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Long term stability estimates and existence of a global attractor in a finite element approximation of the Navier-Stokes equations with numerical sub-grid scale modeling.

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Resumen

Variational multiscale methods lead to stable finite element approximations of the Navier-Stokes equations, both dealing with the indefinite nature of the system (pressure stability) and the velocity stability loss for high Reynolds numbers. These methods enrich the Galerkin formulation with a sub-grid component that is modelled. In fact, the effect of the sub-grid scale on the captured scales has been proved to dissipate the proper amount of energy needed to approximate the correct energy spectrum. Thus, they also act as effective large-eddy simulation turbulence models and allow to compute flows without the need to capture all the scales in the system. In this article, we consider a dynamic sub-grid model that enforces the sub-grid component to be orthogonal to the finite element space in L^2 sense. We analyze the long-term behavior of the algorithm, proving the existence of appropriate absorbing sets and a compact global attractor. The improvements with respect to a finite element Galerkin approximation are the long-term estimates for the sub-grid component, that are translated to effective pressure and velocity stability. Thus, the stabilization introduced by the sub-grid model into the finite element problem is not deteriorated for infinite time intervals of computation.

1. Introduction

The dynamics of Newtonian incompressible flows are governed by the Navier-Stokes equations, a dynamical system that consists in a set of nonlinear partial differential equations with a dissipative structure. For two-dimensional problems, the energy of this system has been proved to be bounded by the data (external forces and boundary conditions) for all times. It is also possible to bound the $H^1(\Omega)$-norm of the fluid velocity, which, together with the Rellich-Kondrachov theorem, allows to prove that any fluid velocity orbit converges to a finite dimensional set, the so-called global attractor, as the time variable goes to infinity (see [16, 31]). Fractal and Hausdorfd dimensions of the global attractor have been estimated using Lyapunov exponents in dimension 2 and 3 [13, 17].

An accurate numerical approximation of the Navier-Stokes equations should mimic their long-term behavior. For direct numerical simulation (DNS), a crude Galerkin approximation using inf-sup stable finite elements admits a numerical global attractor, whose dimension has been estimated in [27]. The convergence of the numerical global attractor to the one of the Navier-Stokes equations has been analyzed in [21]. Similar results have been proved for finite differences [32].

The finite element approximation of the Navier-Stokes equations for large Reynolds numbers (Re) presents two main difficulties that can make their numerical approximations meaningless: one is the indefinite nature of the system, and the other the stability loss due to convection dominant regimes. The first problem can be cured by using appropriate velocity-pressure finite element spaces satisfying a discrete version of the Ladyzhenskaya-Babuška-Brezzi condition (see [4]). These finite element pairs are usually called inf-sup stable elements, and do not include many spaces that would be interesting for their simplicity and/or efficiency. When using Galerkin approximations and finite elements, the only way to solve the velocity stability loss is to capture all the spatial scales of the flow, i.e. to reduce the computational mesh size up to the Kolmogorov microscale λ_K, below which there are the smallest dissipative structures of the flow. This approach, known as direct numerical simulation, requires in dimension 3 $\mathcal{O}(\mathrm{Re}^{2,25})$ mesh nodes. Unsurprisingly, this dimension is also related to the dimension of the continuous global attractor (see [13, 17, 31]). The memory usage grows so fast with respect to Re that DNS computations are unaffordable in most industrial applications, even at moderate Reynolds numbers. Anyway, DNS is a valuable tool in theoretical turbulence research: it allows a deeper understanding of this phenomenon and helps to validate turbulence models.

Both pressure instability and velocity stability loss for convection dominant regimes can be solved by using finite element stabilization techniques (see e.g. [5, 24, 7, 9, 12, 2]). In fact, stabilization is essential for the finite element approximation of high Re flows. The common feature of this family of algorithms is to introduce consistent terms to the formulation that would improve the stability properties of the numerical system without spoiling accuracy. Initially, these stabilization techniques were developed without a sound motivation till they were justified by a multiscale decomposition of the continuous solution into resolved (finite element) and unresolved (sub-grid) scales. Using this decomposition in the variational form of the problem, and modelling the effect of the subscales into the finite element problem, we end up with numerical methods that exhibit enhanced stability properties. We refer to [23, 25] for a detailed exposition of this approach, coined the variational multiscale (VMS) method. Applied to the Navier-Stokes equations, stabilized finite elements lead to stable formulations without the need of representing all the scales of the flow. Thus, coarser meshes can be used, drastically reducing the computational effort of DNS.

