Confidence Level-Based Interval-Valued *p*, *q*, *r*-Spherical Fuzzy Sets with Application to Industrial Waste Management

Umme Habiba[1], Muhammad Amad Sarwar[2,*] and Amel Ali Alhussan[3]

[1] School of Accounting, Tongling University, Tongling, 244061, China

[2] School of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, China

[3] Department of Computer Sciences, College of Computer and Information Sciences, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh, 11671, Saudi Arabia

INFORMATION

Keywords:
Fuzzy sets
aggregation operators
industrial waste management
decision-making

DOI: 10.23967/j.rimni.2026.10.76086

SCIPEDIA Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería

Revista Internacional Métodos numéricos para cálculo y diseño en ingeniería RIMNI

in cooperation with CIMNE

Confidence Level-Based Interval-Valued p, q, r-Spherical Fuzzy Sets with Application to Industrial Waste Management

Umme Habiba[1], **Muhammad Amad Sarwar**[2,*] **and Amel Ali Alhussan**[3]

[1]School of Accounting, Tongling University, Tongling, 244061, China

[2]School of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing, 210016, China

[3]Department of Computer Sciences, College of Computer and Information Sciences, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh, 11671, Saudi Arabia

ABSTRACT

Industrialization plays a substantial role in a country's development and economic growth. As a developing nation, Pakistan heavily relies on its steel industry to drive economic progress. Pakistan Steel Mills Corporation (PkMC) is the largest steel producer in the country, manufacturing thousands of tonnes of steel annually. However, this high steel production has led to a significant increase in industrial solid waste. While various methods are available for managing industrial waste, the selection of the appropriate technology is a complex process due to the wide range of strategies and the multiple factors involved in the decision-making process. Existing research in the interval-valued p, q, r-spherical fuzzy sets (IV$p, q, r$-SFS) environment assumes 100% confidence-level from decision makers in evaluating scenarios, but real-world situations often differ from ideal situations. To address this limitation, this study incorporates confidence-level with IV$p, q, r$-SFS, to identify effective and sustainable waste management strategies for PkMC. Decision-makers can evaluate alternatives by assigning corresponding confidence values using this model to better capture uncertainty. This study develops and analyzes basic operational laws, averaging, and geometric operators, outlining their desirable properties. This study presents a step-by-step algorithm for the proposed MCDM methodology, demonstrating its reliability and efficiency through an industrial waste management example. Model results indicate energy recovery as the most suitable alternative. Comparative analysis further validates the effectiveness of the proposed model. This research offers helpful tips for improving decision-making in waste management and points out the importance of a robust methodological framework in tackling complex, uncertainty-driven challenges.

OPEN ACCESS

Received: 13/11/2025

Accepted: 19/12/2025

Published: 20/03/2026

DOI
10.23967/j.rimni.2026.10.76086

Keywords:
Fuzzy sets
aggregation operators
industrial waste management
decision-making

1 Introduction

Industrial waste refers to any unwanted materials, by-products, chemicals, or substances discharged as a result of industrial activities. Due to its significant environmental impact, industrial

*Correspondence: Muhammad Amad Sarwar (m.amadsarwar26@gmail.com).

waste poses a major challenge for governments worldwide. The contamination of landscapes and water sources, such as aquifers, by industrial effluents has been linked to neurotoxic, genotoxic, and carcinogenic effects. This type of waste includes a wide range of materials, whether hazardous or non-hazardous, disposable, or non-reusable, generated through various manufacturing processes [1,2]. Although regulatory frameworks and monitoring systems have been introduced to manage industrial waste, its production continues to rise. Consequently, the issue has garnered growing attention, becoming a focal point of research across multiple disciplines, including chemical, environmental, and civil engineering [3].

