \\ &S_2\lambda_a := \varepsilon \frac{\partial \mathscr{L}_2\lambda_a}{\partial n_2} - \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2)\mathscr{L}_2\lambda_a \quad (\text{evaluated on } \Gamma_a). \end{aligned}$$

Note that $\mathscr{L}_2\lambda_a = \lambda_a$ on Γ_a. We define $\tilde{S}_b$, a Steklov–Poincaré-like operator acting on Γ_b, as

$$\begin{aligned} &\tilde{S}_b : \Lambda_b \to H^{-1/2}(\Gamma_a), \\ &\tilde{S}_b\lambda_b := -\varepsilon \frac{\partial \mathscr{L}_1\lambda_b}{\partial n_2} + \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2)\mathscr{L}_1\lambda_b \quad (\text{evaluated on } \Gamma_a). \end{aligned}$$

We also define $\tilde{T}_b$, the trace on Γ_b of the L-extension of λ_a into Ω_2:

$$\begin{aligned} &\tilde{T}_b : \Lambda_a \to \Lambda_b, \\ &\tilde{T}_b\lambda_a := T_b\mathscr{L}_2\lambda_a. \end{aligned}$$

Finally, χ and χ' are defined as follows:

$$\begin{aligned} &\chi = \varepsilon \frac{\partial \mathscr{G}_1 f}{\partial n_2} - \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2)\mathscr{G}_1 f - \varepsilon \frac{\partial \mathscr{G}_2 f}{\partial n_2} + \frac{1}{2}(\boldsymbol{a} \cdot \boldsymbol{n}_2)\mathscr{G}_2 f, \\ &\chi' = T_b\mathscr{G}_2 f, \end{aligned}$$

where we have $\chi \in H^{-1/2}(\Gamma_a)$ and $\chi' \in \Lambda_b$. Owing to the previous definitions, the system of two equations for the interface unknowns reads

$$S_2\lambda_a = -\tilde{S}_b\lambda_b + \chi \quad \text{in } H^{-1/2}(\Gamma_a),$$
$$\lambda_b = \tilde{T}_b\lambda_a + \chi' \quad \text{in } \Lambda_b. \tag{21}$$

Let us introduce now the operator

$$\tilde{S}_1 : \Lambda_a \to H^{-1/2}(\Gamma_a),$$
$$\tilde{S}_1\lambda_a := \tilde{S}_b\tilde{T}_b\lambda_a$$

and define S as

$$S = \tilde{S}_1 + S_2.$$

After substituting λ_b given by Eq. $(21)_2$ into Eq. $(21)_1$, we finally obtain the following system of equations for the interface unknowns:

$$S\lambda_a = \chi - \tilde{S}_b\chi' \quad \text{in } H^{-1/2}(\Gamma_a),$$
$$\lambda_b = \tilde{T}_b\lambda_a + \chi' \quad \text{in } \Lambda_b. \tag{22}$$

Once λ_a and λ_b are obtained, we can solve the two Dirichlet problems (18) to obtain the L-homogeneous parts u_1^0 and u_2^0. The Dirichlet-homogeneous parts u_1^* and u_2^* are obtained by solving Eqs. (19) for $i = 1, 2$. Hence, the solutions u_1 and u_2 are calculated by adding up their respective L and Dirichlet-homogeneous contributions. Eq. $(22)_1$ should formally be understood in a weak sense, i.e.,

$$\langle (S_2 + \tilde{S}_1)\lambda_a, \mu_a\rangle_{\Gamma_a} = \langle \chi - \tilde{S}_b\chi', \mu_a\rangle_{\Gamma_a} \quad \forall \mu_a \in \Lambda_a. \tag{23}$$

Lemma 6. *The variational counterpart of the Steklov–Poincaré operators are*

$$\langle \tilde{S}_1\lambda_a, \mu_a\rangle_{\Gamma_a} = a_3(\mathscr{L}_1\tilde{T}_b\lambda_a, E_3\mu_a), \tag{24}$$

$$\langle S_2\lambda_a, \mu_a\rangle_{\Gamma_a} = a_2(\mathscr{L}_2\lambda_a, E_2\mu_a) \quad \forall \mu_a \in \Lambda_a \tag{25}$$

for any extension operators E_2 and E_3.

Proof. The lemma follows from the definitions of $\tilde{S}_1$ and S_2 and Green's formula. □

We can also show that the two right hand-side terms of Eq. $(22)_1$ satisfy

$$\langle \chi, \mu_a\rangle_{\Gamma_a} = \langle f, E_2\mu_a\rangle_{\Omega_2} - a_2(\mathscr{G}_2 f, E_2\mu_a) + \langle f, E_3\mu_a\rangle_{\Omega_3} - a_3(\mathscr{G}_1 f, E_3\mu_a),$$
$$\langle \tilde{S}_b\chi', \mu_a\rangle_{\Gamma_a} = a_3(\mathscr{L}_1 T_b\mathscr{G}_2 f, E_3\mu_a) \quad \forall \mu_a \in \Lambda_a$$

for any extension operators E_2 and E_3. This completes the definition of the variational form of Eq. $(22)_1$, i.e.,

$$a_2(\mathscr{L}_2\lambda_a, E_2\mu_a) + a_3(\mathscr{L}_1\tilde{T}_b\lambda_a, E_3\mu_a)$$
$$= \langle f, E_2\mu_a\rangle_{\Omega_2} - a_2(\mathscr{G}_2 f, E_2\mu_a) + \langle f, E_3\mu_a\rangle_{\Omega_3} - a_3(\mathscr{G}_1 f, E_3\mu_a) - a_3(\mathscr{L}_1 T_b\mathscr{G}_2 f, E_3\mu_a) \quad \forall \mu_a \in \Lambda_a.$$

Remark 7. Eq. (23) can be also obtained by formulating problems (17) in a variational form.

Let us go back to system (22). We first state some useful properties of operators S_2 and $\tilde{S}_1$.

Lemma 8. *S_2 is both continuous and coercive and $\tilde{S}_1$ is continuous and nonnegative.*

Proof. We have shown that Eqs. (25) and (24) hold for any extension operators E_2 and E_1. This leaves us the choice to find appropriate expressions for S_2 and $\tilde{S}_1$ to facilitate their analysis. A straightforward choice consists of taking $E_2 = \mathscr{L}_2$, and $E_1 = \mathscr{L}_1\tilde{T}_b$. Thus, we have

$$\langle S_2\lambda_a, \mu_a\rangle_{\Gamma_a} = a_2(\mathscr{L}_2\lambda_a, \mathscr{L}_2\mu_a),$$

$$\langle \tilde{S}_1\lambda_a, \mu_a\rangle_{\Gamma_a} = a_3(\mathscr{L}_1\tilde{T}_b\lambda_a, \mathscr{L}_1\tilde{T}_b\mu_a) \quad \forall \mu_a \in \Lambda_a.$$

We first show that S_2 is both continuous and coercive. Using the definition of a_2 given by Eq. (3) and applying the Cauchy–Schwartz inequality, we obtain

$$\langle S_2\eta_a, \mu_a\rangle_{\Gamma_a} \leqslant \kappa_{\Omega_2}\|\mathscr{L}_2\eta_a\|_{1,\Omega_2}\|\mathscr{L}_2\mu_a\|_{1,\Omega_2} \quad \forall \eta_a, \mu_a \in \Lambda_a, \tag{26}$$

where $\kappa_{\Omega_2} = \varepsilon + \|\boldsymbol{a}\|_{\infty,\Omega_2} + \|\sigma_0\|_{\infty,\Omega_2}$. According to the a priori estimate given by Eq. (6), we have that $\|\mathscr{L}_2\mu_a\|_{1,\Omega_2} \leqslant C_2\|\mu_a\|_{1/2,(\Gamma_a)}$ for $\mu_a \in \Lambda_a$, and therefore Eq. (26) gives

$$\langle S_2\eta_a, \mu_a\rangle_{\Gamma_a} \leqslant M_{S_2}\|\eta_a\|_{1/2,\Gamma_a}\|\mu_a\|_{1/2,\Gamma_a} \quad \forall \eta_a, \mu_a \in \Lambda_a, \tag{27}$$

which states that S_2 is continuous, with $M_{S_2} = \kappa_{\Omega_2}C_2^2$ the continuity constant. We now show the coercivity of S_2. Owing to the skew-symmetry of the convective term of a_2, for any $\mu_a \in \Lambda_a$ we have

$$\begin{aligned}
&\langle S_2\mu_a, \mu_a\rangle_{\Gamma_a}\\
&\quad = a_2(\mathscr{L}_2\mu_a, \mathscr{L}_2\mu_a) = \varepsilon\,\|\nabla\mathscr{L}_2\mu_a\|_{0,2}^2 + \int_{\Omega_2}\sigma_0(\mathscr{L}_2\mu_a)^2\,\mathrm{d}\Omega\\
&\quad \geqslant \varepsilon\|\nabla\mathscr{L}_2\mu_a\|_{0,\Omega_2}^2 \quad (\sigma_0 \geqslant 0 \text{ almost everywhere}).
\end{aligned} \tag{28}$$

From the trace inequality (see Eq. (5)), we know that there exists a constant $C_2^* > 0$ such that

$$\|\mathscr{L}_2\mu_{a|\partial\Omega_2}\|_{1/2,\partial\Omega_2} \leqslant C_2^*\|\mathscr{L}_2\mu_a\|_{1,\Omega_2} \quad \forall \mu_a \in \Lambda_a,$$

Using the Poincaré–Friedrichs inequality (4), Eq. (28) yields

$$\langle S_2\mu_a, \mu_a\rangle \geqslant N_{S_2}\|\mu_a\|_{1/2,\Gamma_a}^2 \quad \forall \mu_a \in \Lambda_a, \tag{29}$$

where $N_{S_2} := \varepsilon/(C_{\Omega_2}^2 + 1)(C_2^*)^2$ is the coercivity constant.