VMS sub-grid scale models have been motivated by numerical purposes (stability and convergence of the numerical algorithms), but they have also been proved to introduce a numerical dissipation that approximates well the physical dissipation at the unresolved scales [19, 9, 12, 22, 14, 29, 3]. So, these methods can be understood as large-eddy simulation (LES) turbulence models that properly account for the effect of the smaller universal scales onto the large scale motions of the flow that can be captured by the mesh.

The VMS framework is clear for linear stationary problems, leading to effective and accurate numerical methods. In those methods, the sub-grid component is modelled using local problems (the global sub-grid problem is localized at every finite element of the mesh) and the differential operator that defines the problem is replaced by an algebraic one (motivated by Fourier analysis in our case). As a result, the sub-grid component is approximated at every finite element as a closed form in terms of the finite element residual. Enforcing the sub-grid component to be orthogonal to the finite element space we recover the orthogonal sub-grid scale (OSS) model proposed by Codina in [7, 9], otherwise we get the algebraic sub-grid scales (ASGS) model, in the terminology of [9]. OSS has been proved to introduce less numerical dissipation than ASGS in [7].

The extension of this framework to transient and nonlinear problems is not obvious. The main difficulties lie in how to approximate the sub-grid time derivative in the sub-grid problem and how to track the subscale in the nonlinear iterative process. A straightforward choice for the time discrete system is to treat the time derivative of the sub-grid component as a reaction-like term, with reaction coefficient δt^{-1}, δt being the time step size. When $\delta t \to 0$, the algorithm tends to the non-stabilized Galerkin formulation, with the problems pointed out above. In [2, 11, 12] we have devised two cures to this instabilities. The first solution is to use OSS formulations together with a quasi-static approximation of the sub-grid scales, i.e. the sub-grid time derivative is neglected and the steady-state sub-grid model used. A more consistent approach is to consider dynamic sub-grid models that keep the sub-grid time derivative. In this case, the sub-grid model turns into an ordinary differential equation. Dynamic subscales have been proved to exhibit unconditional stability properties in strong norms for convection-diffusion and Stokes systems (both for ASGS and OSS formulations), the semi-discrete problem in space is well-posed and space and time integration commute. We refer to [2, 11, 12] for some works showing the benefits of using dynamic sub-grid scales and numerical analyses that prove the unconditional stability and convergence of the method for some linear problems. With regard to turbulence modelling, the VMS dynamic sub-grid model is able to represent backscatter [29].

1.1. Finite element approximation of the Navier-Stokes equations

From now on, we assume that Ω is a subset of $\mathbb{R}^d$ ($d = 2$ or 3) having a polygonal or polyhedral Lipschitz-continuous boundary, and $\{\mathcal{T}_h\}_{h>0}$ is a regular family of triangulations of $\bar{\Omega}$, that is, $\bar{\Omega} = \cup_{K\in\mathcal{T}_h} K$, with mesh size $h = \text{máx}_{K\in\mathcal{T}_h} h_K$, h_K being the diameter of the triangle K.