Pakistan Steel Mills Corporation (PkMC) stands as one of the nation's primary and most prominent steel producers, historically contributing significantly to Pakistan's industrial output and export economy. Its strategic location near key raw materials, such as iron ore, and access to a cost-effective labor force have supported the rapid development of the steel sector. PkMC employs various steel production methods, including blast furnace reduction, direct reduction, and melting of recycled iron. The steel manufacturing process involves several stages, starting with the use of raw materials like coal and iron ore to produce hot metal, followed by steel production and the storage of finished goods in rolling mills [4]. Typically, steel is manufactured using blast furnaces and basic oxygen furnaces, while scrap steel is processed through electric arc furnaces. Throughout these operations, numerous types of industrial waste are generated, such as carbon and coal dust, slag, acid and oil sludge, fly ash, steel melting shop sludge, refractory waste, and other solid by-products. Key sources of this waste include coke oven by-product plants, clinker plants, blast furnaces, steel melting shops, and rolling mills [5]. The common practice of disposing of this waste in open areas or excavated land poses serious environmental risks through the release of dust and leachate, as well as creating economic burdens due to the demand for land. In response to these challenges, this study aims to identify the most suitable industrial waste management approach for PkMC by applying the multi-criteria decision-making (MCDM) technique.

Pakistan's steel sector currently produces around six million metric tons of steel annually, which in turn generates substantial quantities of industrial solid waste. When industrial operations are located in ecologically sensitive regions, it becomes especially crucial to identify an effective strategy for managing this waste. Although various strategies have been proposed for industrial solid waste management, selecting the most appropriate one remains a significant challenge. From a sustainability standpoint, waste management strategies must assess the comprehensive impacts of industrial waste, considering a wide range of factors along with their relevant criteria and indicators. Consequently, the development of an optimal industrial waste management strategy should be tailored to each country's specific context, integrating its technological, economic, environmental, and social priorities. This necessitates a novel MCDM process. To address this multifaceted issue, the present study applies the aggregation operator-based MCDM approach with confidence levels, which has gained widespread acceptance in environmental decision-making. MCDM offers a structured and rational framework for assessing both discrete and continuous issues, with the objective of identifying the optimal strategy or alternative. This research focuses on pressing issues surrounding industrial solid waste management in Pakistan, a developing nation, and introduces an innovative evaluation framework that combines fuzzy logic with confidence levels, particularly with fuzzy weighting methods. This approach uses qualitative data to prioritize industrial waste management strategies and delivers accurate conclusions through systematic ranking.

MCDM is a process that combines information about various options and uses that data to identify and choose the most appropriate option. Many applications have utilized this evaluation tool, including the planning of sustainable energy, the selection of the best gate security system,

medical waste management, etc. [6–8]. An important step in MCDM is identifying how evaluation information is expressed for different attributes. Decision experts (DE) frequently use crisp numbers to evaluate alternatives. The decision environment can be highly complex, and DE have a limited understanding of opinion targets in terms of crisp values, so crisp numbers can't deliver the full message in all cases. Zadeh [9] addressed this issue by introducing the fuzzy set (FS) theory, which assesses a set element's degree of membership (MB). A fuzzy logic approach and structured elicitation are used to emphasize the critical role of professional expertise and uncertainty modeling in dealing socio-technical systems evaluation [10]. The FS approach is a more nuanced and flexible manner by taking into account the degree of membership. The FS theory does not consider values other than membership degrees. However, when evaluating and understanding decision-making situations, it is crucial to acknowledge the significance of non-membership (NMB). It was Atanassov who developed the framework of intuitionistic fuzzy sets (IFS) [11], which incorporates both MB and NMB. It has been well documented that IFS is an excellent solution for processing a variety of fuzzy information. It is a constraint of IFS that MB grade and NMB grade are constrained to a very small range [0, 1], in other words, that narrow threshold for describing fuzzy knowledge. In response to this observation, during the development of the Pythagorean fuzzy set (PFS) by Yager [12], the restrictions of the IFS were widened so that the sum of the squares of MB and NMB was able to fit within [0, 1]. As a result of the fact that the sum of the squares of the two grades of MB and NMB exceeds 1 in PFS, we are unable to simulate the views of decision experts using information in the form of MB and NMB grades. Yager [13] developed a q-rung orthopair fuzzy set (q-ROFS) as a new way to address uncertainty by upgrading PFS. For q-ROFS, the sum of both $(MB)^q$ and $(NMB)^q$ grades is less than or equal to 1 and $(q = 1, 2, 3, \ldots, n)$.

