A Data-Driven Shrinkage for Efficient Estimation in the Generalized Poisson Regression Model: Simulation and Application

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ABSTRACT

Count data in practice rarely conform to the equidispersion assumption that the standard Poisson regression model imposes. When counts are overdispersed or underdispersed, the generalized Poisson regression model (GPRM) steps in as a more accommodating alternative, extending the classical framework to handle such variation. Like other generalized regression models, GPRM is typically fitted via maximum likelihood estimation (MLE). The problem is that MLE becomes unreliable when predictors are highly correlated, a condition known as multicollinearity, producing unstable coefficients, inflated standard errors, and predictors that appear statistically insignificant even when they are not. This paper introduces a new two-parameter estimator for GPRM specifically designed to handle multicollinearity. We establish its theoretical properties and evaluate it against existing methods through Monte Carlo simulations. We also apply the estimator to Canadian carbon dioxide emissions data. The real-data results mirror the simulation findings: the proposed estimator yields more stable parameter estimates with smaller standard errors, making it a practical tool when correlated predictors threaten inferential quality.

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1 Introduction

Count regression models are utilized to analyze the impact of the predictor variables on the response variables, which are non-negative integers [1]. The Poisson regression model (PRM), the first

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of these models, is based on the equidispersion assumption of the count response variable, wherein the conditional mean is equivalent to the response variable's conditional variance. In reality, count data do not, in most cases, conform to this assumption. PRM results in valid standard error estimation and inference if the response variable's mean is equal to the variance. The response variable's mean is greater than the variance (subdispersion), and the variance is greater than the mean (overdispersion). These models can be implemented in cases of subdispersion and PRM. In the context of overdispersion, the PRM can be substituted with the negative binomial regression model (NBRM) or the Poisson–inverse Gaussian regression model (PIGRM) [2,3]. In the context of overdispersion or underdispersion, the Conway-Maxwell-Poisson regression model (CMPRM) or the generalized Poisson regression model (GPRM) is applied [4,5]. These models offer other mechanisms to relax equidispersion assumptions.

Over- and underdispersion count data are difficult to analyze using standard regression techniques and are common in bioinformatics, electronic engineering, computer science, insurance risk, and medical clinical research. One such model that can handle both over- and underdispersion in count data is the GPRM, which is a generalization of the standard PRM with two parameters. GPORM has been applied across a range of contexts in recent literature. Ortega et al. [6] examined its properties in relation to residual analysis and machine learning. Endawkie et al. [7] applied Bayesian GPORM to model deaths linked to household air pollution in East Africa, while Ref. [8] investigated the asymptotic normality of the MLE using the Fisher scoring algorithm. In applied work, Setia et al. [9] used GPORM to study maternal mortality rates, Al Haris and Arum [10] compared GPORM with NBRM for traffic accident data in Central Java, and Ref. [11] applied the model to stunting cases in East Nusa Tenggara province.

In regression modeling, multicollinearity occurs when two or more predictor variables are approximately linearly correlated with one another. This breaks the assumption that the predictor variables are covariates. In a multicollinearity context, the design matrix will approach a singularity for $\mathbf{X}^T\mathbf{X}$. This causes variance to inflate about the MLE and results in coefficient estimates that are not stable, meaning that small changes in the data produce large changes in the estimates of the parameters. Additionally, this creates standard errors large enough that the truly important covariates are not statistically significant. This leads to bad estimates and practically leads to a loss in interpretability of the fitted model, which also creates fictive inferences [12–14]. A variety of techniques are available for detecting and quantifying multicollinearity. The most popular technique is called the Variance Inflation Factor (VIF), equaling $\mathrm{VIF}_j = 1/(1 - R_j^2)$, where R_j^2 is the coefficient of determination. It is common to consider a VIF of more than 10 to indicate multicollinearity disruption. Condition number κ, which is defined as the square root of the ratio of the largest to the smallest eigenvalue of $\mathbf{X}^\top\mathbf{X}$, is a supportive measure. $\kappa > 30$ is typically indicative of severe multicollinearity [15–17].

To address the challenges posed by multicollinearity, researchers have turned to biased estimators that introduce a controlled amount of bias to stabilize coefficient estimates and reduce inflated variances. The ridge regression estimator [18] was the first of such biased estimators, followed by the Liu estimator [19]. These estimators have been further extended to generalized regression models to improve estimation performance under multicollinearity, such as the ridge regression estimator, which is applied in a GPRM by Alghamdi et al. [20] and the Liu regression estimator, which is developed for the GPRM by Hammad et al. [21]. Later, Özkale and Kaçiranlar [22] suggest that the two-parameter estimator effectively balances the strengths of the ridge and Liu estimators. Since that time, many researchers of two-parameter estimators in GLMs have surfaced. Some of these works are Akram et al. [23], Alrweili [24], Kandemir Çetinkaya and Kaçıranlar [25], Awwad et al. [26], and Sami et al. [27]. Although many GLM applications have surfaced, the usage of two-parameter estimators for the case of GPORM is still very limited.

This paper develops a two-parameter estimate for the GPRM, which provides some control over the harmful effects of multicollinearity. This new estimator has the positives of the ridge and Liu estimators because of the addition of two shrinkage parameters. This provides a more flexible framework and an overall better approach for estimating coefficients. This two-parameter structure better balances the bias and variance framework and thus more efficiently stabilizes the estimate of the two shrinkage parameters as compared to existing single-parameter approaches. Also, analyze the attributes and compare the new estimators via Monte Carlo simulation and a practical data example, and examine the finite-sample behavior of the proposed two-parameter estimator in various multicollinearity situations.

The rest of the paper is organized as follows. Section 2 discusses the GPRM and describes the existing estimators that are dealt with in this study. Section 3 proposes the two-parameter estimator, outlines the properties of the estimator, offers a framework of comparison, and defines the criteria for estimating the shrinkage parameters. Section 4 summarizes a series of Monte Carlo simulations of the proposed estimator and discusses the findings from the study in detail. Section 5 describes a case study in detail and describes the practical use of the two-parameter estimator. Finally, Section 6 sums up the findings of the paper and describes the opportunities for future studies.

2 Methodology

Consul and Jain [28] introduced the generalized Poisson distribution to overcome the limitation of overdispersion or underdispersion in count data, a two-parameter family whose probability mass function is defined as follows:

$$P(Y = y_i; \nu, \tau) = \nu(\nu + \tau y)^{y-1}\frac{\exp(-\nu - \tau y)}{y!}, \tag{1}$$

where $y = 0, 1, 2, \ldots$, $\nu > 0$ and $\max(-1, -\nu/4) \leq \tau < 1$. When $\tau < 0$, the distribution assigns zero probability to values of y beyond a certain threshold m, where m satisfies $\nu + \tau m > 0$. The case $\tau = 0$ reduces the model to the standard Poisson distribution. Using the reparameterization $\tau = \vartheta\nu$, the distribution can be rewritten in terms of ν and ϑ, which is more convenient for regression modeling.

The GPRM assumes that each response y_i, given its covariate vector $\boldsymbol{x}_i$, follows a generalized Poisson distribution with conditional mean $\mu_i = \mathrm{E}(y_i \mid \boldsymbol{x}_i) = c_i f(\boldsymbol{x}_i, \boldsymbol{\beta})$, where c_i is an exposure offset and $f(\boldsymbol{x}_i, \boldsymbol{\beta})$ is a differentiable function of $\boldsymbol{\beta}$. Using the relation $\nu = \mu_i/(1+\vartheta\mu_i)$ and substituting into the original density gives:

$$P(Y = y_i; \mu_i, \vartheta) = \frac{(1+\vartheta y_i)^{y_i-1}}{y_i!}\left(\frac{\mu_i}{1+\vartheta\mu_i}\right)^{y_i}\exp\left(-\frac{\mu_i(1+\vartheta y_i)}{1+\vartheta\mu_i}\right), \tag{2}$$

where $y_i = 0, 1, 2, \ldots$, $\mu_i > 0$ and ϑ is the dispersion parameter. The model reduces to the ordinary Poisson when $\vartheta = 0$, while positive and negative values of ϑ indicate overdispersion and underdispersion, respectively. The mean is $\mathrm{E}(y_i \mid \boldsymbol{x}_i) = \mu_i$ and the variance is $\mathrm{V}(y_i \mid \boldsymbol{x}_i) = \mu_i(1+\vartheta\mu_i)^2$.

Following [29,30], the mean is linked to the covariates through the log link a $\mu_i = \exp(\boldsymbol{x}_i^\top\boldsymbol{\beta})$, where $\boldsymbol{x}_i^\top = (1, x_{i1}, \ldots, x_{i(p-1)})$ and $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_{p-1})^\top$. Model parameters are estimated by maximizing the log-likelihood function [31]:

$$\ell(\vartheta, \boldsymbol{\beta}) = \sum_{i=1}^{n}\left\{y_i\big[\log(\mu_i) - \log(1+\vartheta\mu_i)\big] + (y_i - 1)\log(1+\vartheta y_i) - \log(y_i!) - \frac{\mu_i(1+\vartheta y_i)}{1+\vartheta\mu_i}\right\}. \tag{3}$$

Differentiating with respect to ϑ and β_j yields the score functions:

$$\frac{\partial \ell}{\partial \vartheta} = \sum_{i=1}^{n} \left(\frac{y_i(y_i-1)}{1+\vartheta y_i} - \frac{y_i \mu_i}{1+\vartheta \mu_i} - \frac{\mu_i(y_i-\mu_i)}{(1+\vartheta \mu_i)^2} \right) \quad \text{and} \quad \frac{\partial \ell}{\partial \beta_j} = \sum_{i=1}^{n} \frac{y_i - \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta})}{\left(1+\vartheta \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta})\right)^2}. \tag{4}$$

Since no closed-form solution exists, the Newton–Raphson algorithm is used to obtain the estimates iteratively:

$$\boldsymbol{\beta}^{(s)} = \boldsymbol{\beta}^{(s-1)} + \left(\boldsymbol{V}^{(s-1)}\right)^{-1} \boldsymbol{Z}^{(s-1)}, \tag{5}$$

where $\boldsymbol{Z}^{(s-1)} = \boldsymbol{X}^\top \boldsymbol{W}^{(s-1)} \boldsymbol{u}^{(s-1)}$ is the score vector and $\boldsymbol{V}^{(s-1)}$ is the Hessian. Replacing the Hessian with its expected value leads to the Fisher scoring algorithm with the inverse information matrix $\left(\boldsymbol{I}^{(s-1)}\right)^{-1} = \vartheta \left(\boldsymbol{X}^\top \widehat{\boldsymbol{V}}^{(s-1)} \boldsymbol{X}\right)^{-1}$, where $\widehat{\boldsymbol{V}}$ is diagonal with entries $v_i = \mu_i/(1+\vartheta\mu_i)^2$. At convergence, the MLE is:

$$\hat{\boldsymbol{\beta}}^{ML} = \boldsymbol{A}^{-1} \boldsymbol{X}^\top \widehat{\boldsymbol{V}} \widehat{\boldsymbol{u}}, \tag{6}$$

where $\boldsymbol{A} = \boldsymbol{X}^\top \widehat{\boldsymbol{V}} \boldsymbol{X}$ and $u_i = \vartheta_i + (y_i - \mu_i)/(1+\vartheta\mu_i)$. Using decomposition $\boldsymbol{A} = \boldsymbol{\Gamma}\boldsymbol{\Psi}\boldsymbol{\Gamma}^\top$ holds, with $\boldsymbol{\Psi}$ containing the eigenvalues $\psi_1, \ldots, \psi_p$ and $\boldsymbol{\Gamma}$ the corresponding eigenvectors. The transformed coefficients are defined as $\boldsymbol{\alpha} = \boldsymbol{\Gamma}^\top \hat{\boldsymbol{\beta}}^{ML}$. Then, the asymptotic covariance matrix, mean squared error matrix (MSEM), and scalar mean squared error (MSE) of the MLE are described as follows:

$$\operatorname{Cov}\left(\hat{\boldsymbol{\beta}}^{ML}\right) = \vartheta \boldsymbol{A}^{-1}, \tag{7}$$

$$\operatorname{MSEM}\left(\hat{\boldsymbol{\beta}}^{ML}\right) = \vartheta \boldsymbol{\Gamma}\boldsymbol{\Psi}\boldsymbol{\Gamma}^\top, \tag{8}$$

$$\operatorname{MSE}\left(\hat{\boldsymbol{\beta}}^{ML}\right) = \vartheta \sum_{j=1}^{p} \frac{1}{\psi_j}. \tag{9}$$

When multicollinearity is present, $\boldsymbol{A}$ becomes ill-conditioned, causing the MLE to produce unstable estimates with inflated variances. To address multicollinearity in the GPRM, Alghamdi et al. [20] extended the ridge regression estimator of [18] to this setting. The resulting Generalized Poisson Ridge Regression Estimator (GPRRE) is defined as:

$$\hat{\boldsymbol{\beta}}^{k} = \boldsymbol{A}_k^{-1} \boldsymbol{A} \hat{\boldsymbol{\beta}}^{ML}, \quad k > 0, \tag{10}$$

where $\boldsymbol{A}_k = \boldsymbol{A} + k\boldsymbol{I}$ and $k > 0$ is the ridge parameter. As $k \to 0$, the GPRRE reduces to the MLE, making it a generalization of the standard estimator. Defining $\boldsymbol{\Psi}_k = \boldsymbol{\Psi} + k\boldsymbol{I} = \operatorname{diag}(\psi_1 + k, \ldots, \psi_p + k)$, the bias, covariance, MSEM, and MSE of the GPRRE are given as follows:

$$\operatorname{Bias}\left(\hat{\boldsymbol{\beta}}^{k}\right) = -k\boldsymbol{\Gamma}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\alpha}, \tag{11}$$

$$\operatorname{Cov}\left(\hat{\boldsymbol{\beta}}^{k}\right) = \vartheta \boldsymbol{\Gamma}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Gamma}^\top, \tag{12}$$

$$\operatorname{MSEM}\left(\hat{\boldsymbol{\beta}}^{k}\right) = \vartheta \boldsymbol{\Gamma}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Gamma}^\top + k^2 \boldsymbol{\Gamma}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\alpha}\boldsymbol{\alpha}^\top\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Gamma}^\top, \tag{13}$$

$$\operatorname{MSE}\left(\hat{\boldsymbol{\beta}}^{k}\right) = \vartheta \sum_{j=1}^{p} \frac{\psi_j}{(\psi_j + k)^2} + k^2 \sum_{j=1}^{p} \frac{\alpha_j^2}{(\psi_j + k)^2}. \tag{14}$$