Let us finally prove the continuity and nonnegativeness of $\tilde{S}_1$. Applying the Cauchy–Schwarz inequality to Eq. (24), we obtain

$$\begin{aligned}
\langle \tilde{S}_1\eta_a, \mu_a\rangle_{\Gamma_a} &\leqslant \kappa_{\Omega_3}\|\mathscr{L}_1T_b\mathscr{L}_2\eta_a\|_{1,\Omega_3}\|\mathscr{L}_1T_b\mathscr{L}_2\mu_a\|_{1,\Omega_3}\\
&\leqslant \kappa_{\Omega_3}\|\mathscr{L}_1T_b\mathscr{L}_2\eta_a\|_{1,\Omega_1}\|\mathscr{L}_1T_b\mathscr{L}_2\mu_a\|_{1,\Omega_1} \quad (\Omega_3 \subset \Omega_1)
\end{aligned}$$

for any $\eta_a, \mu_a \in \Lambda_a$ and where $\kappa_{\Omega_3} = \varepsilon + \|\boldsymbol{a}\|_{\infty,\Omega_3} + \|\sigma_0\|_{\infty,\Omega_3}$. From the a priori estimate given by Eq. (6), we have that

$$\begin{aligned}
&\langle \tilde{S}_1\eta_a, \mu_a\rangle_{\Gamma_a}\\
&\quad \leqslant \kappa_{\Omega_3}C_1^2\|T_b\mathscr{L}_2\eta_a\|_{1/2,\partial\Omega_1}\|T_b\mathscr{L}_2\mu_a\|_{1/2,\partial\Omega_1}
\end{aligned}$$

$$
\begin{aligned}
&= \kappa_{\Omega_3} C_1^2 \|T_b \mathscr{L}_2 \eta_a\|_{1/2,\Gamma_b} \|T_b \mathscr{L}_2 \mu_a\|_{1/2,\Gamma_b} \\
&= \kappa_{\Omega_3} C_1^2 \|\gamma_{0,5} \mathscr{L}_2 \eta_a\|_{1/2,\partial\Omega_5} \|\gamma_{0,5} \mathscr{L}_2 \mu_a\|_{1/2,\partial\Omega_5} \\
&\leqslant \kappa_{\Omega_3} C_1^2 C_5^{*2} \|\mathscr{L}_2 \eta_a\|_{1,\Omega_5} \|\mathscr{L}_2 \mu_a\|_{1,\Omega_5} \quad \text{(trace inequality (5))} \\
&\leqslant \kappa_{\Omega_3} C_1^2 C_5^{*2} \|\mathscr{L}_2 \eta_a\|_{1,\Omega_2} \|\mathscr{L}_2 \mu_a\|_{1,\Omega_2} \quad (\Omega_5 \subset \Omega_2) \\
&\leqslant \kappa_{\Omega_3} C_1^2 C_5^{*2} C_2^2 \|\eta_a\|_{1/2,\partial\Omega_2} \|\mu_a\|_{1/2,\partial\Omega_2} \quad \text{(a priori estimate (6))} \\
&= M_{\tilde{S}_1} \|\eta_a\|_{1/2,\Gamma_a} \|\mu_a\|_{1/2,\Gamma_a},
\end{aligned} \tag{30}
$$

which proves the continuity of $\tilde{S}_1$. Finally, owing to the skew-symmetry of a_1, for any $\mu_a \in \Lambda_a$ we have

$$
\begin{aligned}
\langle \tilde{S}_1 \mu_a, \mu_a \rangle_{\Gamma_a} &= a_1(\mathscr{L}_1 \tilde{T}_b \mu_a, \mathscr{L}_1 \tilde{T}_b \mu_a) \\
&= \varepsilon \|\nabla \mathscr{L}_1 \tilde{T}_b \mu_a\|_{0,2}^2 + \int_{\Omega_3} \sigma_0 (\mathscr{L}_1 \tilde{T}_b \mu_a)^2 \, \mathrm{d}\Omega \\
&\geqslant 0 \quad (\sigma_0 \geqslant 0 \text{ almost everywhere}),
\end{aligned}
$$

and the lemma holds. □

The following result is a direct consequence of the previous properties:

Theorem 9. *System* (22) *has a unique solution* $\{\lambda_a, \lambda_b\}$.

Proof. We first prove that S is invertible, showing that it is both continuous and coercive. We have

$$\langle S\eta_a, \mu_a \rangle_{\Gamma_a} = \langle \tilde{S}_1 \eta_a, \mu_a \rangle_{\Gamma_a} + \langle S_2 \eta_a, \mu_a \rangle_{\Gamma_a} \quad \forall \eta_a, \mu_a \in \Lambda_a.$$

Therefore, the continuity of S follows from that of S_2 and $\tilde{S}_1$, i.e.,

$$\langle S\eta_a, \mu_a \rangle \leqslant M_S \|\eta_a\|_{1/2,\Gamma_a} \|\mu_a\|_{1/2,\Gamma_a} \quad \forall \eta_a, \mu_a \in \Lambda_a$$

with continuity constant M_S given by $M_S = M_{\tilde{S}_1} + M_{S_2}$, where $M_{\tilde{S}_1}$ and M_{S_2} are the continuity constants of $\tilde{S}_1$ and S_2 introduced in Eqs. (30) and (27), respectively.

We now show the coercivity of S without trying to obtain sharp estimates. We have already shown the coercivity of S_2 and the nonnegativeness of $\tilde{S}_1$. Therefore,

$$
\begin{aligned}
&\langle S\mu_a, \mu_a \rangle_{\Gamma_a} \\
&\quad = \langle S_2 \mu_a, \mu_a \rangle_{\Gamma_a} + \langle \tilde{S}_1 \mu_a, \mu_a \rangle_{\Gamma_a} \\
&\quad \geqslant \langle S_2 \mu_a, \mu_a \rangle_{\Gamma_a} \geqslant N_S \|\mu_a\|_{1/2,\Gamma_a}^2 \quad \forall \mu_a \in \Lambda_a,
\end{aligned}
$$

where $N_S = N_{S_2}$. Thus, S is a continuous and coercive operator. According to Lax–Milgram Lemma, it is invertible and therefore Eq. $(22)_1$ has a unique solution λ_a. The existence and uniqueness of λ_b follows from that of λ_a, by applying Eq. $(22)_2$. Remember that we have

$$\lambda_b = \tilde{T}_b \mathscr{L}_2 \lambda_a + \chi'.$$

$\mathscr{L}_2\lambda_a$ is the unique solution of problem $(18)_2$. Since the trace operator T_b is well defined, from $H^1(\Omega_2)$ onto Λ_b, we have that λ_b exists and is unique. Inverting S in Eq. $(22)_1$, we find that

$$\lambda_a = S^{-1}(\chi - \tilde{S}_b\chi') \quad \text{in } \Lambda_a,$$
$$\lambda_b = \tilde{T}_b S^{-1}(\chi - \tilde{S}_b\chi') + \chi' \quad \text{in } \Lambda_b,$$

are the solutions of our interface problem. □

4. Iterative scheme

4.1. Relaxed sequential algorithm

In this section, we derive an iterative procedure to solve the domain decomposition problem (7). The sequential version of the iterative overlapping D/R algorithm is defined as follows. Given an initial guess u_2^0 on Γ_b, for each $k \geqslant 0$, find $u_1^{k+1} \in V_1$ and $u_2^{k+1} \in V_2$ such that

$$\begin{aligned}
&a_1(u_1^{k+1}, v_1) = \langle f, v_1\rangle_{\Omega_1} \quad \forall v_1 \in V_1^0,\\
&u_1^{k+1} = u_2^k \quad \text{on } \Gamma_b,\\
&a_2(u_2^{k+1}, v_2) = \langle f, v_2\rangle_{\Omega_2} \quad \forall v_2 \in V_2^0,\\
&a_2(u_2^{k+1}, E_2\mu_a) = -a_3(u_1^{k+1}, E_3\mu_a) + \langle f, E_3\mu_a\rangle_{\Omega_3} + \langle f, E_2\mu_a\rangle_{\Omega_2} \quad \forall \mu_a \in \Lambda_a,
\end{aligned} \tag{31}$$