The picture fuzzy set (PcFS) was developed by Cuong and Kreinovich [14] to deal with situations where previous studies failed to meet criteria. It is required that the sum of the MB grade, the NMB grade, and the neutral (ID) grade in the PcFS be between [0, 1]. Due to its failure when the sum of the MB grade, ID grade, and NMB grade is not between [0, 1], DE is hesitant to use PcFS to solve issues and dilemmas. The spherical fuzzy set (SFS) concept was described by Al-Shamiri et al. [15] to address this issue, easing the requirement that the sum of squares of the aforementioned grades be within [0, 1]. Decisions made by DE in SFS are more flexible than those made by DE in PcFS. Khan et al. [16] presented the t–spherical fuzzy set (t–SFS) as the most generalized form of all fuzzy structures discussed. Dhumras et al. [17] study explores the better interaction between t–SFSs, their corresponding aggregation operators, and their application along with the improvement of the operational laws of t–SFSs. In t–SFSs, the power level of MB $\mathfrak{k}$, ID $\mathfrak{z}$, and NMB $\mathfrak{r}$ is taken the same. In real-life practice, it is possible to have different power levels for $\mathfrak{k}$, $\mathfrak{z}$, and $\mathfrak{r}$, depending on the context. Let's take a scenario: if a DE defines $\mathfrak{k}$ as 0.8, $\mathfrak{z}$ as 0.3, and $\mathfrak{r}$ as 0.6. Then the sum of squares is $(0.8)^2 + (0.4)^2 + (0.5)^2 = 1.05 > 1$. Consequently, we should need to check the next power $(0.8)^3 + (0.3)^3 + (0.6)^3 = 0.701 < 1$. Although, since $(0.8)^3 + (0.3)^2 + (0.6)^2 = 0.922 < 1$, it is therefore possible to capture the situation more effectively by permitting the term levels to differ. Therefore, a novel fuzzy tool with $0 \leq \mathfrak{k}^p + \mathfrak{z}^q + \mathfrak{r}^r \leq 1$ such that $(p, q, r) \geq 1$ in the constraints needs to be designed. An extension of t–SFS, p, q, r - spherical fuzzy set (p, q, r–SFS) has been discussed by Karamaz and Karaaslan [18], and they also explored the distance measures of p, q, r-SFS. Table 1 describes the summary knowledge of some important fuzzy extensions.

Table 1: Summary of fuzzy sets extensions

Study	Year	Fuzzy extension
Zadeh	1965	Fuzzy Set (FS)
Atanassov	1986	Intuitionistic FS (IFS)
Yager	2013	Pythagorean FS (PFS)
Cuong	2013	Picture FS (PcFS)
Yager	2016	q–Rung Orthopair FS (q–ROFS)
Gundogdu	2018	Spherical FS (SFS)
Mahmood	2019	t–SFS
Karamaz and Karaaslan	2024	p, q, r–SFS

It may be difficult for decision makers in the real world when it comes to making real-life decisions because of the lack of adequate information, especially when it comes to the real-life dilemmas in the real world, to express their opinions by using actual numbers. For decision makers, a more appropriate term would be, in this case, to express their opinions in the form of closed intervals [0, 1]. As a result, Gorzałczany introduced IVFS in 1987 to achieve this [19], and IVIFS was introduced in 1989 by Atanassov and Gargov [20]. Alhamzi et al. introduced in 2016 IVPFS, which are generalizations of IFS [21]. Joshi et al. presented the concept of IVq–ROFS as a method for performing fuzzy set predictions [22]. A generalization of picture fuzzy sets is IVPcFS, developed by Cuong and Kreinovich in 2013 [14]. Sarfraz introduced IVSFS, which are generalizations of SFSs [23]. Karaaslan and Karamaz defined the more generalized form of IVSFS as (IVp, q, r-SFS) [24]. Table 2 presents a summary information of the interval-valued fuzzy sets (IVFS). Existing research in the IV$p, q, r$-SFS environment assumes a 100% confidence-level (CL) of decision makers in evaluating scenarios, but real-world situations often differ from these ideal situations. To address this limitation, this study introduces a novel MCDM framework, confidence-level-based IV$p, q, r$-SFS ($CIV_{p,q,r} - SFS$), to identify effective and sustainable waste management strategies for PkMC. Decision-makers can evaluate alternatives by assigning corresponding confidence values using $CIV_{p,q,r} - SFS$ to better capture uncertainty. The advantages of the proposed aggregation operators over the existing ones are summarized in Table 3.