Multicollinearity affects regression models and has pushed researchers to develop biased estimators as alternatives to the MLE. The GPRM, like any generalized linear model, also suffers from this issue. Although Alghamdi et al. [20] proposed the GPRRE to handle multicollinearity, the ridge estimator depends on its nonlinearity shrinkage parameter, which makes choosing the parameter difficult. The Liu estimator is simpler because it combines the ridge and Stein estimators and depends linearly on its shrinkage parameter. Taking advantage of this benefit, Hammad et al. [21] applied the Liu estimator to the GPRM and introduced the generalized Poisson Liu regression estimator (GPLRE) as follows:

$$\hat{\boldsymbol{\beta}}^{\mathrm{d}} = \boldsymbol{A}_I^{-1}\boldsymbol{A}_d\hat{\boldsymbol{\beta}}^{ML}, \qquad 0 < d < 1, \tag{15}$$

where $\boldsymbol{A}_d = (\boldsymbol{A} + d\boldsymbol{I})$, $\boldsymbol{A}_I = (\boldsymbol{A} + \boldsymbol{I})$, and d is the Liu shrinkage parameter. When $d \to 1$, the GPLRE reduces to the MLE, making the MLE a special case of this estimator. Defining $\boldsymbol{\Psi}_I = (\boldsymbol{\Psi} + \boldsymbol{I}) = \mathrm{diag}(\psi_1 + 1, \ldots, \psi_p + 1)$ and $\boldsymbol{\Psi}_d = (\boldsymbol{\Psi} + d\boldsymbol{I}) = \mathrm{diag}(\psi_1 + d, \ldots, \psi_p + d)$, the bias, covariance, MSEM, and MSE of the GPLRE are given as follows:

$$\mathrm{Bias}\left(\hat{\boldsymbol{\beta}}^{d}\right) = (d-1)\boldsymbol{\Gamma}\boldsymbol{\Psi}_I^{-1}\boldsymbol{\alpha}, \tag{16}$$

$$\mathrm{Cov}\left(\hat{\boldsymbol{\beta}}^{d}\right) = \vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Gamma}^{\top}, \tag{17}$$

$$\mathrm{MSEM}\left(\hat{\boldsymbol{\beta}}^{d}\right) = \vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Gamma}^{\top} + (d-1)^2\boldsymbol{\Gamma}\boldsymbol{\Psi}_I^{-1}\boldsymbol{\alpha}\boldsymbol{\alpha}^{\top}\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Gamma}^{\top}, \tag{18}$$

$$\mathrm{MSE}\left(\hat{\boldsymbol{\beta}}^{d}\right) = \vartheta\sum_{j=1}^{p}\frac{(\psi_j + d)^2}{(\psi_j + 1)^2\psi_j} + (d-1)^2\sum_{j=1}^{p}\frac{\hat{\alpha}_j^2}{(\psi_j + 1)^2}. \tag{19}$$

3 Proposed Method

This section proposes a biased estimator for the GPRM to address issues created by multicollinearity. It is suggested that the generalized Poisson two-parameter regression estimator (GPTPRE) is able to use two different shrinkage parameters rather than relying on a single one, which is able to provide better control of the bias/variance trade-off. In this section, we provide a theoretical derivation of the properties of the GPTPRE and also critique the performance of the GPOTPRE based on the MLE, GPRRE, and GPLRE. Finally, we will suggest a set of practices in relation to the choice of the two biasing parameters.

3.1 Generalized Poisson Two-Parameter Regression Estimator

A fundamental limitation of the ridge estimator is that reducing variance requires increasing the parameter k, which inevitably introduces a larger bias. As seen in Eq. (14), these two components move in opposite directions as k grows, making it impossible to minimize both simultaneously with a single parameter. To resolve this conflict, Özkale and Kaçiranlar [22] proposed combining the ridge and Liu estimators into a unified two-parameter estimator. The second parameter d serves to compensate for the bias growth caused by k; this is achieved by augmenting $kd\hat{\beta} = k\beta + \varepsilon$ to the linear regression model (for further details, see [22]). This added flexibility allows the estimator to achieve a better bias–variance trade-off than either single-parameter estimator can offer on its own. Motivated by this advantage, the present paper extends this two-parameter framework to the GPRM, yielding a new estimator termed the GPTPRE.

First, augment the data with pseudo-observations:

$$\boldsymbol{X}^* = \begin{bmatrix}\boldsymbol{X}\\ \sqrt{k}\,\boldsymbol{I}_p\end{bmatrix}, \qquad \widehat{\boldsymbol{u}}^* = \begin{bmatrix}\widehat{\boldsymbol{u}}\\ \sqrt{k}\,d\,\hat{\boldsymbol{\beta}}^{ML}\end{bmatrix}, \qquad \boldsymbol{V}^* = \begin{bmatrix}\widehat{\boldsymbol{V}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I}_p\end{bmatrix},$$

where $k > 0$ and $0 < d < 1$ are the shrinkage parameters. Computing the augmented weighted least-squares solution gives

$$\left(\boldsymbol{X}^{*\top}\boldsymbol{V}^{*}\boldsymbol{X}^{*}\right)\tilde{\boldsymbol{\beta}} = \boldsymbol{X}^{*\top}\boldsymbol{V}^{*}\widehat{\boldsymbol{u}}^{*}.$$

Substituting the augmented matrices yields

$$\boldsymbol{X}^{*\top}\boldsymbol{V}^{*}\boldsymbol{X}^{*} = \boldsymbol{X}^{\top}\widehat{\boldsymbol{V}}\boldsymbol{X} + k\boldsymbol{I}_{p} \quad \text{and} \quad \boldsymbol{X}^{*\top}\boldsymbol{V}^{*}\widehat{\boldsymbol{u}}^{*} = \boldsymbol{X}^{\top}\widehat{\boldsymbol{V}}\widehat{\boldsymbol{u}} + kd\,\hat{\boldsymbol{\beta}}^{ML}.$$

From the definition of $\hat{\boldsymbol{\beta}}^{ML}$, $\boldsymbol{X}^{\top}\widehat{\boldsymbol{V}}\widehat{\boldsymbol{u}} = \boldsymbol{A}\,\hat{\boldsymbol{\beta}}^{ML}$. Thus, the augmented estimator is

$$\tilde{\boldsymbol{\beta}} = \boldsymbol{A}_{k}^{-1}\left(\boldsymbol{A}\hat{\boldsymbol{\beta}}^{ML} + kd\hat{\boldsymbol{\beta}}^{ML}\right) = \boldsymbol{A}_{k}^{-1}\left(\boldsymbol{A} + kd\boldsymbol{I}_{p}\right)\hat{\boldsymbol{\beta}}^{ML}.$$

Denoting $\boldsymbol{A}_{kd} = \boldsymbol{A} + kd\boldsymbol{I}_{p}$, we finally obtain the proposed estimator as follows:

$$\hat{\boldsymbol{\beta}}^{k,d} = \boldsymbol{A}_{k}^{-1}\boldsymbol{A}_{kd}\hat{\boldsymbol{\beta}}^{ML}, \quad k > 0, 0 < d < 1, \tag{20}$$

where $\boldsymbol{A}_{kd} = (\boldsymbol{A} + kd\boldsymbol{I})$, and k and d are the two biasing parameters. The GPTPRE is a general estimation framework that groups many familiar estimation procedures in the following terms:

- When $d = 0$, the GPTPRE approximates the GPRRE, $\hat{\boldsymbol{\beta}}^{k,d} = \hat{\boldsymbol{\beta}}^{k}$.
- When $k = 1$ and $0 < d < 1$, the GPTPRE approximates the GPLRE, $\hat{\boldsymbol{\beta}}^{k,d} = \hat{\boldsymbol{\beta}}^{d}$.
- When $k = d = 0$, the GPTPRE approximates the MLE, $\hat{\boldsymbol{\beta}}^{k,d} = \hat{\boldsymbol{\beta}}^{ML}$.

This nesting structure shows that the GPTPRE is a generalized approach to the MLE, GPRRE, and GPLRE that contains them all within a single systematic framework. The two parameters k and d, therefore, tend to offer much more freedom in terms of the bias-variance trade-off, and the GPTPRE is more versatile regarding the level of multicollinearity than any of the special cases alone. The bias vector and variance-covariance matrix of the GPTPRE are characterized as follows:

$$\operatorname{Bias}\left(\hat{\boldsymbol{\beta}}^{k,d}\right) = \mathrm{E}\left(\hat{\boldsymbol{\beta}}^{k,d}\right) - \boldsymbol{\beta} = -k(d-1)\boldsymbol{\Gamma}\boldsymbol{\Psi}_{k}^{-1}\boldsymbol{\alpha}, \tag{21}$$

$$\operatorname{Cov}\left(\hat{\boldsymbol{\beta}}^{k,d}\right) = \mathrm{E}\left[\left(\hat{\boldsymbol{\beta}}^{k,d} - \mathrm{E}\left(\hat{\boldsymbol{\beta}}^{k,d}\right)\right)\left(\hat{\boldsymbol{\beta}}^{k,d} - \mathrm{E}\left(\hat{\boldsymbol{\beta}}^{k,d}\right)\right)^{\top}\right] = \vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}_{k}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_{k}^{-1}\boldsymbol{\Gamma}^{\top}. \tag{22}$$

Using Eqs. (21) and (22), we can determine the MSEM and MSE for the GPOTPRE as follows:

$$\begin{aligned}\operatorname{MSEM}\left(\hat{\boldsymbol{\beta}}^{k,d}\right) &= \mathrm{E}\left[(\hat{\boldsymbol{\beta}}^{k,d} - \boldsymbol{\beta})(\hat{\boldsymbol{\beta}}^{k,d} - \boldsymbol{\beta})^{\top}\right] = \operatorname{Cov}\left(\hat{\boldsymbol{\beta}}^{k,d}\right) + \operatorname{Bias}\left(\hat{\boldsymbol{\beta}}^{k,d}\right)\operatorname{Bias}\left(\hat{\boldsymbol{\beta}}^{k,d}\right)^{\top}\\ &= \vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}_{k}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_{k}^{-1}\boldsymbol{\Gamma}^{\top} + k(d-1)^{2}\boldsymbol{\Gamma}\boldsymbol{\Psi}_{k}^{-1}\boldsymbol{\alpha}\boldsymbol{\alpha}^{\top}\boldsymbol{\Psi}_{k}^{-1}\boldsymbol{\Gamma}^{\top},\end{aligned} \tag{23}$$

$$\operatorname{MSE}\left(\hat{\boldsymbol{\beta}}^{k,d}\right) = \operatorname{Tr}\left[\operatorname{MSEM}\left(\hat{\boldsymbol{\beta}}^{k,d}\right)\right] = \vartheta\sum_{j=1}^{p}\frac{(\psi_{j} + kd)^{2}}{\psi_{j}(\psi_{j} + k)^{2}} + (k(d-1))^{2}\sum_{j=1}^{p}\frac{\alpha_{j}^{2}}{(\psi_{j} + k)^{2}}. \tag{24}$$

3.2 Theoretical Comparisons among Estimators

In this subsection, we use the following lemmas to compare the estimators within the context of the GPRM, where we compare our proposed GPTPRE with the existing estimators MLE, GPRRE, and GPLRE.

Lemma 1: *Let $\boldsymbol{S}$ be a positive definite matrix and $\boldsymbol{c}$ be a conformable vector. Then the matrix difference $\boldsymbol{S}-\boldsymbol{c}\boldsymbol{c}^\top$ is positive semidefinite if and only if $\boldsymbol{c}^\top\boldsymbol{S}^{-1}\boldsymbol{c}\le 1$ [32, 33].*

Lemma 2: *Let $\hat{\boldsymbol{\beta}}_1$ and $\hat{\boldsymbol{\beta}}_2$ be two competing estimators with MSEMs defined as $\mathrm{MSEM}(\hat{\boldsymbol{\beta}}_j)=\mathrm{Cov}(\hat{\boldsymbol{\beta}}_j)+\boldsymbol{R}_j\boldsymbol{R}_j^\top$, where $\boldsymbol{R}_j$ denotes the bias vector of the j-th estimator. Then $\hat{\boldsymbol{\beta}}_2$ is superior to $\hat{\boldsymbol{\beta}}_1$ in the MSEM sense, that is: $\mathrm{MSEM}(\hat{\boldsymbol{\beta}}_1)-\mathrm{MSEM}(\hat{\boldsymbol{\beta}}_2)=\sigma^2\boldsymbol{D}+\boldsymbol{R}_1\boldsymbol{R}_1^\top-\boldsymbol{R}_2\boldsymbol{R}_2^\top>0$, if and only if the following condition holds: $\boldsymbol{R}_2^\top\left[\sigma^2\boldsymbol{D}+\boldsymbol{R}_1\boldsymbol{R}_1^\top\right]^{-1}\boldsymbol{R}_2<1$ [34].*

Theorem 1: *For the GPRM, denote the bias vector of the GPTPRE by $\boldsymbol{R}_{k,d}$. The GPTPRE outperforms the MLE if and only if the MSEM of the MLE exceeds that of the GPTPRE; that is, $\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{ML})-\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{k,d})>0$, provided that $\boldsymbol{R}_{k,d}{}^\top\boldsymbol{\Gamma}\left[\vartheta\boldsymbol{\Psi}^{-1}-\vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}\right]\boldsymbol{\Gamma}^\top\boldsymbol{R}_{k,d}<1$, for all $k>0$ and $0<d<1$.*

Proof: The MSEM difference between the MLE and the GPTPRE is given by

$$\begin{aligned}\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{ML})-\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{k,d})&=\vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}^{-1}\boldsymbol{\Gamma}^\top-\vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Gamma}^\top-\boldsymbol{R}_{k,d}\boldsymbol{R}_{k,d}{}^\top\\&=\boldsymbol{\Gamma}\left[\vartheta\boldsymbol{\Psi}^{-1}-\vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}\right]\boldsymbol{\Gamma}^\top-\boldsymbol{R}_{k,d}\boldsymbol{R}_{k,d}{}^\top,\end{aligned}\tag{25}$$

The difference between the MSEM in Eq. (25) can be described by MSE as follows:

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}^{ML})-\mathrm{MSE}(\hat{\boldsymbol{\beta}}^{k,d})=\boldsymbol{\Gamma}\,\mathrm{diag}\left\{\frac{\vartheta}{\psi_j}-\frac{\vartheta(\psi_j-d)^2}{\psi_j(\psi_j+k)^2}\right\}_{j=1}^{p}\boldsymbol{\Gamma}^\top-\boldsymbol{R}_{k,d}{}^\top\boldsymbol{R}_{k,d}.\tag{26}$$

The term $\boldsymbol{R}_{k,d}\boldsymbol{R}_{k,d}{}^\top$ in (25) is non-negative definite. Therefore, the difference $\vartheta\boldsymbol{\Psi}^{-1}-\vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}$ becomes positive definite if and only if $(\psi_j+k)^2>(\psi_j+kd)^2$ satisfied for every j. Hence, for any $k>0$ and $0<d<1$, Lemma 1 directly establishes the theorem. □

Theorem 2: *For the GPRM, denote the bias vector of the GPTPRE by $\boldsymbol{R}_{k,d}$ and the bias vector of the GPRRE by $\boldsymbol{R}_k$. The GPTPRE outperforms the GPRRE if and only if the MSEM of the GPRRE exceeds that of the GPTPRE; that is, $\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{k})-\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{k,d})>0$, provided that $\boldsymbol{R}_{k,d}^\top\boldsymbol{\Gamma}\left[\vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}\boldsymbol{\Psi}_k^{-1}-\vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}+\boldsymbol{R}_k\boldsymbol{R}_k^\top\right]\boldsymbol{\Gamma}^\top\boldsymbol{R}_{k,d}<1$, for all $k>0$ and $0<d<1$.*

Proof: The MSEM difference between the GPRRE and the GPTPRE is given by

$$\begin{aligned}\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{k})-\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{k,d})&=\vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Gamma}^\top+\boldsymbol{R}_k\boldsymbol{R}_k^\top-\vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Gamma}^\top-\boldsymbol{R}_{k,d}\boldsymbol{R}_{k,d}^\top\\&=\boldsymbol{\Gamma}\left[\vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}\boldsymbol{\Psi}_k^{-1}-\vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}+\boldsymbol{R}_k\boldsymbol{R}_k^\top\right]\boldsymbol{\Gamma}^\top-\boldsymbol{R}_{k,d}\boldsymbol{R}_{k,d}^\top,\end{aligned}\tag{27}$$

The difference between the MSEM in Eq. (27) can be described by MSE as follows:

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}^{k})-\mathrm{MSE}(\hat{\boldsymbol{\beta}}^{k,d})=\boldsymbol{\Gamma}\,\mathrm{diag}\left\{\frac{\vartheta\psi_j}{(\psi_j+k)^2}-\frac{\vartheta(\psi_j+kd)^2}{\psi_j(\psi_j+k)^2}\right\}_{j=1}^{p}\boldsymbol{\Gamma}^\top+\boldsymbol{R}_k\boldsymbol{R}_k^\top-\boldsymbol{R}_{k,d}^\top\boldsymbol{R}_{k,d}.\tag{28}$$

The term $\boldsymbol{R}_{k,d}\boldsymbol{R}_{k,d}^\top$ and $\boldsymbol{R}_k\boldsymbol{R}_k^\top$ in (27) is non-negative definite. Therefore, the difference $\vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}\boldsymbol{\Psi}_k^{-1}-\vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}$ becomes positive definite if and only if $\psi_j^2(\psi_j+k)^2>(\psi_j+k)^2(\psi_j+kd)^2$ satisfied for every j. Hence, for any $k>0$ and $0<d<1$, Lemma 1 directly establishes the theorem. □

Theorem 3: *For the GPRM, denote the bias vector of the GPTPRE by* $\boldsymbol{R}_{k,d}$ *and the bias vector of the GPLRE by* $\boldsymbol{R}_d$. *The GPTPRE outperforms the GPLRE if and only if the MSEM of the GPLRE exceeds that of the GPTPRE; that is,* $\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{d}) - \mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{k,d}) > 0$, *provided that* $\boldsymbol{R}_{k,d}^{\top}\boldsymbol{\Gamma}\left[\vartheta\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}_I^{-1} - \vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1} + \boldsymbol{R}_d\boldsymbol{R}_d^{\top}\right]\boldsymbol{\Gamma}^{\top}\boldsymbol{R}_{k,d} < 1$, *for all* $k > 0$ *and* $0 < d < 1$.

Proof: The MSEM difference between the GPLRE and the GPTPRE is given by

$$\begin{aligned}\mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{d}) - \mathrm{MSEM}(\hat{\boldsymbol{\beta}}^{k,d}) &= \vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Gamma}^{\top} + \boldsymbol{R}_d\boldsymbol{R}_d^{\top} - \vartheta\boldsymbol{\Gamma}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Gamma}^{\top} - \boldsymbol{R}_{k,d}\boldsymbol{R}_{k,d}^{\top}\\ &= \boldsymbol{\Gamma}\left[\vartheta\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}_I^{-1} - \vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1} + \boldsymbol{R}_d\boldsymbol{R}_d^{\top}\right]\boldsymbol{\Gamma}^{\top} - \boldsymbol{R}_{k,d}\boldsymbol{R}_{k,d}^{\top},\end{aligned} \tag{29}$$

The difference between the MSEM in Eq. (29) can be described by MSE as follows:

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}^{d}) - \mathrm{MSE}(\hat{\boldsymbol{\beta}}^{k,d}) = \boldsymbol{\Gamma}\,\mathrm{diag}\left\{\frac{\vartheta(\psi_j+d)^2}{\psi_j(\psi_j+1)^2} - \frac{\vartheta(\psi_j+kd)^2}{\psi_j(\psi_j+k)^2}\right\}_{j=1}^{p}\boldsymbol{\Gamma}^{\top} + \boldsymbol{R}_d\boldsymbol{R}_d^{\top} - \boldsymbol{R}_{k,d}^{\top}\boldsymbol{R}_{k,d}. \tag{30}$$

The term $\boldsymbol{R}_{k,d}\boldsymbol{R}_{k,d}^{\top}$ and $\boldsymbol{R}_k\boldsymbol{R}_k^{\top}$ in (29) is non-negative definite. Therefore, the difference $\vartheta\boldsymbol{\Psi}_I^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_d\boldsymbol{\Psi}_I^{-1} - \vartheta\boldsymbol{\Psi}_k^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}_{kd}\boldsymbol{\Psi}_k^{-1}$ becomes positive definite if and only if $(\psi_j+d)^2(\psi_j+k)^2 > (\psi_j+1)^2(\psi_j+kd)^2$ satisfied for every j. Hence, for any $k > 0$ and $0 < d < 1$, Lemma 1 directly establishes the theorem. ☐

3.3 Selection of the Biasing Parameters

Choosing suitable values for the biasing parameters k and d is crucial for the practical efficacy of the GPTPRE. As the true coefficients $\boldsymbol{\beta}$ remain unknown, the most appropriate values for k and d cannot be obtained analytically. A common solution in literature is to assume those parameters as the MLE and estimate the unknown parameters as their observed counterparts $\hat{\boldsymbol{\beta}}^{ML}$ and $\hat{\vartheta}$. In this paper, we estimate biasing parameters, k and d, by minimizing the scalar MSE of the GPTPRE for each parameter. The biasing parameters we obtain are then used in the proposed estimator. The performance of these parameters is evaluated in the Monte Carlo simulation study in Section 4.

The initial value of d is computed as $\hat{d} = \min\left(\hat{\alpha}_j^2\Big/\left(\hat{\alpha}_j^2 + \frac{1}{\lambda_j}\right)\right)$. Then, calculated the shrinkage parameters k using this initial value of d as follows:

$$\hat{k}_1 = (p)\mathrm{mean}\left(\frac{\lambda_j}{\hat{\alpha}_j^2\lambda_j(1-d)-d}\right), \quad \text{and} \quad \hat{k}_2 = \max\left(\sqrt{\frac{\lambda_j}{\max(\hat{\alpha}_j^2\lambda_j)(1-d)-d}}\right). \tag{31}$$

For each estimated value of k, the shrinkage parameter d_i is calculated as:

$$\hat{d}_i = \sum_{j=1}^{p}\left(\frac{k_i\hat{\alpha}_j^2\lambda_j - \min(\lambda_j)}{k_i + k_i\hat{\alpha}_j^2\lambda_j}\right), \qquad i = 1,2. \tag{32}$$

To ensure that d_i remains within a valid range, it is adjusted according to the condition:

$$\hat{d}_i = \begin{cases}\hat{d}_i, & \text{if } 0 < d_i < 1,\\ \hat{d}, & \text{otherwise.}\end{cases} \tag{33}$$

To compare with GPRRE, following Alghamdi et al. [20], we use two estimators that are described as $\hat{k}_1 = \min\left(\hat{\eta}/\hat{\alpha}_j^2\right)$ and $\hat{k}_2 = \sum_{j=1}^{p+1}((p+1)q_j)$ where $q_j = \hat{\eta}/(2\lambda_j\hat{\alpha}_j^2 + \hat{\eta})$.

Concerning the shrinkage parameter d in GPLRE, we utilize the two estimators introduced by Hammad et al. [21], which are $\hat{d}_1 = \max\left(0, \min\left(\frac{\hat{\alpha}_j - \hat{\eta}}{\frac{\hat{\eta}}{\lambda_j} + \hat{\alpha}_j^2}\right)\right)$ and $\hat{d}_2 = \max\left(0, \min\left(\frac{\frac{\hat{\alpha}_j^2 + \hat{\eta}}{(\hat{\alpha}_j^2 + \lambda_j)^2}}{\frac{\hat{\alpha}_j^2 \lambda_j + \hat{\eta}}{\lambda_j(\hat{\alpha}_j^2 + \hat{\eta})^2}}\right)\right)$.

4 Simulation Study

This section discusses a Monte Carlo simulation study aimed at assessing the performance of the proposed estimator. It details the simulation design, the empirical results, and the estimator's relative efficiency in comparison to other competing estimators.

Simulation Design

A Monte Carlo simulation study was conducted under the GPRM framework to assess the GPTPRE method against MLE, GPRRE, and GPLRE in terms of finite-sample performance. In the framework, the response variable y_i was sampled from the generalized Poisson distribution [35] with the conditional mean $\mu_i = \exp(\boldsymbol{x}_i^\top \boldsymbol{\beta})$. Here, $\boldsymbol{x}_i$ is the covariate vector of the i-th observation.

To create multicollinearity, the predictor variables were formed as in [36,37]:

$$x_{ij} = (1-\rho^2)^{\frac{1}{2}}\, o_{ij} + \rho\, o_{i,p+1}, \quad i = 1, \ldots, n; \quad j = 1, \ldots, p, \tag{34}$$

with $e_{ij} \sim N(0,1)$ independently. Multicollinearity was controlled by the parameter ρ, with the coefficients designed as $\sum_{j=1}^{p} \beta_j^2 = 1$ with a uniform contribution from each coefficient. The simulation was done in R (version 4.5.2) with E = 1000 replications for each configuration. The empirical MSE and scalar mean absolute error (MAE) were calculated for each replication as in [38]:

$$\mathrm{MSE}(\hat{\boldsymbol{\beta}}^*) = \frac{1}{E}\sum_{l=1}^{E}\left(\hat{\boldsymbol{\beta}}_l - \boldsymbol{\beta}\right)^\top \left(\hat{\boldsymbol{\beta}}_l - \boldsymbol{\beta}\right), \tag{35}$$

$$\mathrm{MAE}(\hat{\boldsymbol{\beta}}^*) = \frac{1}{E}\sum_{l=1}^{E}\sum_{j=1}^{p} |\hat{\boldsymbol{\beta}}_{lj} - \boldsymbol{\beta}_j|, \tag{36}$$

where $\hat{\boldsymbol{\beta}}_l$ is the l-th replication coefficient vector for the estimator being considered. The most efficient estimator for the configuration was the one with the smallest MSE over the replications. Table 1 shows all the simulation factors and their levels.

Table 1: Summary of simulation design factors and levels

Factor	Symbol	Levels
Sample size	n	30, 75, 150, 200, 300, 400
Number of predictors	p	3, 6, 9
Correlation level	ρ	0.80, 0.85, 0.90, 0.95, 0.99
Dispersion parameter	ϑ	0.01, 0.25, 0.60
Number of replications	E	1000

Before analyzing the results, a brief description of Tables 2–10 and Fig. 1 is warranted. Tables 1–9 include results for set values for the number of predictors p and the dispersion parameter set to ϑ. These values are organized in the tables with the number of predictors set to p, and the multicollinearity parameter ρ is the number of predictors. The rows are configured in columns to report the MLE, the two versions of the approaches GPRRE (one with $\hat{k}_1$ and the other with $\hat{k}_2$), the two versions of the GPLRE (one with $\hat{d}_1$ and the other with $\hat{d}_2$), and the two versions of the proposed GPTPRE (one with $(\hat{k}_1, \hat{d}_1)$ and the other with $(\hat{k}_2, \hat{d}_2)$). Four main trends are evident in the nine tables, and these are presented in the next section.