for any extension operators E_3 and E_2. If this algorithm converges, the solutions on both subdomains satisfy Eqs. $(7)_{1\text{–}4}$. The corresponding algorithm for the differential problem given by Eqs. $(16)_{1\text{–}6}$ is straightforward and reads: given an initial guess u_2^0 on Γ_b, for each $k \geqslant 0$, find u_1^{k+1} and u_2^{k+1} such that

$$\begin{aligned}
&Lu_1^{k+1} = f && \text{in } \Omega_1,\\
&u_1^{k+1} = 0 && \text{on } \partial\Omega_1 \setminus \Gamma_b,\\
&u_1^{k+1} = u_2^k && \text{on } \Gamma_b,\\
&Lu_2^{k+1} = f && \text{in } \Omega_2,\\
&u_2^{k+1} = 0 && \text{on } \partial\Omega_2 \setminus \Gamma_a,\\
&\varepsilon\frac{\partial u_2^{k+1}}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2^{k+1} = \varepsilon\frac{\partial u_1^{k+1}}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1^{k+1} && \text{on } \Gamma_a.
\end{aligned} \tag{32}$$

If this algorithm converges, the solutions on both subdomains satisfy Eqs. $(16)_{1\text{–}6}$. For the sake of clarity, we have omitted the relaxation of the transmission conditions; for example, the Dirichlet condition $(32)_2$ could be replaced by

$$u_1^{k+1} = \theta u_2^k + (1-\theta)u_1^k,$$

where $\theta > 0$ is the relaxation parameter.

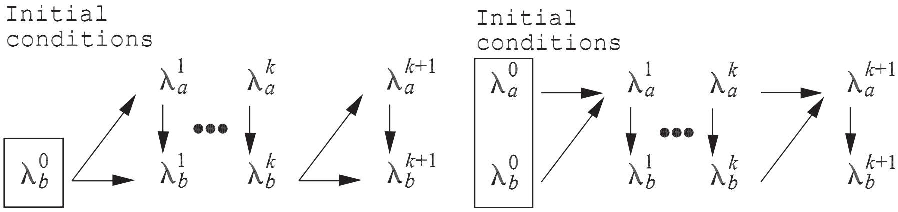


Fig. 3. Relaxed sequential algorithms. (Left) D_θ/R method. (Right) D/R_θ method.

We now investigate the interface iterates produced by this relaxed iterative procedure. The set of equations for the w_i's is the following:

$$\begin{aligned} &Lw_1^{k+1} = f \quad \text{in } \Omega_1, && Lw_2^{k+1} = f \quad \text{in } \Omega_2,\\ &w_1^{k+1} = 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega, && w_2^{k+1} = 0 \quad \text{on } \partial\Omega_2 \cap \partial\Omega,\\ &w_1^{k+1} = \lambda_b^k \quad \text{on } \Gamma_b, && w_2^{k+1} = \lambda_a^{k+1} \quad \text{on } \Gamma_a. \end{aligned}$$

The choice of taking as Dirichlet conditions λ_b at iteration k for w_1^{k+1}, and λ_a at iteration $k+1$ for w_2^{k+1} is arbitrary. According to this choice, we can set

$$w_1^{k+1} = \mathscr{L}_1\lambda_b^k + \mathscr{G}_1 f, \quad w_2^{k+1} = \mathscr{L}_2\lambda_a^{k+1} + \mathscr{G}_2 f.$$

We have that $w_i^k = u_i^k$ for $i = 1, 2$ if and only if the w_i^k's satisfy the transmission conditions $(32)_3$ and $(32)_6$. By noting that the Dirichlet-homogeneous solutions $\mathscr{G}_1 f$ and $\mathscr{G}_2 f$ do not change along the iterative process, the Dirichlet-relaxed iterative scheme, denoted D_θ/R, is given for any $k \geqslant 0$ by

$$\begin{aligned} &S_2\lambda_a^{k+1} = -\tilde{S}_b\lambda_b^k + \chi,\\ &\lambda_b^{k+1} = \theta(\tilde{T}_b\lambda_a^{k+1} + \chi') + (1-\theta)\lambda_b^k. \end{aligned} \tag{33}$$

The Robin transmission condition can be also relaxed by replacing Eq. $(32)_6$ by

$$\varepsilon\frac{\partial u_2^{k+1}}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2^{k+1} = \theta\left[\varepsilon\frac{\partial u_1^{k+1}}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_1^{k+1}\right] + (1-\theta)\left[\varepsilon\frac{\partial u_2^{k}}{\partial n_2} - \frac{1}{2}(\boldsymbol{a}\cdot\boldsymbol{n}_2)u_2^{k}\right].$$

In terms of the interface unknowns, the Robin-relaxed iterative scheme, denoted D/R_θ, produces the following iterates for any $k \geqslant 0$:

$$\begin{aligned} &S_2\lambda_a^{k+1} = \theta(-\tilde{S}_b\lambda_b^k + \chi) + (1-\theta)S_2\lambda_a^k,\\ &\lambda_b^{k+1} = \tilde{T}_b\lambda_a^{k+1} + \chi'. \end{aligned} \tag{34}$$

The dependence of λ_a^{k+1} and λ_b^{k+1} on the values at previous iterations, given two initial values λ_a^0 and λ_b^0, is sketched in Fig. 3; note that the value of λ_a^0 is only needed when using the D/R_θ method.

The continuity and coercivity of S_2 has been proven in last section. According to Lax–Milgram Lemma, S_2 is invertible. We can therefore reformulate the system for the interface unknowns (21) as follows:

$$\begin{aligned} &Q_a\lambda_a = \chi_a,\\ &Q_b\lambda_b = \chi_b, \end{aligned} \tag{35}$$

where we have defined Q_a, Q_b, χ_a and χ_b by

$$Q_a = I_a + S_2^{-1}\tilde{S}_b\tilde{T}_b = I_a + S_2^{-1}\tilde{S}_1,$$
$$Q_b = I_b + \tilde{T}_b S_2^{-1}\tilde{S}_b,$$
$$\chi_a = S_2^{-1}\chi - S_2^{-1}\tilde{S}_b\chi',$$
$$\chi_b = \tilde{T}_b S_2^{-1}\chi + \chi'$$

and where I_a is the identity on Λ_a and I_b is the identity on Λ_b. By solving the Dirichlet- and Robin-relaxed systems for λ_a^{k+1} and λ_b^{k+1}, we can show that both schemes lead to the same following iterates for any $k \geqslant 1$:

$$\begin{aligned} \lambda_a^{k+1} &= \theta(\chi_a - Q_a\lambda_a^k) + \lambda_a^k, \\ \lambda_b^{k+1} &= \theta(\chi_b - Q_b\lambda_b^k) + \lambda_b^k. \end{aligned} \tag{36}$$

We recognize here two *stationary Richardson procedures* for solving Eqs. $(35)_1$ and $(35)_2$. The Richardson procedure for solving λ_a is similar to that produced by the classical Dirichlet/Neumann method; in fact, we obtain the following equivalent iterate:

$$\lambda_a^{k+1} = \theta S_2^{-1}[(\chi - \tilde{S}_b\chi') - S\lambda_a^k] + \lambda_a^k,$$

which is a preconditioned Richardson method for solving Eq. $(22)_1$, using S_2 as preconditioner for S.

Remark 10. As pointed out above, the Richardson procedures (36) are valid only for $k \geqslant 1$. The $\mathrm{D}_\theta/\mathrm{R}$ and $\mathrm{D}/\mathrm{R}_\theta$ methods are therefore not completely equivalent, as the first iterative values λ_a^1 and λ_b^1 may differ, even if λ_a^0 and λ_b^0 are chosen to be equal. Therefore, even though they have the same behavior (given by Eq. (36)) they may yield different values of the iterates λ_a^k, λ_b^k, $k = 1, 2, 3, \ldots$.