Table 2: Summary of interval-valued fuzzy sets extensions

Study	Year	Interval-valued fuzzy extension
Gorzalczany	1987	Interval-valued FS
Atanassov	1989	Interval-valued IFS
Cuong	2013	Interval-valued PcFS
Peng	2016	Interval-valued PFS
Joshi	2018	Interval-valued q–ROFS
Ullah	2019	Interval-valued SFS (IVSFS)
Ullah	2019	Interval-valued t–SFS
Karaaslan	2024	IVp, q, r–SFS

Table 3: Research gaps and advantages

	Parameters							
Operator	MB	NMB	ID	IV	p	q	r	CL
IFS operators [11]	✓	✓	×	×	×	×	×	×
PFS operators [12]	✓	✓	×	×	×	×	×	×
PicFS operators [14]	✓	✓	×	×	×	×	×	×
t-SFS operators [16]	✓	✓	×	×	×	✓	×	×
IVt-SFS operators [23]	✓	✓	✓	✓	×	✓	×	×
p,q,r-SFS operators [18]	✓	✓	✓	×	✓	✓	✓	×
IVp,q,r-SFS operators [24]	✓	✓	✓	✓	✓	✓	✓	×
Proposed method	✓	✓	✓	✓	✓	✓	✓	✓

1.1 Motivations

To facilitate data fusion and decision-making processes, confidence level-based aggregation operators are essential tools. In addition to incorporating confidence into their processes, these operators also consider the accuracy of each information source, thus extending conventional aggregation strategies. Taking into account the differing confidence levels of different data sources, decision makers can evaluate the importance of each input. Several averaging operators (AO) are presented with confidence levels to deal with ambiguous data [25–27], but are unable to deal with interval-values and parametric evaluation. In 2024, Karaaslan and Karamaz [24] defined the more generalized form of IVSFS as (IVp, q, r-SFS). The current exploration of aggregation operators within IV$p, q, r$-SFS environments neglects attribute confidence levels. Existing research on aggregation operators in IV$p, q, r$-SFS environments has operated under the assumption that decision makers possess 100% confidence in evaluated objects. However, real-world scenarios often deviate from these ideal conditions. This limitation needs to be overcome, and decision-makers may need to evaluate alternatives based on IV$p, q, r$-SFS with their corresponding confidence levels rather than giving full confidence to accurately gauge the evaluation process and their familiarity. By doing this, we can minimize uncertainty in decision-making.

Pakistan's steel industry, especially major producers like Pakistan Steel Mills Corporation (PkMC), has intensified industrial waste generation. Improper disposal of by-products, such as slag, sludge, and fly ash has led to environmental contamination, health hazards, and pressure on land resources. Despite global advances in industrial waste management, Pakistan still lacks an integrated, context-specific framework capable of addressing its environmental, economic, and social constraints. Selecting an optimal waste management strategy is complex, as it requires balancing multiple and often conflicting criteria. While environmental studies have used MCDM methods, limited research employs aggregation operators with fuzzy logic to evaluate industrial waste strategies in developing economies. To our best knowledge, no study applies a confidence level-based IV$p, q, r-SFS$ to industry problems. This study fills this gap by introducing a confidence level-based MCDM framework enhanced with fuzzy weighting to identify effective, sustainable waste management strategies for PkMC, supporting industrial policy and environmental decision-making.

1.2 Contributions

There are four main contributions of the study:

- Defined confidence level based IVp, q, r-SFS and basic operational laws.
- Confidence level based interval-valued p, q, r averaging aggregation operator ($CIV_{p,q,r}$-SFWA) and confidence level based interval-valued p, q, r geometric aggregation operator ($CIV_{p,q,r}$-SFWG), are introduced and their desired properties are discussed.
- MCDM method utilizing these operators is proposed to enhance problem solving effectiveness.
- Since industrial waste management is vital, our study determines the most effective method for industrial waste management. As a result, we integrate our formulated decision-making approach within the (IVp, q, r)-SFS paradigm.