Table 2: Simulated MAE and MSE values for different estimators in the GPRM when $p = 3$ with $\vartheta = 0.01$

ρ	n	MSE							MAE						
		MLE	GPRRE		GPLRE		GPTPRE		MLE	GPRRE		GPLRE		GPTPRE	
		–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$	–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$
0.80	30	0.15839	0.15272	0.15017	0.15301	0.15268	0.12838	0.14756	0.61956	0.60926	0.60372	0.60976	0.60918	0.55764	0.59905
	75	0.05711	0.05622	0.05548	0.05623	0.05622	0.05223	0.05492	0.37589	0.37292	0.37036	0.37295	0.37291	0.35874	0.36805
	150	0.04240	0.04229	0.04219	0.04229	0.04229	0.04124	0.04227	0.32984	0.32943	0.32909	0.32944	0.32943	0.32515	0.32935
	200	0.02952	0.02946	0.02942	0.02946	0.02946	0.02894	0.02945	0.27344	0.27320	0.27299	0.27320	0.27320	0.27036	0.27309
	300	0.01811	0.01809	0.01807	0.01809	0.01809	0.01788	0.01809	0.21047	0.21036	0.21027	0.21036	0.21036	0.20908	0.21034
	400	0.01211	0.01210	0.01209	0.01210	0.01210	0.01200	0.01210	0.17647	0.17641	0.17635	0.17641	0.17641	0.17560	0.17639
0.85	30	0.19723	0.18814	0.18380	0.18865	0.18807	0.14945	0.17904	0.70303	0.68716	0.67820	0.68804	0.68702	0.60725	0.66801
	75	0.05271	0.05247	0.05229	0.05248	0.05247	0.05021	0.05241	0.36330	0.36250	0.36187	0.36251	0.36249	0.35355	0.36209
	150	0.04088	0.04076	0.04066	0.04076	0.04076	0.03940	0.04070	0.31943	0.31901	0.31865	0.31901	0.31901	0.31351	0.31880
	200	0.03871	0.03865	0.03860	0.03865	0.03865	0.03803	0.03863	0.29936	0.29915	0.29898	0.29916	0.29915	0.29630	0.29908
	300	0.02149	0.02147	0.02145	0.02147	0.02147	0.02116	0.02147	0.22911	0.22898	0.22886	0.22898	0.22898	0.22709	0.22894
	400	0.01585	0.01583	0.01582	0.01583	0.01583	0.01567	0.01583	0.19746	0.19735	0.19726	0.19735	0.19735	0.19625	0.19733
0.90	30	0.27734	0.25630	0.24911	0.25833	0.25626	0.18162	0.23612	0.81846	0.78810	0.77479	0.79116	0.78821	0.65677	0.75356
	75	0.09517	0.09421	0.09356	0.09425	0.09421	0.08620	0.09387	0.48740	0.48514	0.48352	0.48523	0.48513	0.46400	0.48416
	150	0.06108	0.06083	0.06063	0.06083	0.06083	0.05844	0.06078	0.38859	0.38779	0.38715	0.38780	0.38779	0.37938	0.38749
	200	0.05075	0.05056	0.05041	0.05056	0.05056	0.04886	0.05055	0.34484	0.34425	0.34377	0.34426	0.34425	0.33804	0.34414
	300	0.02588	0.02584	0.02581	0.02584	0.02584	0.02546	0.02583	0.25798	0.25780	0.25764	0.25780	0.25780	0.25574	0.25776
	400	0.01800	0.01798	0.01796	0.01798	0.01798	0.01773	0.01797	0.20655	0.20644	0.20635	0.20645	0.20644	0.20497	0.20643
0.95	30	0.32630	0.31611	0.31349	0.31755	0.31598	0.26675	0.31576	0.84174	0.82978	0.82577	0.83167	0.82960	0.75201	0.82811
	75	0.14862	0.14738	0.14679	0.14746	0.14737	0.13825	0.14754	0.59006	0.58784	0.58657	0.58797	0.58782	0.56434	0.58747
	150	0.11382	0.11272	0.11203	0.11276	0.11271	0.10538	0.11281	0.50987	0.50756	0.50601	0.50766	0.50756	0.48772	0.50734
	200	0.10304	0.10233	0.10183	0.10235	0.10233	0.09671	0.10221	0.51132	0.50966	0.50843	0.50972	0.50966	0.49527	0.50933
	300	0.04344	0.04328	0.04316	0.04329	0.04328	0.04201	0.04330	0.31785	0.31732	0.31687	0.31733	0.31731	0.31135	0.31725
	400	0.03937	0.03926	0.03918	0.03927	0.03926	0.03828	0.03927	0.29783	0.29744	0.29713	0.29745	0.29744	0.29297	0.29736
0.99	30	2.51974	2.18353	2.13131	2.22655	2.16659	1.51154	2.10324	2.37339	2.18082	2.15555	2.21822	2.16940	1.68204	2.11063
	75	0.58513	0.55455	0.55025	0.56031	0.55390	0.40855	0.54978	1.12055	1.09052	1.08380	1.09680	1.08990	0.89603	1.08088
	150	0.44911	0.43151	0.42800	0.43417	0.43114	0.35424	0.42855	0.99816	0.97950	0.97402	0.98256	0.97905	0.86088	0.97401
	200	0.38525	0.36356	0.35841	0.36598	0.36339	0.26739	0.35354	0.89870	0.87338	0.86451	0.87643	0.87317	0.72908	0.85689
	300	0.23334	0.22930	0.22828	0.22964	0.22928	0.21492	0.23132	0.69370	0.68850	0.68645	0.68895	0.68846	0.65626	0.69009
	400	0.13443	0.13295	0.13214	0.13304	0.13294	0.12450	0.13345	0.54029	0.53747	0.53569	0.53765	0.53746	0.51352	0.53770

Table 3: Simulated MAE and MSE values for different estimators in the GPRM when $p = 3$ with $\vartheta = 0.25$

		MSE							MAE						
ρ	n	MLE	GPRRE		GPLRE		GPTPRE		MLE	GPRRE		GPLRE		GPTPRE	
		–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$	–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$
0.80	30	0.46053	0.31883	0.30160	0.34626	0.34535	0.23409	0.23718	1.02318	0.85420	0.82311	0.89361	0.89247	0.72910	0.73365
	75	0.18442	0.13607	0.12218	0.15858	0.15856	0.12272	0.10078	0.67622	0.58114	0.54924	0.62908	0.62905	0.55071	0.49904
	150	0.15122	0.12208	0.11477	0.13810	0.13805	0.11470	0.10681	0.60375	0.54772	0.53255	0.57934	0.57922	0.53034	0.51744
	200	0.11435	0.09538	0.08743	0.10468	0.10466	0.08824	0.07721	0.53423	0.48978	0.46921	0.51275	0.51270	0.47186	0.44429
	300	0.06183	0.05546	0.05245	0.05876	0.05875	0.05311	0.04961	0.37970	0.36190	0.35247	0.37090	0.37088	0.35341	0.34262
	400	0.05735	0.05091	0.04781	0.05508	0.05508	0.05064	0.04488	0.35924	0.33785	0.32749	0.35256	0.35255	0.33835	0.31580
0.85	30	0.98293	0.51338	0.47566	0.57083	0.56966	0.39778	0.35278	1.51141	1.06672	1.02539	1.15877	1.15729	0.93356	0.87499
	75	0.28724	0.21682	0.20577	0.24288	0.24251	0.18782	0.18680	0.81569	0.70608	0.68319	0.75188	0.75140	0.65665	0.65006
	150	0.17572	0.14987	0.14527	0.15655	0.15637	0.12385	0.13711	0.65181	0.61115	0.60167	0.61967	0.61936	0.55454	0.58723
	200	0.16350	0.13927	0.13304	0.14800	0.14790	0.11978	0.12074	0.60312	0.55688	0.54403	0.57587	0.57571	0.52053	0.52090
	300	0.08385	0.07220	0.06767	0.07881	0.07880	0.06798	0.06353	0.44414	0.41393	0.40149	0.43106	0.43102	0.40082	0.38839
	400	0.07528	0.06636	0.06235	0.07074	0.07074	0.06381	0.05919	0.41848	0.39340	0.38153	0.40650	0.40650	0.38525	0.37090
0.90	30	0.93742	0.52338	0.49869	0.57782	0.57287	0.39201	0.39108	1.49683	1.08569	1.06069	1.16967	1.16394	0.90693	0.91312
	75	0.40053	0.26453	0.24832	0.29977	0.29910	0.20662	0.19952	0.91069	0.74497	0.71684	0.79491	0.79382	0.64130	0.63163
	150	0.29997	0.20992	0.19513	0.24293	0.24253	0.16618	0.15004	0.85040	0.71145	0.67975	0.77141	0.77083	0.63632	0.60221
	200	0.19031	0.14727	0.13536	0.16450	0.16439	0.13181	0.11718	0.65303	0.57209	0.54377	0.61036	0.61019	0.54123	0.50159
	300	0.11093	0.08617	0.07756	0.09874	0.09873	0.07767	0.06744	0.50849	0.44908	0.42635	0.48181	0.48178	0.42945	0.40031
	400	0.08978	0.08055	0.07948	0.08327	0.08325	0.07143	0.07944	0.44326	0.42078	0.41452	0.42786	0.42783	0.39599	0.41189
0.95	30	1.39434	0.68568	0.61503	0.73949	0.72640	0.45941	0.49403	1.75569	1.16272	1.11183	1.24383	1.22666	0.91296	0.96249
	75	0.69756	0.35523	0.33207	0.42833	0.42243	0.24879	0.24121	1.26140	0.86158	0.82935	0.99036	0.98316	0.71784	0.71149
	150	0.37120	0.24514	0.22668	0.27554	0.27456	0.18067	0.17592	0.91539	0.73927	0.70294	0.79132	0.78987	0.62488	0.61076
	200	0.41531	0.23425	0.21640	0.30635	0.30560	0.21250	0.17398	0.96205	0.72767	0.69324	0.83489	0.83374	0.67412	0.61403
	300	0.27496	0.17123	0.15825	0.22178	0.22162	0.16534	0.13223	0.76195	0.59903	0.56922	0.68955	0.68926	0.58407	0.51310
	400	0.17297	0.11311	0.09903	0.14661	0.14658	0.10519	0.07679	0.63523	0.52082	0.48843	0.58883	0.58876	0.50485	0.43700
0.99	30	10.29727	5.32634	4.36780	5.36866	5.12326	4.35925	4.00430	4.47136	2.82694	2.51654	2.84944	2.75923	2.41725	2.25060
	75	3.15282	1.40036	1.22424	1.45542	1.40813	1.08569	1.05365	2.57919	1.51317	1.42112	1.59936	1.56902	1.30382	1.24845
	150	2.29237	1.16845	1.03466	1.22105	1.19520	0.82008	0.81423	2.19237	1.40828	1.32950	1.50336	1.48103	1.12120	1.10060
	200	2.06271	0.81493	0.69730	0.89270	0.88734	0.62708	0.42346	2.08101	1.18045	1.10247	1.30363	1.29980	1.01925	0.80377
	300	0.97230	0.46448	0.43442	0.54880	0.54507	0.33742	0.31343	1.38051	0.87839	0.84036	1.01608	1.01110	0.73882	0.69181
	400	0.84326	0.39721	0.37043	0.49354	0.49256	0.31237	0.23957	1.28257	0.83394	0.79473	0.97848	0.97736	0.73482	0.63666

Table 4: Simulated MAE and MSE values for different estimators in the GPRM when $p = 3$ with $\vartheta = 0.6$

		MSE							MAE						
ρ	n	MLE	GPRRE		GPLRE		GPTPRE		MLE	GPRRE		GPLRE		GPTPRE	
		–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$	–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$
0.80	30	1.28523	0.56982	0.49232	0.66483	0.66264	0.47518	0.45441	1.72158	1.13137	1.08250	1.25503	1.25297	1.04324	1.11148
	75	0.48727	0.18794	0.18986	0.32642	0.32642	0.21199	0.21048	1.08032	0.68888	0.69698	0.90373	0.90373	0.72181	0.77842
	150	0.28669	0.20109	0.20199	0.22999	0.22989	0.17363	0.20788	0.83429	0.71396	0.71533	0.75364	0.75347	0.64908	0.73647
	200	0.25123	0.14779	0.14132	0.20281	0.20281	0.13468	0.14292	0.77510	0.60487	0.59125	0.70370	0.70369	0.57455	0.61708
	300	0.14159	0.09909	0.09887	0.12481	0.12481	0.09290	0.11004	0.57845	0.49757	0.49497	0.54561	0.54560	0.47128	0.52664
	400	0.12292	0.08998	0.09118	0.11207	0.11207	0.08947	0.11918	0.53865	0.47612	0.47882	0.51644	0.51644	0.46702	0.55203
0.85	30	4.08508	1.45169	1.04593	1.54094	1.52900	0.99917	0.85019	2.64834	1.47085	1.34464	1.60735	1.59797	1.34127	1.29655
	75	0.66090	0.34931	0.33069	0.43844	0.43767	0.32047	0.34396	1.22712	0.90483	0.88794	1.00886	1.00796	0.83993	0.92340
	150	0.42584	0.27376	0.26681	0.32170	0.32100	0.24283	0.25092	0.97458	0.80275	0.79543	0.86064	0.85988	0.75181	0.78562
	200	0.31286	0.21515	0.20876	0.23922	0.23882	0.17709	0.18647	0.83216	0.69651	0.68668	0.73874	0.73823	0.62862	0.65878
	300	0.19873	0.14622	0.14689	0.17106	0.17100	0.13178	0.16747	0.67168	0.59027	0.59473	0.62800	0.62795	0.54847	0.64982
	400	0.17834	0.10233	0.09972	0.15421	0.15420	0.10717	0.10558	0.65242	0.51369	0.50897	0.61056	0.61055	0.51137	0.52451
0.90	30	2.79195	1.27089	1.04774	1.31347	1.30807	1.16750	1.09644	2.33764	1.47485	1.36142	1.52733	1.52312	1.38582	1.42347
	75	0.96513	0.43061	0.38727	0.55282	0.55106	0.38210	0.36509	1.41977	0.91360	0.88247	1.07629	1.07429	0.85406	0.91304
	150	0.56921	0.21713	0.20244	0.34250	0.34240	0.22002	0.21354	1.09100	0.68889	0.68470	0.87854	0.87839	0.67380	0.76719
	200	0.42166	0.21958	0.22002	0.29694	0.29678	0.17023	0.21274	0.98565	0.71925	0.71999	0.83596	0.83571	0.62190	0.73453
	300	0.30828	0.12411	0.12514	0.23271	0.23271	0.14427	0.14675	0.83424	0.54366	0.54477	0.73355	0.73355	0.57159	0.62038
	400	0.23307	0.14939	0.13924	0.19217	0.19217	0.14447	0.12074	0.73502	0.59064	0.56577	0.67288	0.67288	0.57597	0.52393
0.95	30	4.58559	1.84576	1.43812	1.88252	1.79607	1.37403	1.24508	3.02796	1.74470	1.57370	1.77870	1.72786	1.49948	1.49719
	75	2.26596	0.97032	0.83861	1.05887	1.04662	0.80246	0.69954	2.08555	1.20535	1.14774	1.32715	1.31653	1.09095	1.09547
	150	0.94105	0.36023	0.34208	0.48064	0.47819	0.31118	0.28361	1.45299	0.87781	0.84801	1.04916	1.04580	0.78016	0.77620
	200	0.89216	0.29874	0.28785	0.46127	0.46078	0.29238	0.22757	1.43011	0.78827	0.77634	1.04118	1.04088	0.76650	0.69395
	300	0.59068	0.22703	0.22856	0.37415	0.37414	0.23664	0.20981	1.12457	0.68431	0.68924	0.90933	0.90930	0.67927	0.68047
	400	0.43897	0.16828	0.16652	0.30795	0.30790	0.18506	0.13964	0.96537	0.59582	0.59293	0.81884	0.81877	0.62061	0.57998
0.99	30	22.23685	7.52851	5.22570	7.49389	6.93652	6.56864	4.65921	6.68774	3.36205	2.80027	3.39400	3.18867	3.11890	2.53666
	75	8.17051	2.67251	2.12089	2.72224	2.53485	2.85975	2.27679	4.06588	1.93983	1.71538	1.99474	1.92656	2.00424	1.73580
	150	4.99828	1.85817	1.46932	1.89702	1.83051	1.74659	1.32920	3.14402	1.63282	1.46834	1.71115	1.68120	1.56625	1.38672
	200	3.87747	0.89143	0.64200	0.99830	0.96467	1.06874	0.53838	2.78207	1.10380	0.98383	1.28343	1.27110	1.22883	0.95819
	300	2.68694	1.19489	1.02189	1.26243	1.25483	1.11606	0.95897	2.26363	1.21771	1.14973	1.36984	1.36509	1.12264	1.06232
	400	2.01113	0.64556	0.52344	0.80169	0.79514	0.72862	0.44971	2.00054	0.96233	0.89116	1.22507	1.22233	0.99607	0.79034