4.2. Convergence

This section studies the convergence of the $\mathrm{D}_\theta/\mathrm{R}$ and $\mathrm{D}/\mathrm{R}_\theta$ iterative schemes given by Eqs. $(32)_{1–6}$ at the differential level, or $(31)_{1–4}$ at the variational level. Rather than directly studying the whole system of equations for u_1 and u_2, we base our analysis on the equivalent interface equations systems, i.e., Eqs. $(33)_{1-2}$ for the $\mathrm{D}_\theta/\mathrm{R}$ method and Eqs. $(34)_{1-2}$ for the $\mathrm{D}/\mathrm{R}_\theta$ method. The result we can prove is

Theorem 11. *Assume that ε is large enough so that*

$$\kappa^* := 2N_{S_2} - 2\|\boldsymbol{a}\|_{\infty,\Gamma_a} C_2^2 \frac{M_{\tilde{S}_1} + M_{S_2}}{N_{S_2}} > 0, \tag{37}$$

where the constants N_{S_2}, $M_{\tilde{S}_1}$ and M_{S_2} have been introduced in Eqs. (29), (30) *and* (27), *respectively. Then, there exists $\theta_{\max}$ such that for any given $\lambda_a^0 \in \Lambda_a$ and $\lambda_b^0 \in \Lambda_b$ and for all $\theta \in (0, \theta_{\max})$, the sequences $\{\lambda_a^k\}$ and $\{\lambda_b^k\}$ given by* (36) *converge in Λ_a and Λ_b, respectively. The upper bound of the relaxation parameter $\theta_{\max}$ can be estimated by*

$$\theta_{\max} = \frac{\kappa^* N_{S_2}^2}{M_{S_2}(M_{\tilde{S}_1} + M_{S_2})^2}. \tag{38}$$

More precisely, convergence is linear, the convergence factor being $K_\theta < 1$ defined in the proof below.

Proof. The proof is split into two steps. We first show the Richardson procedure for the sequence $\{\lambda_a^k\}$ given by Eq. $(36)_1$ converges. The proof is based on the abstract Theorem 3.1 of [2], or Theorem 4.2.2 of [26]. Secondly, we show that if the sequence $\{\lambda_a^k\}$ converges, then $\{\lambda_b^k\}$ does as well.

Let us start with the first step and define R_a the Richardson iteration operator as

$$R_a : \Lambda_a \to H^{-1/2}(\Gamma_a),$$

$$R_a\mu_a := (I_a - \theta Q_a)\mu_a = (I_a - \theta S_2^{-1}S)\mu_a.$$

If we define $e_a^k = \lambda_a^k - \lambda_a$ as the error with respect to λ_a at iteration k, λ_a being solution of problem $(35)_1$, the error equation reads

$$e_a^{k+1} = R_a e_a^k.$$

The Richardson procedure $(36)_1$ is therefore convergent if the operator R_a is a contraction with respect to some norm. Let us introduce the following application:

$$(\cdot,\cdot)_{S_2} : \Lambda_a \times \Lambda_a \to \mathbb{R},$$

$$(\eta_a, \mu_a)_{S_2} := \tfrac{1}{2}(\langle S_2\eta_a, \mu_a\rangle_{\Gamma_a} + \langle S_2\mu_a, \eta_a\rangle_{\Gamma_a}).$$

It is easy to check that this application is a scalar product, and that it induces the following S_2-norm:

$$\|\mu_a\|_{S_2} := \langle S_2\mu_a, \mu_a\rangle_{\Gamma_a}^{1/2},$$

which, owing to both the coercivity and continuity of S_2, is equivalent to the natural norm on Λ_a, i.e.,

$$N_{S_2}^{1/2}\|\mu_a\|_{1/2,\Gamma_a} \leqslant \|\mu_a\|_{S_2} \leqslant M_{S_2}^{1/2}\|\mu_a\|_{1/2,\Gamma_a} \quad \forall \mu_a \in \Lambda_a. \tag{39}$$

By definition we have

$$\|R_a\mu_a\|_{S_2}^2 = \|\mu_a\|_{S_2}^2 + \theta^2\langle S\mu_a, S_2^{-1}S\mu_a\rangle_{\Gamma_a} - \theta(\langle S_2\mu_a, S_2^{-1}S\mu_a\rangle_{\Gamma_a} + \langle S\mu_a, \mu_a\rangle_{\Gamma_a}). \tag{40}$$

Using the same strategy as in [26], it can be checked that

$$\langle S_2\mu_a, S_2^{-1}S\mu_a\rangle_{\Gamma_a} + \langle S\mu_a, \mu_a\rangle_{\Gamma_a} \geqslant \kappa^*\|\mu_a\|_{1/2,\Gamma_a}^2 \quad \forall \mu_a \in \Lambda_a$$

with κ^* defined in Eq. (37). Since the norm of S_2^{-1} is $1/N_{S_2}$, and owing to the continuity of S_2 and $\tilde{S}_1$ and to the assumption of the theorem, Eq. (40) yields

$$\|R_a\mu_a\|_{S_2}^2 \leqslant K_\theta\|\mu_a\|_{S_2}^2$$

with K_θ given by

$$K_\theta = 1 + \theta^2\frac{(M_{S_2} + M_{\tilde{S}_1})^2}{N_{S_2}^2} - \theta\frac{\kappa^*}{M_{S_2}}.$$

The Richardson procedure is a contraction in the S_2-norm if $K_\theta < 1$, i.e., if $0 < \theta < \theta_{\max}$, with $\theta_{\max}$ given by Eq. (38).

Let us now go on to the second step of the proof, i.e., the convergence of the sequence $\{\lambda_a^k\}$ implies that of the sequence $\{\lambda_b^k\}$. Although the Dirichlet and Robin-relaxed methods lead to the same Richardson procedure for λ_b (Eq. $(36)_2$) for $k \geqslant 1$, we have to treat their convergence separately. We define $e_b^k = \lambda_b^k - \lambda_b$. Since the converged solution satisfies $\lambda_b = \tilde{T}_b\lambda_a + \chi'$, Eq. $(34)_2$ for the Robin-relaxed scheme gives for any $k \geqslant 1$,

$$e_b^k = \tilde{T}_b e_a^k.$$

Therefore, we have

$$\begin{aligned}
&\|e_b^k\|_{1/2,\Gamma_b}\\
&\quad = \|T_b\mathscr{L}_2 e_a^k\|_{1/2,\Gamma_b}\\
&\quad \leqslant C_2^*\|\mathscr{L}_2 e_a^k\|_{1,\Omega_2} \quad \text{(trace inequality (5))}\\
&\quad \leqslant C_2^* C_2\|e_a^k\|_{1/2,\Gamma_a} \quad \text{(a priori estimate (6))}\\
&\quad \leqslant \frac{C_2^* C_2}{N_{S_2}^{1/2}}\|e_a^k\|_{S_2} \quad \text{(norm equivalence (39))}\\
&\quad \leqslant K_\theta^k\,\frac{C_2^* C_2}{N_{S_2}^{1/2}}\|e_a^0\|_{1/2,\Gamma_a},
\end{aligned}$$

which shows that the sequence $\{\lambda_b^k\}$ converges whenever $K_\theta < 1$.

Now we study the convergence of the Dirichlet-relaxed algorithm (for $\theta \neq 1$). From Eq. $(33)_2$, we have that, for any $k \geqslant 1$,

$$e_b^k = \theta\tilde{T}_b e_a^k + (1-\theta)e_b^{k-1}.$$

According to this equation, we can generate the following sequence:

$$\begin{array}{rcrcl}
e_b^k & = & \theta\ \tilde{T}_b e_a^k & + & (1-\theta)\ e_b^{k-1},\\
(1-\theta)\ e_b^{k-1} & = & \theta(1-\theta)\ \tilde{T}_b e_a^{k-1} & + & (1-\theta)^2\ e_b^{k-2},\\
(1-\theta)^2\ e_b^{k-2} & = & \theta(1-\theta)^2\ \tilde{T}_b e_a^{k-2} & + & (1-\theta)^3\ e_b^{k-3},\\
\vdots & & & \vdots & \\
(1-\theta)^{k-2}\ e_b^2 & = & \theta(1-\theta)^{k-2}\ \tilde{T}_b e_a^2 & + & (1-\theta)^{k-1}\ e_b^1,\\
(1-\theta)^{k-1}\ e_b^1 & = & \theta(1-\theta)^{k-1}\ \tilde{T}_b e_a^1 & + & (1-\theta)^k\ e_b^0.
\end{array}$$

Adding up all the terms, we find the following equality:

$$e_b^k = (1-\theta)^k e_b^0 + \theta(1-\theta)^k \sum_{n=1}^{k} (1-\theta)^{-n}\tilde{T}_b e_a^k,$$

which gives

$$
\begin{aligned}
\|e_b^k\|_{1/2,\Gamma_b} &\leqslant |1-\theta|^k \|e_b^0\|_{1/2,\Gamma_b} + \theta|1-\theta|^k \sum_{n=1}^{k} |1-\theta|^{-n} \|T_b \mathscr{L}_2 e_a^n\|_{1/2,\Gamma_b} \\
&\leqslant |1-\theta|^k \, \|e_b^0\|_{1/2,\Gamma_b} + \frac{\theta}{K_\theta}\,|1-\theta|^{k-1}\,\frac{C_2^* C_2}{N_{S_2}^{1/2}}\,\|e_a^1\|_{S_2} \sum_{n=1}^{k} \left(\frac{K_\theta}{|1-\theta|}\right)^n .
\end{aligned}
$$