In order to get a conforming finite element approximation of the Navier-Stokes problem, we consider conforming finite element spaces $V_h \subset \boldsymbol{H}_0^1(\Omega)$ and $Q_h \subset L^2(\Omega)/\mathbb{R}$ for velocity and pressure respectively, with optimal interpolation properties. To simplify the exposition, we will consider $Q_h \in \mathcal{C}^0(\Omega)$. Then, the semi-discrete problem in space consists of finding e.g. $[\mathbf{u}_h, p_h] \in L^2(0,T;V_h) \times L^1(0,T;Q_h)$ such that

$$\left\{ \begin{array}{l} (\partial_t \mathbf{u}_h, \mathbf{v}_h) + \langle (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h, \mathbf{v}_h \rangle + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = \langle \mathbf{f}, \mathbf{v}_h \rangle, \\ (q_h, \nabla \cdot \mathbf{u}_h) = 0, \end{array} \right. \tag{1}$$

almost everywhere in time. Analogously to the continuous problem, it is easy to prove that the semi-discrete system (1) satisfies

$$\|\mathbf{u}_h(\mathbf{x},t)\|^2 + \nu \int_0^t \|\nabla \mathbf{u}_h(\mathbf{x},s)\|^2 \mathrm{d}s \lesssim \frac{1}{\nu} \int_0^t \|\mathbf{f}\|_{-1}^2 \mathrm{d}s + \|\mathbf{u}_h(\mathbf{x},0)\|^2.$$

Even for high Re, the viscous dissipative term of the continuous problem becomes dominant at the smallest scales of the flow; viscous effects extract energy to the system at the smallest scales, killing any fluctuation under a certain level, the Kolmogorov microscale λ_K (see [26, 28]). λ_K is obviously related to the number of nodes that are needed in a DNS computational mesh, since all the scales of the flow must be captured in such computations. When the computational mesh is substantially coarser than a DNS one, the smallest scales are $\mathcal{O}(h) \gg \lambda_K$, i.e. they belong to the inertial range. On the other hand, following the energy cascade, the energy from larger scales is transfered to the smallest scales. Since eddies in the range $\mathcal{O}(h)$ are much larger than the dissipative eddies that exist at Kolmogorov scales, kinetic energy is essentially not dissipated in this range. The viscous dissipation term never becomes important and, as a result, the smallest scales exhibit an energy pile-up (see [20]), leading to space instabilities.

Pressure stability for the Galerkin approximation of the Navier-Stokes equations cannot be attained from energy bounds. In order to mimic the mathematical structure of the continuous problem, we can build velocity-pressure finite element spaces satisfying a discrete inf-sup condition

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v}_h \in V_h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{\|q_h\| \|\mathbf{v}_h\|_1} \geq \beta^* > 0$$

where β^* is uniform with respect to h. Obviously, the discrete inf-sup condition is not a direct consequence of the continuous inf-sup condition. In fact, some interesting velocity-pressure pairs, like equal-order velocity pressure approximations, fail to satisfy this condition, leading to pressure instabilities.

Using VMS stabilized finite element approximations, we get numerical methods with enhanced stability properties for which there is the hope that they can act as turbulence models. Pressure stability does not rely on a discrete inf-sup condition and fluid velocity bounds remain effective at high Re for mesh sizes $h \gg \lambda_K$, placed in the inertial range. Furthermore, the effect of the unresolved scales, i.e. scales in the range $(h, \lambda_K]$, into the captured scales is properly modelled; in particular, the viscous dissipation that takes place at the smallest unresolved scales. In fact, it has been proved that the energy spectra of VMS-based algorithms approximate accurately the continuous spectra till $\mathcal{O}(h)$ scales (see [19, 12, 29, 3]).

We do not include here the motivation of these algorithms, that can be found elsewhere (see [23, 25]). In particular, we consider the sub-grid scales to be orthogonal to the finite element velocity space and dynamic. In order to state the problem, we introduce the sub-grid velocity component $\tilde{\mathbf{u}}$, which is modelled. We assume the sub-grid pressure $\tilde{p} = 0$, since the terms obtained from this component are not essential for the good performance of the algorithm (see e.g. [8]). The sub-grid velocity belongs to the sub-grid space $\tilde{V}$, to be defined. The finite element approximation of the Navier-Stokes equations using a VMS dynamic orthogonal sub-grid model reads as follows: find $\mathbf{u}_h \in L^2(0,\infty; \boldsymbol{V}_h)$, $p_h \in L^1(0,\infty; Q_h)$, and $\tilde{\mathbf{u}} \in L^2(0,\infty; \tilde{\boldsymbol{V}})$ such that