Table 5: Simulated MAE and MSE values for different estimators in the GPRM when $p = 6$ with $\vartheta = 0.01$

		MSE							MAE						
ρ	n	MLE	GPRRE		GPLRE		GPTPRE		MLE	GPRRE		GPLRE		GPTPRE	
		–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$	–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$
0.80	30	0.32266	0.31838	0.31730	0.31862	0.31833	0.23408	0.31157	1.17008	1.16374	1.16179	1.16412	1.16366	1.00744	1.15371
	75	0.10414	0.10387	0.10368	0.10388	0.10387	0.09308	0.10348	0.65479	0.65401	0.65343	0.65402	0.65400	0.61988	0.65286
	150	0.05567	0.05560	0.05554	0.05560	0.05560	0.05291	0.05551	0.49869	0.49836	0.49808	0.49836	0.49836	0.48555	0.49788
	200	0.03762	0.03759	0.03756	0.03759	0.03759	0.03665	0.03756	0.41172	0.41156	0.41142	0.41156	0.41156	0.40609	0.41137
	300	0.03410	0.03408	0.03406	0.03408	0.03408	0.03326	0.03405	0.39008	0.38997	0.38988	0.38997	0.38997	0.38503	0.38984
	400	0.02244	0.02243	0.02242	0.02243	0.02243	0.02200	0.02242	0.31580	0.31573	0.31567	0.31573	0.31573	0.31257	0.31567
0.85	30	0.56050	0.54220	0.53861	0.54437	0.54147	0.31495	0.51821	1.48517	1.46145	1.45529	1.46453	1.46047	1.09936	1.42765
	75	0.09110	0.09085	0.09067	0.09086	0.09085	0.08458	0.09047	0.64215	0.64126	0.64057	0.64127	0.64126	0.61641	0.63975
	150	0.08035	0.08024	0.08015	0.08024	0.08024	0.07530	0.08013	0.58962	0.58923	0.58891	0.58923	0.58922	0.57047	0.58882
	200	0.06666	0.06656	0.06648	0.06656	0.06656	0.06300	0.06640	0.53603	0.53565	0.53534	0.53566	0.53565	0.52078	0.53508
	300	0.04069	0.04060	0.04053	0.04060	0.04060	0.03921	0.04048	0.41709	0.41668	0.41631	0.41668	0.41668	0.40915	0.41602
	400	0.02316	0.02314	0.02312	0.02314	0.02314	0.02261	0.02311	0.32514	0.32500	0.32487	0.32501	0.32501	0.32103	0.32478
0.90	30	0.35381	0.35023	0.34921	0.35049	0.35015	0.25620	0.34420	1.24561	1.23987	1.23776	1.24027	1.23974	1.05812	1.23009
	75	0.15929	0.15827	0.15771	0.15832	0.15826	0.13142	0.15662	0.82579	0.82345	0.82205	0.82355	0.82342	0.74888	0.81971
	150	0.13216	0.13164	0.13129	0.13165	0.13164	0.12096	0.13095	0.75269	0.75121	0.75015	0.75124	0.75121	0.71800	0.74894
	200	0.08926	0.08914	0.08906	0.08914	0.08914	0.08357	0.08901	0.62418	0.62382	0.62355	0.62382	0.62382	0.60396	0.62339
	300	0.05611	0.05600	0.05591	0.05600	0.05600	0.05360	0.05582	0.48997	0.48952	0.48913	0.48952	0.48951	0.47889	0.48873
	400	0.03577	0.03569	0.03561	0.03570	0.03570	0.03446	0.03553	0.39130	0.39087	0.39048	0.39092	0.39091	0.38377	0.39001
0.95	30	0.83619	0.81843	0.81780	0.82114	0.81759	0.51524	0.80399	1.89939	1.87964	1.87782	1.88286	1.87861	1.44913	1.86043
	75	0.43423	0.42951	0.42862	0.42989	0.42941	0.31476	0.42354	1.33015	1.32360	1.32184	1.32415	1.32345	1.12394	1.31485
	150	0.16827	0.16776	0.16745	0.16778	0.16775	0.14990	0.16742	0.84112	0.83985	0.83904	0.83989	0.83984	0.78847	0.83885
	200	0.12311	0.12282	0.12261	0.12282	0.12282	0.10895	0.12244	0.72201	0.72114	0.72049	0.72116	0.72113	0.67854	0.71995
	300	0.08045	0.08016	0.07995	0.08018	0.08017	0.07337	0.07966	0.58894	0.58794	0.58715	0.58799	0.58798	0.56255	0.58609
	400	0.06966	0.06951	0.06938	0.06951	0.06951	0.06477	0.06929	0.53801	0.53742	0.53693	0.53744	0.53744	0.51864	0.53651
0.99	30	2.33072	2.18136	2.19171	2.20013	2.16277	1.00822	2.01811	3.11608	3.01480	3.02213	3.03076	3.00010	1.89603	2.88189
	75	1.40098	1.37531	1.37614	1.37875	1.37364	0.92361	1.35751	2.37395	2.35050	2.35049	2.35419	2.34892	1.85947	2.33087
	150	1.02784	1.00510	1.00485	1.00811	1.00426	0.59424	0.97830	2.05200	2.02910	2.02828	2.03237	2.02822	1.51385	2.00113
	200	0.62404	0.61562	0.61475	0.61646	0.61543	0.41471	0.60355	1.60375	1.59304	1.59129	1.59416	1.59281	1.27933	1.57691
	300	0.42633	0.41200	0.40775	0.41356	0.41285	0.29000	0.39242	1.31790	1.29422	1.28542	1.29745	1.29637	1.07268	1.25705
	400	0.30737	0.30445	0.30337	0.30461	0.30442	0.23501	0.30127	1.10221	1.09733	1.09518	1.09759	1.09728	0.96220	1.09173

Table 6: Simulated MAE and MSE values for different estimators in the GPRM when $p = 6$ with $\vartheta = 0.25$

		MSE							MAE						
ρ	n	MLE	GPRRE		GPLRE		GPTPRE		MLE	GPRRE		GPLRE		GPTPRE	
		–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$	–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$
0.80	30	2.20176	1.33793	1.33343	1.47896	1.47298	0.58655	0.78127	3.00949	2.24056	2.26538	2.44030	2.43404	1.48355	1.66820
	75	0.61946	0.42703	0.42127	0.49179	0.49087	0.22374	0.28319	1.65964	1.38582	1.37084	1.48872	1.48739	1.00781	1.14136
	150	0.50791	0.33898	0.32692	0.42649	0.42629	0.22024	0.23803	1.46937	1.18486	1.15629	1.35044	1.35012	0.97609	1.00717
	200	0.34826	0.27773	0.26469	0.30839	0.30822	0.19128	0.21068	1.23729	1.10299	1.07642	1.16801	1.16773	0.92573	0.96846
	300	0.23146	0.19321	0.18297	0.21401	0.21396	0.15121	0.15203	0.98843	0.89991	0.87266	0.95142	0.95133	0.80200	0.79050
	400	0.14812	0.12626	0.11733	0.14078	0.14078	0.11267	0.09800	0.79689	0.73260	0.70309	0.77768	0.77768	0.69927	0.64568
0.85	30	2.72998	1.62947	1.62337	1.70828	1.68107	0.72916	0.98185	3.31395	2.44493	2.45894	2.57651	2.55744	1.60947	1.83105
	75	0.89046	0.55682	0.56469	0.65255	0.65178	0.27374	0.32145	1.94401	1.52113	1.52178	1.67548	1.67446	1.07674	1.17925
	150	0.54605	0.36607	0.35897	0.44850	0.44795	0.22168	0.25237	1.51423	1.23939	1.22330	1.37902	1.37809	0.97638	1.04567
	200	0.42342	0.32136	0.30956	0.36485	0.36448	0.20975	0.24625	1.36447	1.18350	1.16090	1.26940	1.26880	0.96140	1.03814
	300	0.29298	0.25424	0.24236	0.26644	0.26634	0.17821	0.19690	1.11671	1.03746	1.00891	1.06700	1.06680	0.87665	0.91132
	400	0.21153	0.17726	0.16610	0.19654	0.19652	0.14313	0.13618	0.95449	0.87033	0.83729	0.92159	0.92154	0.79107	0.75812
0.90	30	3.26771	1.78294	1.73898	1.95177	1.91511	0.94008	1.05407	3.52108	2.41068	2.43514	2.67646	2.65175	1.69791	1.74990
	75	1.19252	0.74111	0.75160	0.85867	0.85279	0.32277	0.42526	2.23297	1.70863	1.71317	1.89196	1.88518	1.12704	1.26683
	150	0.79913	0.53940	0.54011	0.61837	0.61706	0.25961	0.33770	1.87180	1.52263	1.51258	1.65344	1.65161	1.07035	1.22810
	200	0.65983	0.44224	0.42913	0.52800	0.52744	0.25373	0.28265	1.64289	1.30029	1.27027	1.47300	1.47214	1.01307	1.03864
	300	0.37213	0.26993	0.25509	0.32428	0.32408	0.18410	0.18611	1.25705	1.04510	1.01206	1.17602	1.17569	0.89458	0.87852
	400	0.28200	0.23418	0.22126	0.25460	0.25453	0.16770	0.17639	1.06993	0.96839	0.93642	1.01788	1.01775	0.82788	0.83968
0.95	30	10.46077	4.95744	4.37445	5.01590	4.83062	2.03434	2.34372	5.88320	3.81590	3.73900	3.98354	3.89291	2.54980	2.57500
	75	2.73392	1.42655	1.45238	1.63548	1.61052	0.69380	0.82879	3.30463	2.24653	2.27872	2.52880	2.50814	1.48122	1.61663
	150	1.64490	0.77025	0.81469	1.05008	1.04354	0.40296	0.42343	2.57502	1.63250	1.67802	2.04744	2.04116	1.19996	1.16419
	200	1.15852	0.60032	0.61815	0.78614	0.78448	0.31245	0.33220	2.12831	1.48467	1.49249	1.76209	1.76018	1.07870	1.11261
	300	0.83959	0.48105	0.48181	0.63502	0.63403	0.26185	0.26075	1.81975	1.33403	1.31857	1.58699	1.58583	1.02194	0.96657
	400	0.54057	0.34283	0.33072	0.44331	0.44288	0.21701	0.21363	1.48251	1.15965	1.12420	1.34802	1.34738	0.95138	0.91119
0.99	30	32.85335	18.97613	17.24432	18.60052	17.08825	8.26936	10.64229	10.91261	7.61755	7.32946	7.58801	7.10360	4.87401	5.43487
	75	11.47258	5.72426	5.53832	5.74280	5.26777	2.16309	2.77531	6.66701	4.22181	4.21323	4.33003	4.11368	2.55449	2.73967
	150	7.67509	3.61700	3.62639	3.73693	3.55372	1.63476	1.80822	5.49584	3.43267	3.49349	3.63093	3.54178	2.25073	2.26678
	200	5.56585	2.61049	2.67901	2.75209	2.66065	1.09721	1.15029	4.72322	2.96581	3.03701	3.19690	3.14557	1.88534	1.87109
	300	3.57700	2.25285	2.32605	2.28765	2.24567	0.93155	1.28558	3.64945	2.73053	2.79765	2.79735	2.77036	1.61286	1.91282
	400	2.56116	0.83453	0.94009	1.26284	1.25199	0.42888	0.32420	3.21869	1.67302	1.78195	2.25263	2.24389	1.22816	0.99960

Table 7: Simulated MAE and MSE values for different estimators in the GPRM when $p = 6$ with $\vartheta = 0.6$