The geometric progression is

$$
\sum_{n=1}^{k} \left(\frac{K_\theta}{|1-\theta|}\right)^n = \begin{cases} \frac{1}{2}\,k(k+1) & \text{if } K_\theta = |1-\theta|, \\ \dfrac{K_\theta}{|1-\theta|^k}\,\dfrac{|1-\theta|^k - K_\theta^k}{|1-\theta| - K_\theta} & \text{otherwise,} \end{cases}
$$

and thus we find the following two expressions for the norm of the error:

$$
\|e_b^k\|_{1/2,\Gamma_b} \leqslant \begin{cases} |1-\theta|^k \left(\|e_b^0\|_{1/2,\Gamma_b} + \dfrac{1}{2}\,k(k+1)\theta\,\dfrac{C_2^* C_2}{N_{S_2}^{1/2}}\,\|e_a^0\|_{S_2} \right) & \text{if } K_\theta = |1-\theta|, \\ |1-\theta|^k \|e_b^0\|_{1/2,\Gamma_b} + \theta\,\dfrac{|1-\theta|^k - K_\theta^k}{|1-\theta| - K_\theta}\,\dfrac{C_2^* C_2}{N_{S_2}^{1/2}}\,\|e_a^0\|_{S_2} & \text{otherwise.} \end{cases}
$$

Owing to these inequalities and since $\theta < 2$ (see Eqs. (37)–(38)), we conclude that if $K_\theta < 1$ the sequence $\{\lambda_b^k\}$ converges. □

Note that once $\lambda_a = \lim_{k\to\infty} \lambda_a^k$ and $\lambda_b = \lim_{k\to\infty} \lambda_b^k$ are found, the solutions in Ω_1 and Ω_2 are obtained by solving the two Dirichlet problems given by Eqs. (17). As a consequence, the convergences of sequences $\{\lambda_a^k\}$ and $\{\lambda_b^k\}$ imply the convergence of the whole algorithm.

Remark 12. This result carries over to the discrete variational problems provided the stability and continuity properties of the continuous case are inherited. In particular, the rate of convergence will be independent of the number of degrees of freedom.

5. Numerical examples

We present four numerical examples to test the overlapping D/R method in the diffusion as well as in the advection dominated limits. We apply the DD method to the discrete problem, rather to the continuous one considered up to now. However, the decomposition given by Eq. (7) and the results we have proven related to it apply also to the discrete (finite-dimensional) setting resulting from a *finite element discretization*, that is what we consider in what follows. In particular, in all the cases we consider a finite element partition of the computational domain made of piecewise bilinear quadrilateral elements.

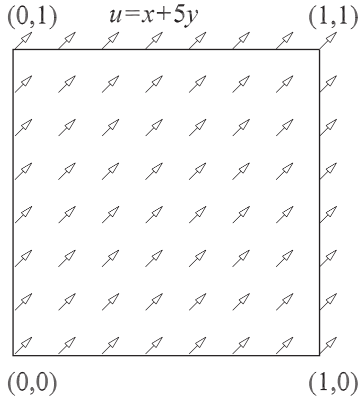


Fig. 4. Computational domain and boundary conditions.

5.1. Skew advection

Through this example, which was used as a first test case of the classical γ-D/R method in [2], we want to compare the disjoint and overlapping versions of the D/R method for a skew advection field. As an additional indication when using overlapping grids, we will systematically give the results of the Schwarz method (D/D) for overlapping subdomains, and that of the adaptive D/N method (A-D/N) for both disjoint and overlapping subdomains. The overlapping version of the A-D/N method uses a Neumann interface at outflow and a Dirichlet interface at inflow, as in the classical disjoint case. We propose to solve the equation

$$-\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u = f \quad \text{in } \Omega = (0,1)\times(0,1)$$

with a skew advection field $\boldsymbol{a} = [1,1]^{\mathrm{t}}$, and look for the exact solution $u = u(x,y) = x + 5y$, which belongs to the finite element space of work. According to this choice, we impose $f = 6$, and exact Dirichlet conditions on the boundary; see Fig. 4.

We define three different meshes, with h=1/10, 1/20 and 1/40. In addition, we define three different partitionings. The splitting of the two subdomains is always performed vertically and symmetrically with respect to the line $x = 0.5$. The first partition splits Ω into two disjoint subdomains, the second into two overlapping subdomains with horizontal overlapping length $\delta = 0.2$, and the third one with $\delta = 0.4$. As for the numerical strategy, we use the variational subgrid scale model (indispensable for small ε), as described in [8]. In order to introduce as few extrinsic errors to the DD methods themselves as possible, all the matrices involved in the Schur complement system are inverted using a direct solver. When considering disjoint subdomains, the convergence criterion is

$$100\,\frac{\|\mathbf{u}_{\Gamma_a}^{k+1} - \mathbf{u}_{\Gamma_a}^{k}\|_2}{\|\mathbf{u}_{\Gamma_a}^{k}\|_2} \leqslant 10^{-10},$$

while for overlapping subdomains it is given by

$$100\,\frac{\|\mathbf{u}_{\Gamma_a}^{k+1} - \mathbf{u}_{\Gamma_a}^{k}\|_2 + \|\mathbf{u}_{\Gamma_b}^{k+1} - \mathbf{u}_{\Gamma_b}^{k}\|_2}{\|\mathbf{u}_{\Gamma_a}^{k}\|_2 + \|\mathbf{u}_{\Gamma_b}^{k}\|_2} \leqslant 10^{-10},$$

Table 1
Number of iterations ($\theta = 0.5$, $\delta = 0$)

$\varepsilon \setminus h$	D/R			A-D/N		
	1/10	1/20	1/40	1/10	1/20	1/40
10^{1}	2	2	2	8	8	8
10^{0}	5	4	2	15	15	15
10^{-1}	7	6	5	31	31	31
10^{-2}	8	8	7	39	39	39
10^{-3}	8	8	8	40	40	40
10^{-4}	9	8	8	41	41	41
10^{-5}	9	8	8	41	41	41

Table 2
θ_{opt} and number of iterations ($\delta = 0$)

ε	D/R		A-D/N	
	θ_{opt}	No.	θ_{opt}	No.
10^{1}	0.50	2	0.50	8
10^{0}	0.50	4	0.54	11
10^{-1}	0.50	6	0.65	19
10^{-2}	0.50	8	0.81	17
10^{-3}	0.50	8	0.90	17
10^{-4}	0.50	9	0.93	18
10^{-5}	0.50	9	0.93	18

where $\mathbf{u}^k_{\Gamma_a}$ and $\mathbf{u}^k_{\Gamma_b}$ are the arrays of nodal unknowns corresponding to the nodes on the interfaces Γ_a and Γ_b, respectively, computed at iteration k. Tables 1 and 2 present the already known results of the disjoint D/R and adaptive D/N methods. The former confirms the mesh independence of both methods, while the latter gives the optimum relaxation parameter θ_{opt} and the corresponding numbers of iterations needed to achieve convergence. Possible values of θ have been limited to two decimal places. As expected, we note that θ_{opt} for the D/R method is always 0.5, while that of the A-D/N method it is somewhere between 0.5 and 1, and depends on ε.

Tables 3–6 present the same results for the overlapping methods. The tables show that the overlapping D/R method behaves like the classical D/N method for ε high, and like the D/D method for ε small. We observe that when $\varepsilon \ll 1$, the convergence of the D/R will improve with decreasing h.

We also note that for all the DD methods tested, the optimum θ is close to unity in the diffusion dominated range, while it is exactly one in the advection-dominated range. This contrasts completely with the disjoint counterparts of the DD methods.

Table 7 gives the number of iterations needed to achieve convergence for the different methods, as a function of the overlapping length, and for the second finest mesh $h = 1/20$. We observe that

Table 3
Number of iterations ($\theta = 1.0$, $\delta = 0.2$)

$\varepsilon \setminus h$	D/R			A-D/N			D/D		
	1/10	1/20	1/40	1/10	1/20	1/40	1/10	1/20	1/40
10^{1}	23	23	23	23	23	23	21	21	21
10^{0}	23	23	23	19	19	19	21	21	21
10^{-1}	10	11	11	7	8	8	10	11	11
10^{-2}	10	6	3	7	4	3	10	6	3
10^{-3}	12	7	5	7	5	4	11	7	5
10^{-4}	12	7	5	7	5	4	12	7	5
10^{-5}	12	7	5	7	5	4	12	7	5

Table 4
θ_{opt} and number of iterations ($\delta = 0.2$)

ε	D/R		A-D/N		D/D	
	θ_{opt}	No.	θ_{opt}	No.	θ_{opt}	No.
10^{1}	0.87	14	0.87	14	1.14	16
10^{0}	0.87	14	0.90	13	1.14	15
10^{-1}	0.98	9	1.00	8	1.02	9
10^{-2}	1.00	6	1.00	4	1.00	6
10^{-3}	1.00	7	1.00	5	1.00	7
10^{-4}	1.00	7	1.00	5	1.00	7
10^{-5}	1.00	7	1.00	5	1.00	7

Table 5
Number of iterations ($\theta = 1.0$, $\delta = 0.4$)

$\varepsilon \setminus h$	D/R			A-D/N			D/D		
	1/10	1/20	1/40	1/10	1/20	1/40	1/10	1/20	1/40
10^{1}	12	12	12	12	12	12	11	11	11
10^{0}	12	12	12	11	11	11	11	11	11
10^{-1}	6	6	6	5	5	5	6	6	6
10^{-2}	6	4	2	5	3	2	6	4	2
10^{-3}	7	4	2	5	4	3	7	4	3
10^{-4}	7	4	3	5	4	3	7	4	3
10^{-5}	7	4	3	5	4	3	7	4	3

for $\varepsilon = 10^1$ and 10^0, the overlapping does not improve convergence. This is rather a coincidence than a rule. For example, locating the interface at $x = 0.75$, the disjoint D/R method converges in 14 iterations at least in both cases!