$$\left\{ \begin{array}{l} (\partial_t \mathbf{u}_h, \mathbf{v}_h) + b\,(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + \nu\,(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) \\ -\,(p_h, \nabla \cdot \mathbf{v}_h) - b\,(\mathbf{u}_h, \mathbf{v}_h, \tilde{\mathbf{u}}) = \langle \mathbf{f}, \mathbf{v}_h \rangle\,, \end{array} \right. \tag{2a}$$

$$(q_h, \nabla \cdot \mathbf{u}_h) - (\tilde{\mathbf{u}}, \nabla q_h) = 0, \tag{2b}$$

$$(\partial_t \tilde{\mathbf{u}}, \tilde{\mathbf{v}}) + \tau^{-1}\,(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \langle \mathbf{f}, \tilde{\mathbf{v}} \rangle - b\,(\mathbf{u}_h, \mathbf{u}_h, \tilde{\mathbf{v}}) - (\nabla p_h, \tilde{\mathbf{v}})\,, \tag{2c}$$

and

$$\mathbf{u}_h(0) = \mathbf{u}_{0h}, \quad \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}_0. \tag{3}$$

A proper initialization of the problem is obtained by using $\mathbf{u}_{0h}$ and $\tilde{\mathbf{u}}_0$ solution of the projection problem

$$\begin{aligned}(\mathbf{u}_{0h}, \mathbf{v}_h) - (\xi_h, \nabla\cdot\mathbf{v}_h) &= (\mathbf{u}_0, \mathbf{v}_h),\\ (\nabla\cdot\mathbf{u}_{0h}, q_h) - (q_h, \tilde{\mathbf{u}}_0) &= 0,\\ (\tilde{\mathbf{u}}_0, \tilde{\mathbf{v}}) + (\nabla\xi_h, \tilde{\mathbf{v}}) &= (\mathbf{u}_0, \tilde{\mathbf{v}}).\end{aligned}$$

The nice feature of this choice is the fact that the initial velocity components satisfy the stabilized mass conservation equation, which can have important effects on the stability of the fully discrete problem (see [6]).

The so-called stabilization parameter is

$$\tau = \left(\frac{C_{\mathrm{s}}\nu}{h^2} + \frac{C_{\mathrm{c}}\|\mathbf{u}_h\|_\ell}{h|\Omega|^{\frac{1}{\ell}}}\right)^{-1}. \tag{4}$$

C_{s} are C_{c} are algorithmic constants independent of physical and numerical parameters that are usually motivated from the analysis of one-dimensional tests (see e.g. [8]). For practical purposes, a non-constant $\tau(\mathbf{x})$ is usually implemented, in which the global velocity norm is replaced by its pointwise modulus. The use of a variable stabilization parameter introduce some technical complications in the numerical analysis that have been faced in [10] for the linearized Oseen problem.

In the following, we assume that $2 \leq \ell \leq \infty$. Furthermore, we use the skew-symmetric form of the convective trilinear form (see [30])

$$b(\mathbf{u}_h, \mathbf{v}_h, \mathbf{w}) = \langle(\mathbf{u}_h\cdot\nabla)\mathbf{v}_h, \mathbf{w}\rangle + \frac{1}{2}\langle\nabla\cdot\mathbf{u}_h, \mathbf{v}_h\cdot\mathbf{w}\rangle.$$

For the shake of conciseness in the following exposition, let us introduce the operator

$$\mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) : V_h \times V_h \longrightarrow \boldsymbol{L}^1(\Omega), \qquad \mathcal{N}(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{u}_h\cdot\nabla)\mathbf{v}_h + \frac{1}{2}(\nabla\cdot\mathbf{u}_h)\mathbf{v}_h.$$

The weak form of the sub-grid model is not standard. We refer to [2, 11, 12] for stability and convergence analyses for dynamic orthogonal sub-grid models applied to linear problems, namely convection-diffusion-reaction systems and the Stokes problem. The linearized stationary problem is fully analyzed in [10]. In the next sections we will analyze the stability of this nonlinear finite-dimensional problem (2), with special emphasis on its long-term behavior.