		MSE							MAE						
ρ	n	MLE	GPRRE		GPLRE		GPTPRE		MLE	GPRRE		GPLRE		GPTPRE	
		–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$	–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$
0.80	30	12.79985	5.10854	3.96413	5.26533	4.96207	1.90845	2.15569	5.81262	3.64503	3.40323	3.81331	3.7255	2.42047	2.48208
	75	2.10764	1.17071	1.14847	1.28833	1.27831	0.54734	0.64658	2.94692	2.14073	2.13844	2.29698	2.29029	1.44197	1.59342
	150	1.25042	0.72404	0.74091	0.86645	0.86604	0.38888	0.44659	2.20635	1.65728	1.67557	1.84168	1.84118	1.20589	1.31636
	200	0.84223	0.47098	0.47914	0.60108	0.60072	0.23507	0.26444	1.89038	1.39319	1.39703	1.6086	1.60814	0.99109	1.07063
	300	0.48196	0.30074	0.30462	0.39612	0.39608	0.19455	0.23774	1.40546	1.11636	1.12165	1.28405	1.28399	0.911	1.03922
	400	0.35493	0.2049	0.19707	0.30971	0.30971	0.17708	0.16708	1.24143	0.93357	0.91505	1.16453	1.16453	0.88988	0.88874
0.85	30	18.7516	8.48646	6.02062	8.3763	8.32762	2.75327	3.24319	6.55716	4.28115	3.91105	4.38697	4.35907	2.79211	2.91357
	75	2.69529	1.41991	1.4108	1.55332	1.54019	0.65818	0.74508	3.28963	2.30837	2.32852	2.47117	2.46095	1.5366	1.64545
	150	1.2901	0.62943	0.65664	0.82407	0.82245	0.31616	0.37225	2.31721	1.57871	1.61297	1.85502	1.85282	1.11256	1.24934
	200	1.07773	0.61368	0.64617	0.74358	0.74263	0.28626	0.33332	2.10789	1.56285	1.59639	1.76144	1.76061	1.08081	1.15712
	300	0.68426	0.46859	0.47811	0.53665	0.53648	0.24392	0.28472	1.72223	1.41942	1.42353	1.53326	1.53303	1.03128	1.11493
	400	0.45579	0.31318	0.30049	0.38152	0.38147	0.2011	0.1984	1.39121	1.14474	1.10993	1.2774	1.27732	0.933	0.91969
0.90	30	18.34144	8.62224	6.42433	8.51493	8.36152	3.1847	3.48231	7.45891	4.84758	4.44024	4.94215	4.85837	3.19758	3.20786
	75	5.79933	2.64092	2.21749	2.75805	2.72835	0.94848	1.14117	4.18476	2.77278	2.70811	2.98022	2.95927	1.76351	1.90972
	150	1.95712	0.9804	1.0288	1.16728	1.16137	0.471	0.53862	2.88261	1.97927	2.03413	2.21894	2.2142	1.3406	1.45947
	200	1.51439	0.6866	0.72299	0.94085	0.93899	0.34618	0.4125	2.50649	1.62682	1.66302	1.98961	1.98796	1.17794	1.32106
	300	0.91968	0.50685	0.5218	0.65762	0.65724	0.25271	0.2741	1.94139	1.41987	1.43254	1.65392	1.65348	1.02315	1.06423
	400	0.61824	0.39695	0.39874	0.47994	0.47981	0.21698	0.24349	1.56458	1.24715	1.23942	1.3886	1.38841	0.92863	0.98902
0.95	30	52.76507	20.77579	14.9065	19.72043	18.76241	7.46303	8.07673	11.98992	7.15231	6.41411	7.05628	6.69905	4.75713	4.59488
	75	6.31593	3.01761	2.89355	3.15366	3.01645	1.34472	1.58502	5.04071	3.11171	3.10701	3.3285	3.25439	2.08523	2.18609
	150	5.6668	2.67997	2.57673	2.8473	2.76997	1.23662	1.31131	4.76066	2.99873	2.96883	3.25925	3.2136	2.06156	2.10238
	200	3.0076	1.24131	1.2912	1.50206	1.48963	0.60756	0.60536	3.44378	2.05813	2.11256	2.40839	2.40055	1.43926	1.44678
	300	1.823	0.796	0.88203	1.01423	1.01028	0.34725	0.37087	2.69027	1.71608	1.81002	2.01343	2.01032	1.12831	1.18493
	400	1.47	0.80532	0.85416	0.96334	0.96233	0.35719	0.42836	2.438	1.74861	1.80005	1.97331	1.97234	1.1545	1.26196
0.99	30	139.26762	68.28907	54.19958	66.88903	59.09277	26.02929	29.89937	21.08029	13.88972	12.68552	13.688	12.41662	8.95999	9.04796
	75	35.96965	14.72925	12.96426	14.34536	12.59783	6.60884	6.67136	12.00021	6.79055	6.4063	6.73553	6.21365	4.67956	4.24289
	150	20.69076	8.4711	7.94495	8.50273	7.61343	4.54996	4.02322	8.94527	5.1346	4.99246	5.25927	4.96634	3.76019	3.26925
	200	13.32748	4.49027	4.5357	4.6703	4.20384	2.23756	1.93821	7.26364	3.69288	3.73413	3.95432	3.76084	2.67641	2.3224
	300	8.13705	4.3249	4.30193	4.30702	4.10642	1.90377	2.26197	5.59118	3.81311	3.83748	3.82846	3.72657	2.28426	2.51309
	400	6.54874	2.37388	2.38196	2.67197	2.58860	1.23003	1.05251	5.12639	2.64574	2.70269	3.06260	3.02030	1.93603	1.62650

Table 8: Simulated MAE and MSE values for different estimators in the GPRM when $p = 9$ with $\vartheta = 0.01$

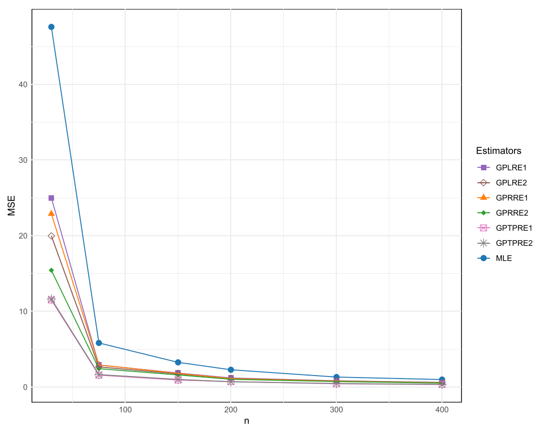
ρ	n	MSE							MAE						
		MLE	GPRRE		GPLRE		GPTPRE		MLE	GPRRE		GPLRE		GPTPRE	
		–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1,\hat{d}_1$	$\hat{k}_2,\hat{d}_2$	–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1,\hat{d}_1$	$\hat{k}_2,\hat{d}_2$
0.80	30	0.47112	0.46266	0.46127	0.46334	0.46246	0.42635	0.44953	1.68912	1.67594	1.67306	1.67700	1.67557	1.61276	1.65422
	75	0.22918	0.22430	0.22236	0.22446	0.22430	0.21768	0.21472	1.18537	1.17428	1.16888	1.17470	1.17433	1.15652	1.15055
	150	0.11828	0.11782	0.11748	0.11784	0.11783	0.11624	0.11670	0.86798	0.86614	0.86471	0.86624	0.86622	0.86053	0.86156
	200	0.07134	0.07125	0.07118	0.07126	0.07125	0.07064	0.07110	0.67220	0.67180	0.67146	0.67180	0.67180	0.66891	0.67104
	300	0.03695	0.03687	0.03680	0.03687	0.03687	0.03674	0.03667	0.48234	0.48182	0.48134	0.48184	0.48184	0.48092	0.48054
	400	0.03052	0.03047	0.03042	0.03048	0.03047	0.03037	0.03035	0.43883	0.43843	0.43807	0.43846	0.43845	0.43770	0.43750
0.85	30	28.16163	5.22351	4.09001	24.85113	3.27526	5.91025	3.08561	4.51887	3.06312	2.83790	4.01763	2.71391	3.08980	2.47656
	75	0.27195	0.26818	0.26652	0.26834	0.26818	0.25698	0.26098	1.29880	1.28998	1.28557	1.29035	1.28999	1.26328	1.27228
	150	0.12705	0.12575	0.12483	0.12577	0.12574	0.12363	0.12288	0.89945	0.89487	0.89156	0.89494	0.89485	0.88741	0.88455
	200	0.08180	0.08129	0.08089	0.08139	0.08138	0.08071	0.07998	0.70906	0.70673	0.70488	0.70726	0.70725	0.70446	0.70070
	300	0.04659	0.04654	0.04649	0.04654	0.04654	0.04632	0.04644	0.54168	0.54139	0.54113	0.54140	0.54140	0.54014	0.54080
	400	0.03566	0.03560	0.03555	0.03560	0.03560	0.03547	0.03545	0.46847	0.46807	0.46770	0.46807	0.46807	0.46721	0.46708
0.90	30	3.21948	2.24823	1.96932	2.23651	2.20852	2.12012	1.37762	3.43336	3.11731	3.07170	3.13255	3.09946	2.86152	2.66300
	75	0.26709	0.26578	0.26516	0.26581	0.26577	0.25874	0.26334	1.29487	1.29161	1.28993	1.29169	1.29158	1.27400	1.28523
	150	0.15198	0.15158	0.15131	0.15158	0.15158	0.14925	0.15084	0.97718	0.97591	0.97504	0.97593	0.97591	0.96813	0.97343
	200	0.10113	0.10089	0.10072	0.10090	0.10089	0.09966	0.10039	0.79596	0.79508	0.79441	0.79509	0.79508	0.79036	0.79325
	300	0.06649	0.06630	0.06614	0.06631	0.06631	0.06578	0.06586	0.64222	0.64129	0.64050	0.64135	0.64134	0.63880	0.63909
	400	0.04982	0.04962	0.04945	0.04962	0.04962	0.04935	0.04915	0.55966	0.55854	0.55756	0.55854	0.55854	0.55702	0.55582
0.95	30	1.86518	1.77747	1.78449	1.78568	1.76913	1.49363	1.64898	3.36847	3.29184	3.29868	3.30070	3.28348	3.01140	3.16938
	75	0.50289	0.48989	0.48680	0.49085	0.49003	0.46391	0.46550	1.74756	1.72354	1.71620	1.72572	1.72424	1.67746	1.67394
	150	0.29407	0.29225	0.29136	0.29232	0.29223	0.28058	0.28879	1.33436	1.33009	1.32786	1.33025	1.33005	1.30343	1.32173
	200	0.20040	0.19963	0.19920	0.19965	0.19963	0.19496	0.19827	1.08929	1.08731	1.08611	1.08735	1.08730	1.07460	1.08363
	300	0.11711	0.11614	0.11543	0.11620	0.11619	0.11454	0.11377	0.85563	0.85218	0.84957	0.85246	0.85243	0.84642	0.84362
	400	0.10012	0.09946	0.09897	0.09950	0.09949	0.09829	0.09793	0.78936	0.78684	0.78495	0.78700	0.78698	0.78205	0.78093
0.99	30	132.57193	99.64527	80.78307	131.38801	98.03568	102.95215	56.26627	11.28248	10.47966	10.23177	10.89552	10.31702	9.40448	9.33441
	75	2.09801	2.02689	2.04257	2.03460	2.02206	1.72912	1.91344	3.53378	3.46889	3.48222	3.47699	3.46435	3.18335	3.35452
	150	1.40669	1.29302	1.30487	1.29923	1.28868	1.16141	1.12062	2.89215	2.77017	2.77958	2.77770	2.76536	2.61997	2.55723
	200	0.90835	0.87927	0.87880	0.88157	0.87883	0.78970	0.83164	2.31092	2.27294	2.27008	2.27635	2.27275	2.15287	2.20458
	300	0.63823	0.61400	0.61121	0.61519	0.61362	0.57044	0.56809	1.98141	1.94481	1.93904	1.94663	1.94421	1.87323	1.86956
	400	0.42478	0.41327	0.40971	0.41360	0.41319	0.39285	0.38909	1.57633	1.55533	1.54778	1.55594	1.55517	1.51584	1.50824

Table 9: Simulated MAE and MSE values for different estimators in the GPRM when $p = 9$ with $\vartheta = 0.25$

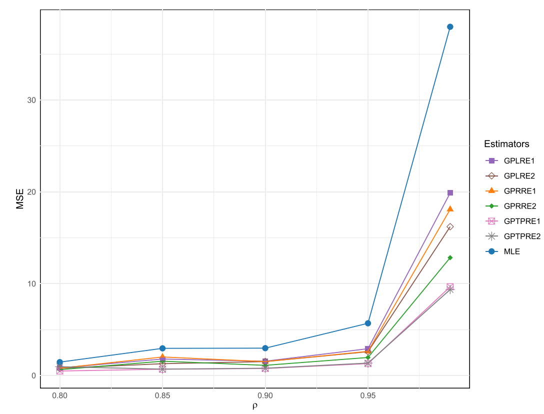
ρ	n	MSE							MAE						
		MLE	GPRRE		GPLRE		GPTPRE		MLE	GPRRE		GPLRE		GPTPRE	
		–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$	–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1, \hat{d}_1$	$\hat{k}_2, \hat{d}_2$
0.80	30	4.18427	2.61547	2.23457	2.73693	2.69724	1.86226	1.93027	4.93057	3.80278	3.58916	3.97358	3.94333	3.15063	3.21664
	75	1.47559	1.00994	0.93637	1.09763	1.09607	0.67582	0.73441	3.01163	2.44931	2.35583	2.59033	2.58854	1.89688	2.01516
	150	1.06469	0.73267	0.66994	0.83081	0.82979	0.53376	0.54792	2.48177	2.04085	1.93508	2.19958	2.19834	1.70311	1.72012
	200	0.55394	0.41911	0.37308	0.48107	0.48090	0.36146	0.34706	1.86074	1.61647	1.52074	1.73805	1.73773	1.49598	1.46012
	300	0.37027	0.32811	0.29552	0.33860	0.33854	0.26831	0.27715	1.50181	1.41260	1.33491	1.43760	1.43748	1.27527	1.29093
	400	0.30356	0.26999	0.24162	0.28282	0.28279	0.23834	0.23157	1.36678	1.28810	1.21212	1.32054	1.32048	1.20428	1.17740
0.85	30	13.65715	8.30759	5.72444	8.39265	8.24428	4.53927	4.75953	7.75769	5.74048	4.97841	5.95822	5.88082	4.26637	4.29264
	75	1.87840	1.25410	1.14593	1.32829	1.32553	0.85984	0.91664	3.30584	2.66477	2.55900	2.76704	2.76386	2.10682	2.20154
	150	1.11049	0.82424	0.75218	0.86660	0.86554	0.60448	0.65038	2.60600	2.23293	2.11587	2.30290	2.30153	1.83072	1.91183
	200	0.67451	0.50392	0.44857	0.56912	0.56880	0.35223	0.35236	2.03975	1.76328	1.65183	1.87834	1.87785	1.46035	1.44735
	300	0.51656	0.36496	0.30962	0.45694	0.45689	0.29335	0.23667	1.79563	1.49546	1.36955	1.69172	1.69161	1.35924	1.19784
	400	0.30976	0.27960	0.25209	0.28659	0.28655	0.21573	0.22656	1.40536	1.33694	1.26766	1.35318	1.35311	1.17109	1.20070
0.90	30	13.00338	7.49223	5.94145	7.38345	7.06939	5.48947	5.61119	8.30527	6.01756	5.45623	6.10511	5.92898	4.98848	4.99163
	75	2.49516	1.35346	1.26396	1.60182	1.59186	1.11545	1.10756	3.88751	2.72730	2.63069	3.09762	3.08898	2.35149	2.29745
	150	2.14665	1.15983	1.07239	1.46854	1.46006	0.88939	0.82112	3.52531	2.41687	2.31292	2.91450	2.90735	2.09749	1.93278
	200	1.14769	0.73286	0.67828	0.87002	0.86880	0.55503	0.54353	2.60624	2.05195	1.94770	2.28001	2.27852	1.72606	1.66283
	300	0.82102	0.57943	0.51845	0.67999	0.67990	0.40424	0.38427	2.25230	1.87004	1.74350	2.05412	2.05396	1.54685	1.47137
	400	0.67392	0.56779	0.51542	0.58116	0.58081	0.38199	0.42560	2.04564	1.88034	1.78361	1.90155	1.90104	1.51373	1.60828
0.95	30	20.07183	11.60601	9.24031	11.38805	10.66867	8.31441	8.29139	10.45939	7.51155	6.78968	7.53599	7.25617	6.15861	5.99003
	75	6.31564	3.48643	3.12760	3.60192	3.47935	1.84925	2.00487	6.11231	4.33198	4.12338	4.51762	4.44316	2.99905	3.08652
	150	3.05617	1.60343	1.48906	1.85919	1.83921	1.10837	1.10432	4.30323	2.93883	2.83442	3.32727	3.31161	2.37227	2.32736
	200	2.22024	1.05286	0.97578	1.46215	1.45566	0.83387	0.73335	3.68212	2.33824	2.24187	2.97938	2.97396	2.04887	1.77760
	300	1.55989	0.99140	0.92419	1.11854	1.11663	0.63446	0.64737	3.09790	2.44841	2.34422	2.63203	2.62971	1.89275	1.88999
	400	1.07739	0.61950	0.54981	0.82166	0.82131	0.42480	0.36679	2.54337	1.89845	1.75574	2.22894	2.22840	1.56450	1.37695
0.99	30	122.89461	75.38620	59.48592	76.10639	63.77217	53.83897	56.16945	24.68113	19.26165	17.53703	19.40315	17.05847	16.41830	16.62399
	75	22.53242	11.23621	9.86496	11.20343	9.81577	8.38980	8.30711	11.55454	7.43213	6.91138	7.52645	6.99784	6.08470	5.76807
	150	14.93956	9.03175	7.86472	8.89566	8.19165	3.84252	4.59308	9.12310	6.73334	6.31140	6.67870	6.36185	4.08104	4.50079
	200	11.19563	5.88496	5.14015	5.88073	5.62488	4.97108	4.73324	7.97309	5.35784	5.05231	5.44766	5.30922	4.63373	4.39376
	300	6.56018	4.04702	3.81908	4.02807	3.89702	2.49487	2.75462	6.20734	4.72124	4.60400	4.71795	4.63900	3.28704	3.54450
	400	5.26824	3.19504	3.14326	3.24263	3.17564	2.21197	2.38682	5.60354	4.20952	4.18311	4.27889	4.23569	3.14429	3.33143