Before closing the analysis of this example, let us examine how the error is reduced by the disjoint and overlapping D/R methods ($\delta = 0.2$), for high advection ($\varepsilon = 10^{-4}$). We choose θ such

Table 6
Number of iterations and θ_{opt} ($\delta = 0.4$)

ε	D/R		A-D/N		D/D	
	θ_{opt}	No.	θ_{opt}	No.	θ_{opt}	No.
10^{1}	0.96	10	0.96	10	1.03	9
10^{0}	0.97	10	0.97	9	1.01	10
10^{-1}	1.00	6	1.00	5	1.00	6
10^{-2}	1.00	4	1.00	3	1.00	4
10^{-3}	1.00	4	1.00	4	1.00	4
10^{-4}	1.00	4	1.00	4	1.00	4
10^{-5}	1.00	4	1.00	4	1.00	4

Table 7
Number of iterations ($\theta = \theta_{\mathrm{opt}}$)

$\varepsilon \setminus \delta$	D/R			A-D/N			D/D		
	0	0.2	0.4	0	0.2	0.4	0	0.2	0.4
10^{1}	2	14	10	8	14	10	—	16	10
10^{0}	4	14	10	11	13	9	—	15	10
10^{-1}	6	9	6	19	8	5	—	9	6
10^{-2}	8	6	4	17	4	3	—	6	4
10^{-3}	8	7	4	17	5	4	—	7	4
10^{-4}	8	7	4	18	5	4	—	7	4
10^{-5}	8	7	4	18	5	4	—	7	4

that the rate of convergence of each method is more or less the same, to be able to compare the error reduction using the same scale; this choice corresponds to $\theta = 0.44$ in the case of the disjoint D/R method, and $\theta = 0.9$ in the case of the overlapping D/R method. The initial solution is the exact solution, on which we superimpose an error with respect to the analytical solution somewhere on the interface. In the case of the disjoint D/R method, we introduce the error at point $(0.5, 0.5)$, while for the overlapping version, we introduce the error at point $(0.4, 0.5)$. The magnitude of the error in both cases is 0.5, using as normalization the maximum exact value over the domain, i.e., 6. On the one hand, Fig. 5(top left) and (top right) show how the error is advected along the streamlines of the flow, at iterations 2 and 4, respectively. On the other hand, Fig. 5(bottom left) and (bottom right) show how the error is mostly confined between the interfaces, located at $x = 0.4$ and 0.6.

5.2. Normal and tangential advections

This example studies the solution of a thermal boundary layer, also presented in [2],

$$-\varepsilon \Delta u + \boldsymbol{a} \cdot \nabla u = 0 \quad \text{in } \Omega = (0,1) \times (0,0.5)$$

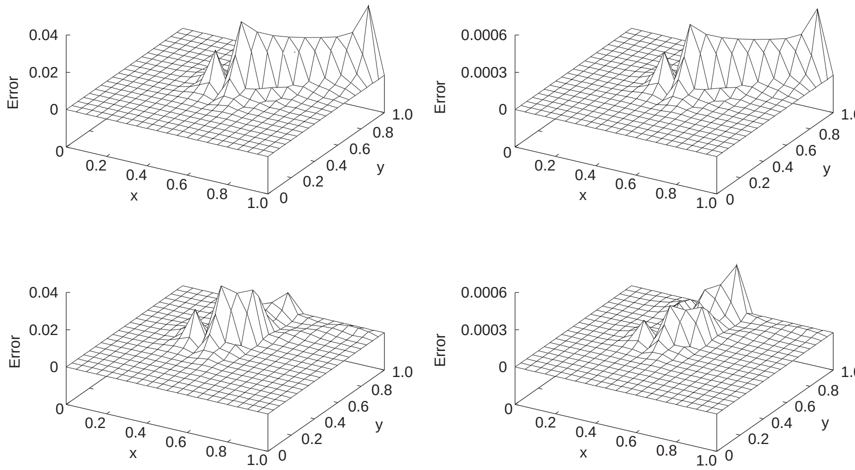


Fig. 5. Error. (Top left) Disjoint D/R, iteration 2. (Top right) Disjoint D/R, iteration 4. (Bottom left) Overlapping D/R, iteration 2. (Bottom right) Overlapping D/R, iteration 4.

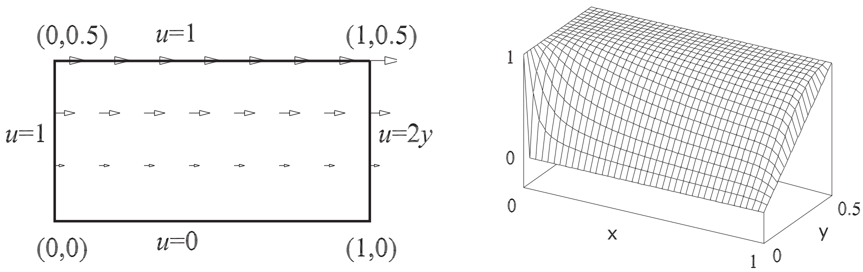


Fig. 6. (Left) Computational domain and boundary conditions. (Right) Solution for $\varepsilon = 10^{-2}$.

with a horizontal advection field $\boldsymbol{a} = [2y, 0]^t$ and the following boundary conditions:

$$u = \begin{cases} 1 & \text{at } x = 0, \text{ and } y = 0.5, \\ 2y & \text{at } x = 1, \\ 0 & \text{elsewhere.} \end{cases}$$

The geometry as well as the boundary conditions are shown in Fig. 6(left).

This example is solved using the same numerical strategy as that of the previous example. The mesh convergence shares sensibly the same characteristics as that of the first example, and so only the results run with a mesh size of $h = 1/20$ are reported here. The solution obtained on this mesh for $\varepsilon = 10^{-2}$ is shown in Fig. 6(right). Two different partitionings are performed. First, we consider a symmetric vertical partitioning of the domain, i.e., the interface is placed normal to the advection field. Tables 8 and 9 compare the optimum relaxation parameters and the associated number of iterations of the disjoint and overlapping versions of the different DD methods. As it was already observed in the previous example, we note that the θ_{opt} of the disjoint D/R method is 0.5, while that of the overlapping D/R is 1. The results of the A-D/N method are more mitigated. On the one

Table 8
Normal advection. θ_{opt} and number of iterations ($\delta = 0$)

ε	D/R		A-D/N	
	θ_{opt}	No.	θ_{opt}	No.
10^1	0.50	5	0.50	7
10^0	0.50	6	0.51	9
10^{-1}	0.50	9	0.61	15
10^{-2}	0.50	10	0.74	21
10^{-3}	0.50	7	0.92	12
10^{-4}	0.50	4	0.99	8
10^{-5}	0.50	3	1.00	6

Table 9
Normal advection. θ_{opt} and number of iterations ($\delta = 0.2$)

ε	D/R		A-D/N		D/D	
	θ_{opt}	No.	θ_{opt}	No.	θ_{opt}	No.
10^1	0.96	10	0.96	10	1.04	10
10^0	0.96	10	0.96	10	1.04	10
10^{-1}	0.97	10	0.98	8	1.03	10
10^{-2}	1.00	5	1.00	5	1.00	6
10^{-3}	1.00	4	1.00	3	1.00	4
10^{-4}	1.00	5	1.00	3	1.00	5
10^{-5}	1.00	5	1.00	3	1.00	5

hand, the θ_{opt} of the disjoint version tends to unity very slowly for decreasing ε. On the other hand, the θ_{opt} of the overlapping version is, as in the case of the overlapping D/R, unity for $\varepsilon \leqslant 10^{-2}$.

We now partition Ω horizontally. In this case, the Neumann and Robin conditions coincide as $\boldsymbol{a} \cdot \boldsymbol{n} = 0$. Table 10 gives the results obtain for the classical D/N method. As in the case of the normal advection, we observe that the optimum relaxation parameter of all methods tends to unity rapidly whenever $\varepsilon \leqslant 10^{-2}$, while that of the disjoint D/N method remains around 0.5.