2. Long-term stability in $L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$

Our first result proves that the VMS finite element approximation of the Navier-Stokes equations (2) exhibits an absorbing set in $\boldsymbol{L}^2(\Omega)$. A key difference with respect to previous analysis is the proof of an $L^2(\Omega)$ absorbing set for the sub-grid component too. We prove the existence of the $\boldsymbol{L}^2(\Omega)$ absorbing set and some long-term stability bounds in the next theorem that holds in 2 and 3 dimensions. When there is no confusion, we will omit the time label for the unknowns.

Let us introduce the nondimensional generalized Grashof number $\mathrm{G} := \frac{|\Omega|^{\frac{2}{d}}}{\nu^2}\|\mathbf{f}\|_{L^\infty(L^2)}$ introduced in [15]; G can also be interpreted as Re^2. In the next theorems, we make use of $\rho := \nu\mathrm{G}$

Theorem 2.1 *Let us assume that the elliptic regularity assumptions hold. Then, the solution of problem (2) for $d = 2,\ 3$ satisfies*

$$\mathbf{u} \in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega)), \qquad \tilde{\mathbf{u}} \in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega)),$$
$$\nabla\mathbf{u} \in L^2_{\rm loc}(0,\infty;\boldsymbol{L}^2(\Omega)), \qquad \tau^{-\frac{1}{2}}\tilde{\mathbf{u}} \in L^2_{\rm loc}(0,\infty;\boldsymbol{L}^2(\Omega)),$$

for $\mathbf{u}_0 \in \boldsymbol{L}^2(\Omega)$ and $\mathbf{f} \in L^\infty(0,\infty;\boldsymbol{L}^2(\Omega))$. On the other hand, the following inequality holds,

$$\limsup_{t\to\infty} \left(\|\mathbf{u}_h(t)\|^2 + \|\tilde{\mathbf{u}}(t)\|^2\right) \lesssim \frac{|\Omega|^{\frac{4}{d}}}{\nu^2}\|\mathbf{f}\|^2_{L^\infty(0,\infty;\boldsymbol{L}^2)}. \tag{5}$$

which implies the existence of an absorbing set in $\boldsymbol{L}^2(\Omega)$.

In the next theorem we translate the sub-grid stability in terms of the finite element components, as it is usual for stabilized methods. The extra estimates for scheme (2) in the next theorem, that the Galerkin finite-element method does not provide, are weighted with a time-independent parameter $\tau_0 = \inf_{t\in(0,\infty)}\tau(t)$, i.e.

$$\tau_0^{-1} = \frac{C_{\rm s}\nu}{h^2} + \frac{C_{\rm c}\sup_{t\in(0,\infty)}\|\mathbf{u}_h(t)\|_\ell}{h|\Omega|^{\frac{1}{\ell}}}.$$

Observe that the paramater τ_0^{-1} is well-defined for a fixed $h > 0$ by using an inverse inequality $\|\mathbf{v}_h\|_{L^\ell(\Omega)} \lesssim h^{-(\frac{1}{2}-\frac{1}{\ell})}\|\mathbf{v}_h\|_{L^2(\Omega)}$ (for $2 \leq \ell \leq \infty$) and estimate (5). Thus, τ_0 does not degenerate to 0. Let us stress the fact that the introduction of the weighting parameter τ_0 comes from technical aspects in the subsequent analysis but the results apply to system (2) with the time-dependent expression of τ in (4).