Table 10: Simulated MAE and MSE values for different estimators in the GPRM when $p = 9$ with $\vartheta = 0.6$

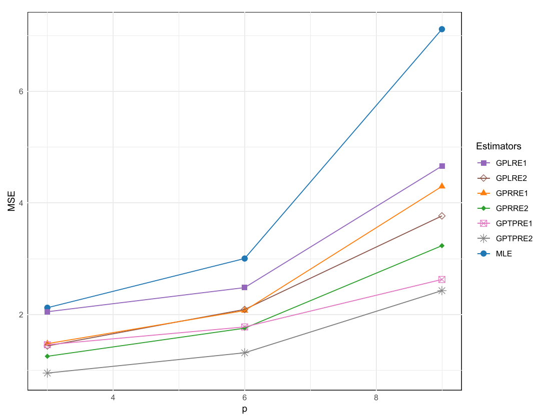
ρ	n	MSE							MAE						
		MLE	GPRRE		GPLRE		GPTPRE		MLE	GPRRE		GPLRE		GPTPRE	
		–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1,\hat{d}_1$	$\hat{k}_2,\hat{d}_2$	–	$\hat{k}_1$	$\hat{k}_2$	$\hat{d}_1$	$\hat{d}_2$	$\hat{k}_1,\hat{d}_1$	$\hat{k}_2,\hat{d}_2$
0.80	30	32.71960	16.17118	10.74567	15.49013	14.16665	10.47086	34.19718	11.50059	7.93578	6.64834	7.93998	7.66589	5.75084	6.56801
	75	4.53292	2.68402	2.17610	2.78684	2.74331	1.43333	1.57954	5.19768	3.88364	3.55945	4.01404	3.98430	2.68182	2.83983
	150	2.66648	1.48038	1.36109	1.69963	1.69205	0.75452	0.81490	4.03843	2.88583	2.78684	3.19881	3.19282	1.99918	2.08215
	200	1.60540	0.96392	0.87119	1.14671	1.14527	0.56357	0.60226	3.15022	2.37129	2.25963	2.66654	2.66483	1.80168	1.86474
	300	1.02243	0.70503	0.64694	0.79307	0.79243	0.36173	0.40017	2.49148	2.07052	1.96834	2.20439	2.20365	1.47802	1.55276
	400	0.70263	0.53879	0.48831	0.59109	0.59091	0.33162	0.35827	2.10338	1.83359	1.72866	1.93555	1.93525	1.44136	1.48674
0.85	30	59.08212	62.55410	46.22021	28.62349	23.16563	9.29284	10.51239	15.13738	11.75568	9.62185	10.48274	9.66177	6.65630	6.87594
	75	7.29500	4.12724	3.17405	4.19927	4.00996	1.83265	2.15912	5.95911	4.37949	4.00461	4.48358	4.39600	2.93739	3.18367
	150	3.41065	2.16075	1.85300	2.24672	2.23784	1.10628	1.26808	4.32207	3.32773	3.14563	3.45111	3.44389	2.25584	2.44362
	200	2.01093	1.22162	1.12248	1.37953	1.37779	0.63203	0.66974	3.49014	2.67632	2.56870	2.89597	2.89436	1.87394	1.94531
	300	1.14217	0.58384	0.54708	0.85566	0.85565	0.31970	0.30303	2.60813	1.82682	1.75196	2.26602	2.26601	1.39370	1.36927
	400	0.80021	0.59801	0.54679	0.64511	0.64498	0.29380	0.34673	2.18830	1.89551	1.80102	1.97115	1.97093	1.33632	1.45261
0.90	30	69.84886	32.45082	17.55670	31.33124	30.04549	11.63361	12.55472	16.87210	11.10941	8.63166	10.94716	10.45927	7.19337	7.11709
	75	9.76983	4.87211	3.79629	4.98281	4.64885	2.28241	2.38686	7.45210	4.97040	4.50802	5.16950	5.00443	3.33930	3.34340
	150	5.37478	2.60867	2.12581	2.86741	2.80485	1.64134	1.62708	5.57607	3.57169	3.27008	3.96131	3.91868	2.75754	2.67881
	200	3.80434	1.98658	1.78074	2.22599	2.20285	1.03772	1.06548	4.74742	3.27227	3.12812	3.59784	3.58246	2.28424	2.31797
	300	1.82203	1.04718	0.96882	1.20771	1.20319	0.56173	0.60653	3.35128	2.45483	2.34294	2.73131	2.72688	1.73638	1.79417
	400	1.40497	0.94822	0.89658	1.00815	1.00651	0.41439	0.50433	2.92282	2.38451	2.30062	2.47145	2.46948	1.50753	1.68756
0.95	30	119.51788	49.48105	27.99457	61.79155	48.20144	21.10240	21.44650	21.88380	14.13013	11.11330	14.89533	13.72660	9.71559	9.42741
	75	23.92313	12.59749	9.59994	12.00691	10.80313	4.76217	5.69728	11.40711	7.96907	7.06506	7.84123	7.33529	4.74448	5.18001
	150	9.39215	5.35451	4.36325	5.38196	5.14404	2.81322	2.97730	7.29668	5.19121	4.72637	5.29464	5.17193	3.64135	3.68215
	200	5.81088	2.53709	2.29722	2.95293	2.90026	1.72045	1.56218	5.96269	3.56115	3.43487	4.14315	4.10992	2.87756	2.67945
	300	4.18947	2.34001	2.17539	2.51647	2.48130	1.20000	1.28826	4.99514	3.56432	3.44827	3.82453	3.79903	2.43685	2.49955
	400	2.67235	1.32482	1.21427	1.60291	1.59264	0.59294	0.57006	4.00388	2.67305	2.57218	3.07308	3.06381	1.76594	1.69714
0.99	30	1169.66924	487.32383	285.41249	554.82727	431.50233	203.04177	221.67485	66.39014	43.85542	35.24430	45.60002	39.40847	30.22848	30.70429
	75	88.23299	43.37665	32.08588	41.26118	36.08168	21.93167	21.57370	21.82276	14.30193	12.44337	13.97089	12.58996	10.18258	9.54456
	150	46.64576	28.17754	23.30137	27.19677	24.33348	10.85667	13.44046	16.10522	12.01028	10.95206	11.82133	10.96134	7.21369	8.03481
	200	33.94591	17.08197	14.54488	16.87117	14.35658	9.89662	9.73627	13.89903	9.15586	8.43906	9.13228	8.31767	6.67905	6.30733
	300	16.42268	9.33222	7.92974	9.15195	8.37049	3.89191	4.37771	9.92247	7.26942	6.69475	7.18609	6.85425	4.43070	4.74878
	400	12.35376	7.08982	6.60365	7.04476	6.62944	3.64164	4.13908	8.43136	6.14006	5.96370	6.14918	5.94400	4.00702	4.33406



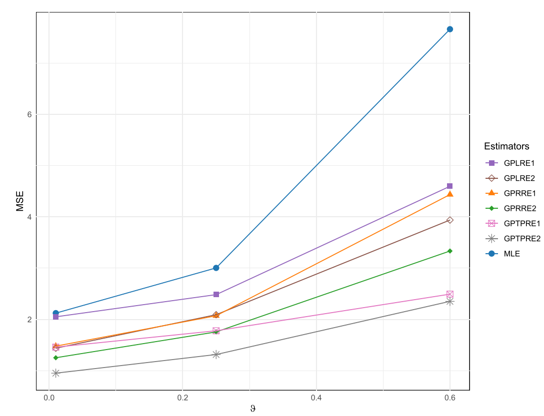
(a) Simulated MSE values with sample size for estimators in the GPORM.



(b) Simulated MSE values with multicollinearity levels for estimators in the GPORM.



(c) Simulated MSE values with explanatory variables for estimators in the GPORM.



(d) Simulated MSE values with dispersion parameter for estimators in the GPORM.

Figure 1: Simulated MSE values with main factors in simulation study

All of the estimators display consistency as the sample size n increases. As demonstrated in Table 2 ($p = 3,\ \vartheta = 0.01,\ \rho = 0.99$), the MLE MSE decreases from 2.51974 with $n = 30$ to 0.13443 with $n = 400$, and the GPTPRE with $(\hat{k}_1, \hat{d}_1)$ decreases from 1.51154 to 0.12450 with the same sample size increase. Although the sample size is increasing, the GPTPRE estimator outperforms its competing estimators under $n = 400$ in the presence of significant multicollinearity.

When increasing ρ from 0.80 to 0.99, the MSE and MAE of each estimator grow. The MLE is the least stable across changes to ρ as the MLE is directly tied to the correlation among predictor variables. When predictor variables are highly correlated, the Fisher information matrix is nearly singular. This expansion of the Fisher information matrix causes instability through severe MLE variance. Regression with regularization improves the MLE performance the most across stable MLE variance, while still allowing the user to control for additional stable variance. Within this context, the GPRRE applies a ridge penalty while the GPLRE applies a Liu penalty. Last, the GPTPRE is the most performance-improving as a dual balanced variable and bias penalty. The items within Table 4 show that the MSE of the MLE begins with a valued estimate of variance 1.28523 when ρ is 0.80 and ends with a valued estimate of variance 22.23685 when ρ is 0.99. Last, the GPTPRE is a balanced penalty through dual separation of variables with controlled bias, variance, and additional penalty variables.

When increasing ϑ, multicollinearity and overdispersion begin to impact the estimators. Each time the weighting for the estimator changes, the information for each observation decreases. When multicollinearity and overdispersion are introduced together, the dual shrinkage of the GPTPRE is especially effective when combined.

As p increases from 3 to 9, all MSE and MAE values increase for two reasons. First, the scalar MSE accumulates errors across more coefficient dimensions. Second, a larger predictor space makes the information matrix more ill-conditioned under the same ρ, further inflating the MLE variance. Comparing Tables 2 and 8 at $\rho = 0.99$, $n = 30$, and $\vartheta = 0.01$, the MLE MSE increases from 2.51974 to 132.57193 as p grows from 3 to 9, while the GPTPRE under $(\hat{k}_2, \hat{d}_2)$ increases from 2.10324 to 56.26627. This confirms that the relative advantage of the GPTPRE is amplified as dimensionality increases.

Across all tables, a clear and consistent ordering emerges among the estimators. The MLE always produces the largest MSE and MAE, particularly under high ρ, large p, high ϑ, and small n. The GPRRE and GPLRE both improve over the MLE by reducing variance through their respective single-penalty shrinkage mechanisms, but their gains are limited because each addresses only one aspect of the regularization problem. The GPTPRE consistently outperforms both by combining ridge and Liu penalties within a unified framework. For example, in Table 7 ($p = 6$, $\vartheta = 0.60$, $\rho = 0.99$, $n = 30$), the MLE MSE is 139.26762, the best GPRRE variant achieves 54.19958, the best GPLRE variant achieves 59.09277, and the GPTPRE under $(\hat{k}_1, \hat{d}_1)$ achieves 26.02929, which is the lowest among all estimators. A similar ordering holds in Table 10 ($p = 9$, $\vartheta = 0.60$, $\rho = 0.99$, $n = 30$), where the MLE MSE reaches 1169.66924, the best GPRRE variant achieves 285.41249, the best GPLRE variant achieves 431.50233, and the GPTPRE under $(\hat{k}_1, \hat{d}_1)$ achieves 203.04177, the smallest value recorded in that configuration. Regarding the choice of the shrinkage parameter pair, the $(\hat{k}_1, \hat{d}_1)$ combination generally produces smaller MSE and MAE than $(\hat{k}_2, \hat{d}_2)$ across most configurations, although the latter proves competitive or superior in several high-dispersion settings, as seen in Table 3 at $\rho = 0.99$.