5.3. Curved advection

We increase a bit the difficulty. We consider a curved advection field and impose a discontinuity in the Dirichlet condition. This example was proposed in [29] and consists in solving

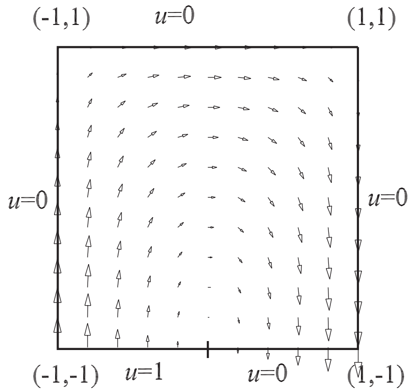
$$-\varepsilon \Delta u + \boldsymbol{a} \cdot \nabla u + \sigma u = 0 \quad \text{in } \Omega = (-1, 1) \times (-1, 1),$$

where the advection field and the source term are given by

$$\boldsymbol{a} = \tfrac{1}{2}[(1 - x^2)(1 + y), -x(4 - (1 + y)^2)]^{\mathrm{t}},$$
$$\sigma = 10^{-4}$$

Table 10
Tangential advection. θ_{opt} and number of iterations

ε	D/N $\delta=0$		D/N $\delta=0.1$		D/N $\delta=0.2$		D/D $\delta=0.1$		D/D $\delta=0.2$	
	θ_{opt}	No.	θ_{opt}	No.	θ_{opt}	No.	θ_{opt}	No.	θ_{opt}	No.
10^{1}	0.50	5	0.79	18	0.89	14	1.24	20	1.08	12
10^{0}	0.50	6	0.79	18	0.89	14	1.24	20	1.07	12
10^{-1}	0.49	10	0.80	18	0.91	13	1.21	19	1.06	11
10^{-2}	0.47	10	0.99	10	1.00	6	1.01	9	1.00	6
10^{-3}	0.48	8	1.00	6	1.00	4	1.00	6	1.00	4
10^{-4}	0.47	8	1.00	7	1.00	4	1.00	6	1.00	4
10^{-5}	0.47	7	1.00	7	1.00	4	1.00	7	1.00	4



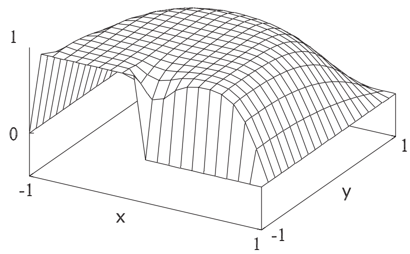


Fig. 7. (Left) Computational domain and boundary conditions. (Right) Solution for $\varepsilon = 10^{-2}$.

and the Dirichlet boundary conditions for u are

$$u = \begin{cases} 1 & \text{at } y=-1,\ 0<x<0.5, \\ 0 & \text{elsewhere.} \end{cases}$$

See Fig. 7(left) for a sketch of the problem. We present here the results obtained on three meshes composed of constant element length h such that $h = 1/10$ for the coarse mesh, $h = 1/20$ for the medium mesh and $h = 1/40$ for the fine mesh. Fig. 7(right) shows the solution obtained on the medium mesh for $\varepsilon = 10^{-2}$.

In this example, we want to compare the results of the overlapping and disjoint D/R method without trying to adjust the relaxation parameter. For the disjoint versions, we take $\theta = 0.5$ and for the overlapping versions we take $\theta = 1.0$. We consider symmetrical horizontal and vertical partitionings, with an overlap of $\delta = 0.4$ for the overlapping partitions. As different results have been found (in the disjoint version) depending on where the Dirichlet and Robin interfaces are imposed, the Dirichlet/Robin method is referred to as D/R method when the Dirichlet condition is imposed on the top and left subdomain interfaces in the case of horizontal and vertical partitionings, respectively. On the contrary, the Dirichlet/Robin method is referred to as R/D method.

Table 11
Number of iterations. Coarse mesh: $h = 1/10$

ε	Disjoint				Overlapping			
	D/R		R/D		D/R		R/D	
	Horiz.	Verti.	Horiz.	Verti.	Horiz.	Verti.	Horiz.	Verti.
10^1	6	7	6	7	23	23	23	23
10^0	10	10	10	10	23	22	23	23
10^{-1}	16	18	16	18	13	15	12	15
10^{-2}	23	16	17	16	8	6	8	6
10^{-3}	40	16	21	16	9	9	9	9
10^{-4}	46	18	22	17	9	10	9	10
10^{-5}	47	18	22	18	9	10	9	10

Table 12
Number of iterations. Medium mesh: $h = 1/20$

ε	Disjoint				Overlapping			
	D/R		R/D		D/R		R/D	
	Horiz.	Verti.	Horiz.	Verti.	Horiz.	Verti.	Horiz.	Verti.
10^1	6	7	6	7	24	23	24	23
10^0	10	10	10	10	23	23	23	23
10^{-1}	17	18	17	18	11	14	12	14
10^{-2}	15	17	14	17	5	5	5	5
10^{-3}	23	16	17	16	6	5	6	5
10^{-4}	26	16	18	16	6	7	6	7
10^{-5}	27	17	18	17	6	7	6	7

Tables 11–13 give the numbers of iterations needed to achieve convergence for all the methods. We notice that in the diffusion range, the disjoint versions converge better than the overlap versions. The tendency is inverted as soon as the advection compensates and overcomes the diffusion, i.e., when $\varepsilon \leqslant 10^{-1}$. In addition, the overlapping version shows much less sensitivity to the positioning of the interface when the mesh is coarse. In all cases, the number of iterations is bounded as the diffusion decreases.

5.4. Rotating advection

We consider once more the exact linear solution $u(x, y) = x + 5y$ of the first test case, but this time using a rotating advection field centered at $(0.6, 0.6)$ and given by

$$\boldsymbol{a} = [-y + 0.6, x - 0.6]^{\mathrm{t}},$$

Table 13
Number of iterations. Fine mesh: $h = 1/40$

ε	Disjoint				Overlapping			
	D/R		R/D		D/R		R/D	
	Horiz.	Verti.	Horiz.	Verti.	Horiz.	Verti.	Horiz.	Verti.
10^{1}	6	7	7	7	24	23	24	23
10^{0}	10	10	10	10	23	23	23	23
10^{-1}	17	19	17	19	11	14	12	14
10^{-2}	16	18	12	18	4	6	4	6
10^{-3}	16	16	14	16	4	3	4	3
10^{-4}	18	16	15	16	4	5	4	5
10^{-5}	18	17	15	17	4	5	4	5

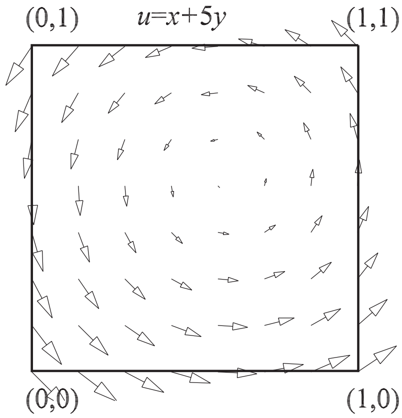


Fig. 8. Computational domain and boundary conditions.

which leads us to choose the force term $f = 5x - y$ (see geometry in Fig. 8). We have chosen this case because of its complicated local behavior. Around the center of the rotating advection field, diffusion dominates. In addition, the interfaces considered are both inflow and outflow. The results presented here have been obtained on a 20×20 element mesh, and the interfaces are the same as that of the first test case.

Table 14 shows the number of iterations needed to achieve convergence for the optimum relaxation parameter.

In this example, we have observed notable differences in the results depending on which interfaces the Robin and Dirichlet conditions are imposed; we denote them D/R when the left subdomain is assigned a Dirichlet condition and R/D when it is assigned a Robin condition. We observe that for the disjoint and overlapping versions with $\delta = 0.2$ the number of iterations blows up when ε decreases. However, the overlapping decreases this figure by approximately one order of magnitude. In addition, we have considered the case of $\delta = 0.4$. The compared results are shown in Fig. 9(left) and (right). They confirm the improvement in convergence when using overlapping.