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^d$ for $d = 2$ or 3. The algorithm (2) with $2 < \ell \leq \infty$ in (4) satisfies, for any $\bar{t} \geq t_0$,*

$$\tau_0^{\frac{1}{2}}\|\nabla p_h + \mathcal{N}(\mathbf{u}_h,\mathbf{u}_h)\|_{H_0^{-1}(t_0,\bar{t};\boldsymbol{L}^{q'}(\Omega))} \leq C$$

for $q' = \frac{2\ell}{\ell-2}$. The case $\ell = 2$ satisfies

$$\tau_0^{\frac{1}{2}}\|\nabla p_h + \mathcal{N}(\mathbf{u}_h,\mathbf{u}_h)\|_{H_0^{-1}(t_0,\bar{t};\boldsymbol{W}_0^{-1,(d+\varepsilon)'})} \leq C$$

for a fixed $\varepsilon > 0$, where $(d+\varepsilon)'$ denotes its conjugate exponent, and C is a constant that depends on $(\mathbf{u}_0,\rho,\Omega)$. In particular, for $t_0 \to \infty$, C only depends on (ρ,Ω)

3. Absorbing set in $\boldsymbol{H}^1(\Omega)$ and the global attractor for $d = 2$

In this section, we prove the existence of an absorbing set in $\boldsymbol{H}^1(\Omega)$, which is the key result for the existence of a global attractor for algorithm (2). In order to get the bounds that lead to the existence of the $\boldsymbol{H}^1(\Omega)$ absorbing set, let us introduce the scalar value

$$\tau_U^{-1} = \frac{C_s\nu}{h^2} + \frac{C_{\rm c}U}{h}$$

where $U > 0$ is a bounded characteristic velocity of the problem. In particular,

$$U = \sup_{t\in(t_0,\infty)} |\Omega|^{-\frac{1}{\ell}} \|\mathbf{u}_h\|$$

is a possible choice, since $\ell \geq 2$ and $\sup_{t\in(t_0,\infty)} \|\mathbf{u}_h\|$ has been bounded in Theorem 2.1. The long-term stability of the sub-grid velocity in the next theorem is weighted by $\tau_0^{\frac{1}{2}}$, whose introduction has been motivated by technical reasons. Again, the introduction of the weighting parameter τ_U is purely technical and the following results apply to system (2) with the time-dependent expression of τ in (4).

Theorem 3.1 ($\boldsymbol{H}^1(\Omega)$ absorbing set) *Let $\Omega \subset \mathbb{R}^2$ have the elliptic regularity assumptions. Then, the solution $(\mathbf{u}_h, p_h, \tilde{\mathbf{u}})$ of problem (2), for $2 \leq \ell < \infty$, satisfies the long-term stability bound*

$$\limsup_{t\to\infty} \left(\nu\|\nabla\mathbf{u}_h\|^2 + \tau_U^{-1}\|\tilde{\mathbf{u}}\|^2\right) \lesssim \left(a_3 + \frac{a_2}{\bar{t}}\right)\exp\left(a_1\right)$$

with

$$a_1 = \int_t^{t+\bar{t}} \left(\|\mathbf{f}\|^2 + U^4\right) \mathrm{d}s \leq \bar{t}\left(\|\mathbf{f}\|^2_{L^\infty(L^2)} + U^4\right), \tag{6a}$$

$$a_2 = \int_t^{t+\bar{t}} \left(\nu\|\nabla\mathbf{u}_h\|^2 + \tau_U^{-1}\|\tilde{\mathbf{u}}\|^2\right) \mathrm{d}s \lesssim \rho^2\left(1 + \frac{\bar{t}\nu}{|\Omega|}\right), \tag{6b}$$

$$a_3 = \int_t^{t+\bar{t}} \nu^{-2}\left(\nu^{-2}\|\mathbf{u}_h\|^2 + 1\right)\left(\nu\|\nabla\mathbf{u}_h\|^2 + \tau_\nu^{-1}\|\tilde{\mathbf{u}}\|^2\right) \mathrm{d}s \lesssim \left(\nu^{-4}\rho^2 + \nu^{-2}\right) a_2 \tag{6c}$$

for any fixed $\bar{t} > 0$. This bound proves the existence of an absorbing set in $\boldsymbol{H}^1(\Omega)$ for the finite element fluid velocity and an absorbing set in $L^2(\Omega)$ for $\tau_U^{-\frac{1}{2}}\tilde{\mathbf{u}}$.

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