5 Real Data Application

Reducing carbon dioxide emissions from vehicles is a crucial and pressing concern. Understanding what contributes to these types of emissions is crucial for informing policy and influencing consumer purchasing decisions. To demonstrate the practical value of the GPTPRE, it will be applied to an example dataset from the Open Canada dataset, which includes data on fuel and emissions for vehicles on the Canadian market. This data set will include only plug-in hybrid electric vehicles (PHEVs) for the years 2020–2025 and include 245 vehicles. The variable of interest is tailpipe emissions for carbon dioxide, which show a vehicle's environmental impact and are expressed in grams per kilometer (y).

This analysis aims to discern which vehicle traits impact CO_2 emissions the most and to determine if the estimator in question is better at producing accurate coefficients when compared to the others in the presence of multicollinearity. The design of the vehicle and the way it uses fuel have the most prominence and can be quantified by six variables that are included in the analysis. They are electric motor power in kW (x_1), engine volume in liters (x_2), number of cylinders (x_3), city fuel economy in liters/100 km (x_4), highway fuel economy in liters/100 km (x_5), and combined fuel economy in liters/100 km (x_6). The combined fuel economy is a hybrid of the city and highway economy, where the city economy is weighted by 55%, and the highway economy is weighted by 45%. The estimates for emissions are based on the combined fuel economy, which is measured and reported in grams/km.

Before fitting the model, the response variable y was inspected for preliminary analysis. The sample mean and variance of the response variable y were $\bar{y} = 104.38$ and $s^2 = 4028.24$. Thus, variance over mean $= 4028.24/104.38 = 38.59$. Since the variance is so much greater than the mean, the response variable is overdispersed. Hence, the GPRM is more suitable than the PRM. Fig. 2 shows the histogram of the response variable y. The histogram shows that the response variable is right-skewed, the box plot shows that the response variable has multiple outliers, and the violin plot shows that the response variable has a long tail. The violin plot also shows high-density regions. These multiple visualizations confirmed that the response variable is not normally distributed. The diagnostics justify why GPRM is appropriate for the response variable.

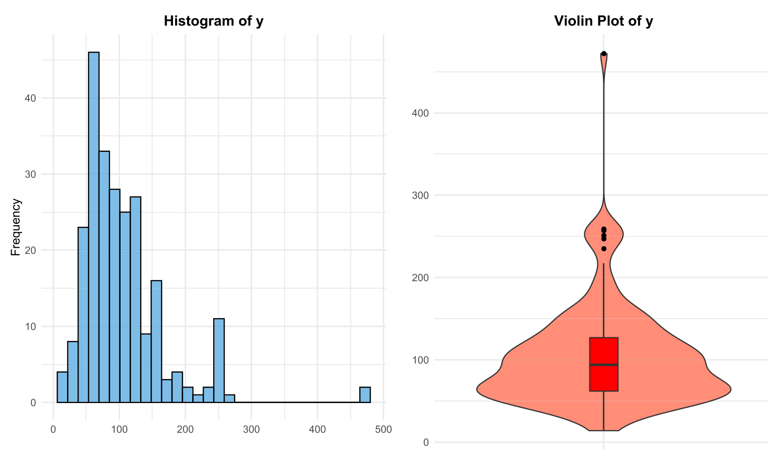


Figure 2: Histogram and violin plot of the carbon dioxide emissions (y)

Utilized three different techniques to assess multicollinearity among the predictors. First, the pairwise Pearson correlation coefficients will be documented, as demonstrated in Fig. 3, which indicates the strong correlation among x_4, x_5, and x_6. Second, a condition number will be established to assess the extent of multicollinearity, as a condition number of 4557.855 is significantly greater than the 30 threshold, which denotes substantial multicollinearity. Finally, the VIF will be determined, with predicted values of 1.28, 8.79, 10.38, 2873.80, 804.81, and 6496.13 for x_1, x_2, x_3, x_4, x_5, and x_6, respectively. Predictors with values of 6496.13, 2873.80, and 804.81 will confirm that the predictor space has severe collinearity. High multicollinearity will result in high inflated standard errors of the MLE and unreliable estimates of the coefficient, which will support the use of the shrinkage-based estimators designed in this paper.

Table 11 presents a comparison of seven count regression models based on four information criteria. The GPRM achieves the lowest values across all criteria: Akaike Information Criterion (AIC) $=$ 2278.556, Bayesian Information Criterion (BIC) $=$ 2310.067, Corrected AIC (AICc) $=$ 2279.166, and the highest log-likelihood (LL) of -1130.28, indicating a superior fit to the data relative to all competing models. The standard PRM, CMPRM, double Poisson regression model (DPRM), and geometric regression model (GERM). perform notably poorly, reflecting their inability to accommodate the overdispersion present in the data. These results confirm that the GPRM is the most appropriate framework for modeling the CO_2 emissions data and justify its selection as the basis for subsequent estimation and inference.

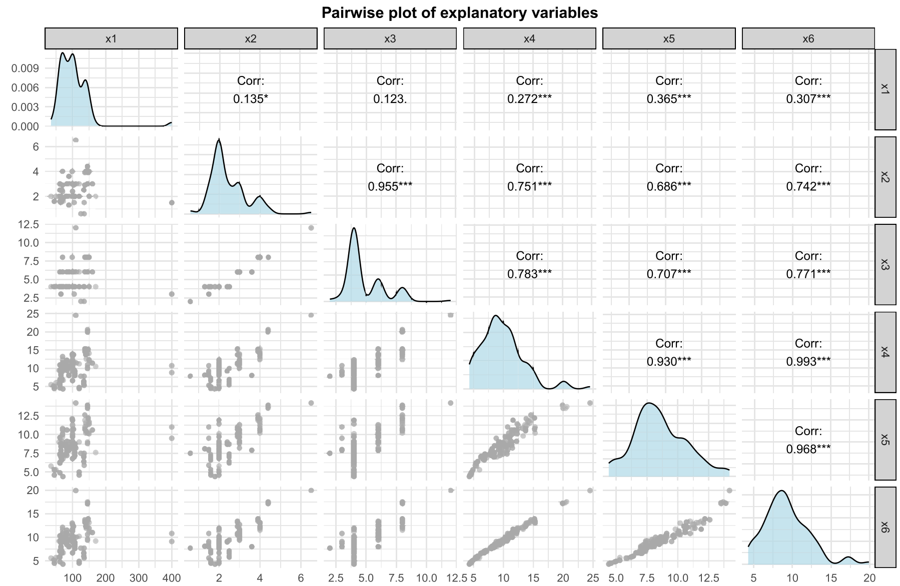


Figure 3: Pairwise plot of variables in the data set

Table 11: Model comparison criteria

Criterion	PRM	NBRM	PIGRM	DPRM	GERM	CMPRM	GPRM
AIC	3533.442	2288.821	2301.358	2328.972	2730.210	5531.507	2278.556
BIC	3561.452	2320.332	2332.870	2360.483	2758.220	5563.018	2310.067
AICc	3534.052	2289.431	2301.969	2329.582	2730.820	5532.273	2279.166
LL	−1758.72	−1135.41	−1141.68	−1155.49	−1357.10	−2756.75	−1130.28

Table 12 presents all the competing estimators' coefficients, SEs, CIs, and MSE results. The results will be explained in the following subsections.

The constant β_0 and electric motor power β_1 are the only two significant predictors, which are the same for all estimators. It is also clear that $\beta_1 \approx -0.003$ implies that more electric motor power is equated to more motor power and consequently more power emitted to the atmosphere. Like the remaining predictors, the MLE also treats the remaining predictors as insignificant, as their standard errors have been blown out as a result of multicollinearity. However, the GPTPRE has detected a significant positive effect of highway fuel consumption β_5, CIs are, and $[0.133, 0.242]$ and $[0.125, 0.237]$ under the two parameter combinations, a relationship that the MLE has failed to capture.

The MLE provides the largest collection of standard errors, ranging from β_4, β_5, and β_6 $(0.347, 0.281, 0.415)$, as a result of a severe negative impact of multicollinearity. The GPLRE is only slightly better, while the GPRRE is better in comparison. The GPTPRE provides the largest improvement in the standard errors of β_4, β_5, and β_6, which are 0.028, 0.027, and 0.029, under $(\hat{k}_2, \hat{d}_2)$, respectively. The estimators of the GPOTPRE have the best performance of the MSE with values of 1.155 in comparison with 2.820 for the MLE, 1.705 for the GPLRE, and 1.817 for the GPRRE.

Table 12: Performance comparison of competing estimators in the GPRM for the data: coefficients, SEs, CIs, and MSE Values

Estimator	Parameter	β_0	β_1	β_2	β_3	β_4	β_5	β_6	MSE
MLE	–	3.1606	−0.00341	−0.01608	−0.01472	0.08637	0.10206	0.01284	29.29298
		(0.08923)	(0.00038)	(0.06741)	(0.03948)	(0.30593)	(0.24832)	(0.55475)	
		[2.98571, 3.3355]	[−0.00415, −0.00267]	[−0.14819, 0.11603]	[−0.0921, 0.06266]	[−0.51324, 0.68598]	[−0.38465, 0.58877]	[−1.07445, 1.10013]	
GPRRE	$\hat{k}_1$	2.98694	−0.00339	−0.02925	0.00353	0.04912	0.11667	0.05078	2.32873
		(0.08399)	(0.00038)	(0.06462)	(0.03772)	(0.07143)	(0.06076)	(0.12621)	
		[2.82232, 3.15156]	[−0.00413, −0.00265]	[−0.15591, 0.09741]	[−0.0704, 0.07747]	[−0.09089, 0.18913]	[−0.00242, 0.23576]	[−0.19659, 0.29815]	
	$\hat{k}_2$	2.94287	−0.00338	−0.03233	0.00808	0.04349	0.1234	0.05353	1.85795
		(0.08267)	(0.00038)	(0.0639)	(0.03732)	(0.05968)	(0.05175)	(0.10421)	
		[2.78084, 3.1049]	[−0.00412, −0.00265]	[−0.15758, 0.09292]	[−0.06507, 0.08122]	[−0.07348, 0.16047]	[0.02198, 0.22482]	[−0.15071, 0.25778]	
GPLRE	$\hat{d}_1$	3.1358	−0.00341	−0.01806	−0.01212	0.07606	0.10014	0.0273	14.18862
		(0.08847)	(0.00038)	(0.06701)	(0.03915)	(0.20969)	(0.17077)	(0.37958)	
		[2.9624, 3.30919]	[−0.00415, −0.00267]	[−0.14941, 0.11328]	[−0.08885, 0.0646]	[−0.33493, 0.48705]	[−0.23456, 0.43485]	[−0.71667, 0.77126]	
	$\hat{d}_2$	3.1358	−0.00341	−0.01806	−0.01212	0.07606	0.10014	0.0273	14.18862
		(0.08847)	(0.00038)	(0.06701)	(0.03915)	(0.20969)	(0.17077)	(0.37958)	
		[2.9624, 3.30919]	[−0.00415, −0.00267]	[−0.14941, 0.11328]	[−0.08885, 0.0646]	[−0.33493, 0.48705]	[−0.23456, 0.43485]	[−0.71667, 0.77126]	
GPTPRE	$\hat{k}_1, \hat{d}_1$	2.54755	−0.00333	−0.05486	0.0462	−0.00043	0.18756	0.06768	1.16503
		(0.07088)	(0.00038)	(0.05719)	(0.03364)	(0.02676)	(0.02771)	(0.03962)	
		[2.40864, 2.68646]	[−0.00407, −0.00259]	[−0.16694, 0.05722]	[−0.01973, 0.11213]	[−0.05287, 0.05201]	[0.13325, 0.24186]	[−0.00997, 0.14534]	
	$\hat{k}_2, \hat{d}_2$	2.58626	−0.00334	−0.05307	0.04269	0.00365	0.18123	0.06657	1.15505
		(0.07203)	(0.00038)	(0.05787)	(0.03401)	(0.02809)	(0.02863)	(0.04245)	
		[2.44509, 2.72743]	[−0.00407, −0.0026]	[−0.16649, 0.06034]	[−0.02397, 0.10934]	[−0.05141, 0.0587]	[0.12511, 0.23735]	[−0.01663, 0.14977]	

Note: Values in parentheses (.) represent Standard Errors (SEs), and values in brackets [.,.] represent 95% Confidence Intervals (CIs).

The detrimental impact electric motor power has on emissions bolsters the argument for more extensive electric drivetrains to decrease vehicular CO_2 output. The GPTPRE has detected a pronounced positive impact from fuel use on the highway, meaning highway travel fuel efficiency is a primary driver of emissions decreases. Once fuel use is controlled, engine size, i.e., displacement and number of cylinders, has no separate impact on emissions. These findings suggest that fuel efficiency is more of a regulatory target than engine design.

6 Conclusion

This study proposes a two-parameter shrinkage estimator called GPTPRE to address the issues that multicollinearity causes with the GPRM. The two biasing parameters used in GPTPRE are combined to provide a balance of bias and variance and are shown to be more flexible than the alternatives GPRRE and GPLRE. Theoretical aspects of the GPTPRE were developed, and the bias vector, covariance matrix, and mean square error were calculated. Theoretical comparisons that are in favor of the GPTPRE were also established to show that it surpasses the MLE, GPRRE, and GPLRE approaches. The performance of the GPOTPRE was assessed using a real CO_2 emissions dataset of PHEVs in Canada, in addition to a Monte Carlo simulation study. Over many aspects of sample size, correlation models, and dispersion, the simulation studies showed consistently that GPOTPRE reaches the lowest MSE among all the competing estimators. The real data application also showed that the GPTPRE significantly outperformed the existing approaches in MSE while also significantly reducing the error and multicollinearity estimators. Limitations of the GPTPRE should also be considered despite its positive aspects. The selection of the biasing parameters k and d depends on sample-based estimates, a situation that may even require a cross-validation step to define the more data-optimal biasing parameters and is crucial in most real-world applications. In addition, GPTPRE is likely to cope with multicollinearity within the GPRM framework, but deals little with outliers in the data. When outliers and multicollinearity occur, the GPTPRE lacks the ability to explain the estimates disrupted by the two sources of instability working in combination. Therefore, the shortcomings of the GPTPRE indicate several promising options for future research: the GPTPRE improvement for robust multicollinearity and outliers [39–41], a two-parameter shrinkage extension within multicollinearity-based integrated count regression models, and employing a data-driven approach with cross-validation and/or the bootstrap framework to optimally determine the parameters k and d.

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Conflicts of Interest: The authors declare no conflicts of interest.

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