Table 14
θ_{opt} and number of iterations

ε	D/R $\delta=0$		R/D $\delta=0$		D/R $\delta=0.2$		R/D $\delta=0.2$	
	θ_{opt}	No.	θ_{opt}	No.	θ_{opt}	No.	θ_{opt}	No.
10^{1}	0.50	5	0.50	5	0.87	14	0.87	14
10^{0}	0.50	8	0.50	8	0.87	14	0.87	14
10^{-1}	0.50	14	0.50	13	0.88	14	0.88	14
10^{-2}	0.49	40	0.50	34	0.97	11	1.07	13
10^{-3}	0.46	243	0.48	200	1.43	37	1.54	49
10^{-4}	0.47	1864!	0.49	1493!	1.85	221	1.87	242
10^{-5}	0.47	8460!	0.51	6257!	1.94	753	1.95	816

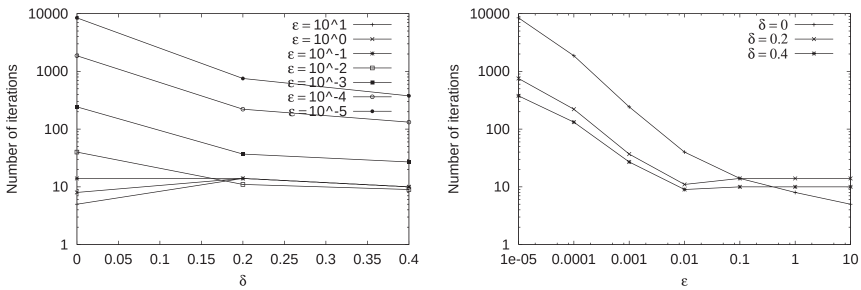


Fig. 9. Number of iterations as a function of δ and ε.

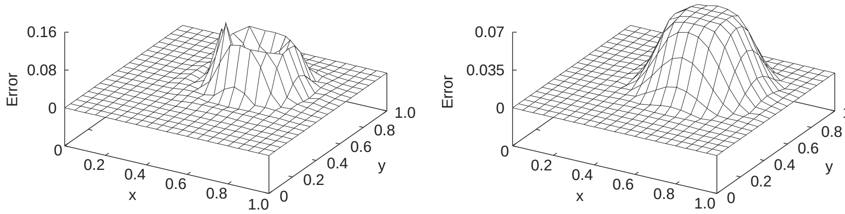


Fig. 10. Error of the unpreconditioned Richardson procedure. (Left) After 1000 iterations. (Right) After 4000 iterations.

As in the first test case, we now introduce a perturbation (an error peak) on the interfaces. The difficulty of solving this case relies in the fact that, for small diffusion coefficients, the error is advected around and around, flowing along the streamlines. If the error is introduced near the center of the vortex, it can remain for a long time within the domain before being diffused and absorbed by the boundary conditions. We consider here the case $\varepsilon=10^{-4}$. As an illustration, we have also solved the unpreconditioned Richardson procedure for the interface unknowns, using disjoint subdomains. The error magnitude is 0.5 (normalized by the maximum value, i.e., 6). Fig. 10 shows the error obtained after 1000 and 4000 iterations, using $\theta=0.50$. After 1000 iterations, we still recognize the error peak introduced at point $(0.5, 0.5)$; we also note that the error has been totally advected around.

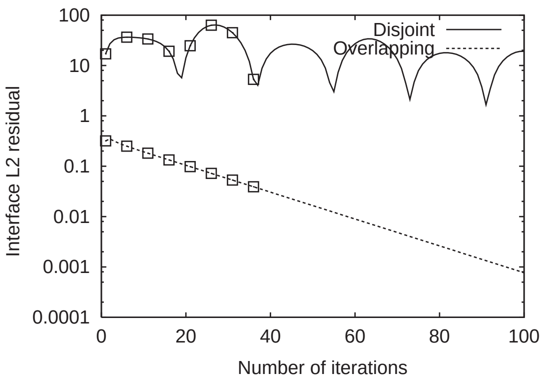


Fig. 11. Convergence histories of the disjoint and overlapping D/R methods.

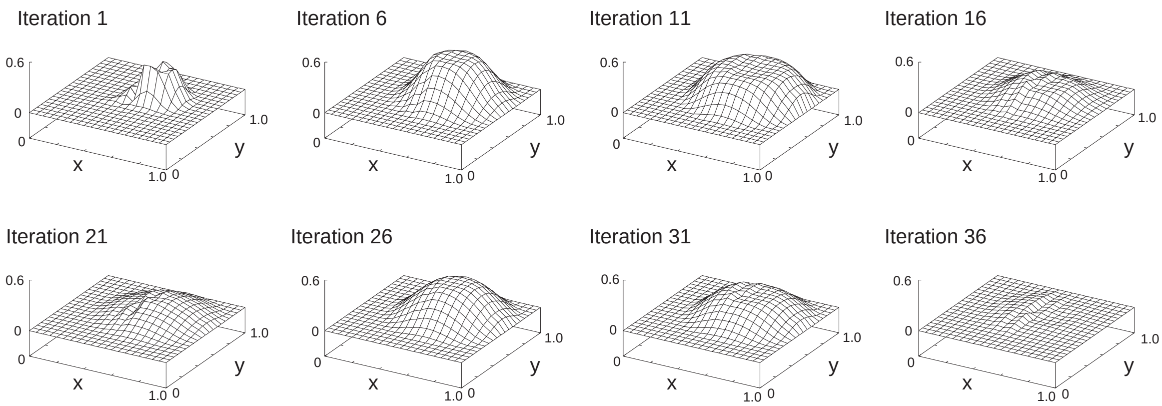


Fig. 12. Error. Disjoint D/R.

After 4000 iterations, the error has been diffused inside and outside the advection circle. Let us now go back to the analysis of the disjoint and overlapping ($\delta = 0.2$) D/R methods. In the case of the disjoint D/R method, we introduce the error at point $(0.5, 0.5)$, while for the overlapping version, we introduce the error at point $(0.6, 0.5)$. Fig. 11 compares the convergence histories of both versions, using $\theta = 0.5$ and 1.0, respectively. We observe that the convergence of the disjoint D/R method is far from monotone.

Fig. 12 represents the error with respect to the exact solution and normalized by the maximum exact solution at iterations 1, 6, 11, 16, 21, 26, 31 and 36. These iterations are labeled in Fig. 11. We notice that after few iterations the error of the disjoint D/R exhibits more or less the same error profile as the unpreconditioned Richardson procedure, although the error is diffused much more rapidly (in terms of iterations). However, after having decreased one order of magnitude, the error bounces up, before decreasing once again, and so on, until convergence. This phenomenon can be clearly identified in the convergence history of the method. The error profiles of the overlapping versions at iterations 1, 6, 11, 16, 21, 26, 31 and 36 are shown in Fig. 13. They confirm the

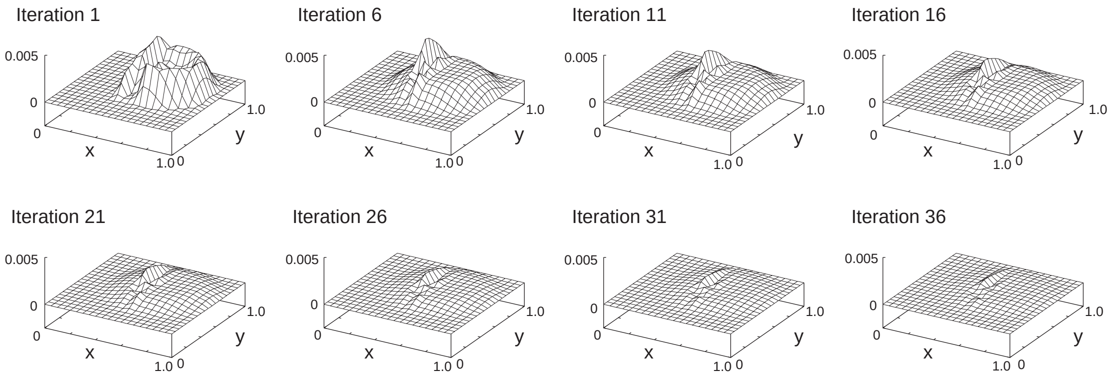


Fig. 13. Error. Overlapping D/R.

improvements achieved by the overlapping method. We conclude that the overlapping can be useful when a vortex passes near the interface.

6. Conclusions

In this paper we have presented an overlapping Dirichlet/Robin method to solve advection–diffusion problems which intends to inherit the robustness properties of the classical Schwarz method, but allowing the limit case of zero (or extremely small) overlapping.

From the analytical point of view, we have extended the analysis of the disjoint Dirichlet/Robin method to the case of overlapping subdomains, showing that the problem is well defined (equivalent to the original one) and proving convergence for the associated iteration-by-subdomain scheme. The key ingredient is the introduction of Steklov–Poincaré-like operators which map a trace space onto the dual of *another* trace space.

From the numerical point of view and using the DD method in conjunction with a stabilized finite element method, we have observed that the overlapping version is certainly more robust than the disjoint one, leading to smaller number of iterations to achieve convergence and damping out errors faster. In the examples presented, no reaction term were present. However, we have observed and shown for simple cases [13] that the presence of a reaction like term (coming from a time integration scheme for example) improves considerably the convergence. In fact, the DD algorithm developed along this work is presently used by the authors to solve the Navier-Stokes equations in domains involving moving objects, and the algorithm has proved to be robust in laminar as well as in turbulent simulations. These simulations use a Chimera method based on Dirichlet/Robin coupling [14].

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