An overview of the paper's structure is provided here. After the introduction, we provide an overview of the preliminary supporting definitions and operations. In addition, we discuss novel aggregation operators for IV$p, q, r$-SFS that incorporate confidence levels and their properties. Thereafter, we present an MCDM approach based on these proposed operators. And we include an example demonstrating the application of the proposed approach to identify the most suitable alternative for industrial waste management. This section also covers sensitivity analysis of the parameters to highlight its advantages. Furthermore, we conduct a comparative analysis of the developed approach and existing methods to highlight its advantages. Finally, it ends with the article's concluding remarks.

2 Preliminaries

In this section, we will summarize some of the most important terms, definitions, and concepts related to this area of study.

2.1 Spherical Fuzzy Set (SFS)

For any universal set $\mathcal{L}$, a Spherical Fuzzy Set (SFS) [15] H can be expressed as:

$$H = \{\mathfrak{g}, (\mathfrak{k}_H(\mathfrak{g}), \mathfrak{z}_H(\mathfrak{g}), \mathfrak{r}_H(\mathfrak{g})) \mid \mathfrak{g} \in \mathcal{L}\} \tag{1}$$

Here, $\mathfrak{k}_H(\mathfrak{g})$ represents membership grade, $\mathfrak{z}_H(\mathfrak{g})$ represents neutral membership grade, and $\mathfrak{r}_H(\mathfrak{g})$ represents non-membership grade of $\mathfrak{g}$ in element $\mathfrak{g} \in \mathcal{L}$ such that $0 \leq \mathfrak{k}_H(\mathfrak{g}), \mathfrak{z}_H(\mathfrak{g}), \mathfrak{r}_H(\mathfrak{g}) \leq 1$ and $(\mathfrak{k}_H(\mathfrak{g}))^2 + (\mathfrak{z}_H(\mathfrak{g}))^2 + (\mathfrak{r}_H(\mathfrak{g}))^2 \leq 1$.

The grade of rejection for spherical fuzzy sets can be expressed as:

$$O(\mathfrak{g}) = \sqrt{1 - (\mathfrak{k}_H(\mathfrak{g}))^2 - (\mathfrak{z}_H(\mathfrak{g}))^2 - (\mathfrak{r}_H(\mathfrak{g}))^2} \tag{2}$$

2.2 t-Spherical Fuzzy Set (t-SFS)

For any universal set $\mathcal{L}$, a t-Spherical Fuzzy Set (t-SFS) [16] T can be expressed as:

$$T = \{\mathfrak{g}, (\mathfrak{k}_T(\mathfrak{g}), \mathfrak{z}_T(\mathfrak{g}), \mathfrak{r}_T(\mathfrak{g})) \mid \mathfrak{g} \in \mathcal{L}\} \tag{3}$$

In this context, $\mathfrak{k}_T(\mathfrak{g})$ represents membership grade, $\mathfrak{z}_T(\mathfrak{g})$ represents neutral membership grade, and $\mathfrak{r}_T(\mathfrak{g})$ represents non-membership grade of $\mathfrak{g}$ in element $\mathfrak{g} \in \mathcal{L}$ such that $0 \leq \mathfrak{k}_T(\mathfrak{g}), \mathfrak{z}_T(\mathfrak{g}), \mathfrak{r}_T(\mathfrak{g}) \leq 1$ and $(\mathfrak{k}_T(\mathfrak{g}))^t + (\mathfrak{z}_T(\mathfrak{g}))^t + (\mathfrak{r}_T(\mathfrak{g}))^t \leq 1$.

The grade of rejection for spherical fuzzy sets can be expressed as:

$$O(\mathfrak{g}) = \sqrt{1 - (\mathfrak{k}_T(\mathfrak{g}))^t - (\mathfrak{z}_T(\mathfrak{g}))^t - (\mathfrak{r}_T(\mathfrak{g}))^t} \tag{4}$$

2.3 p, q, r-Spherical Fuzzy Set (p, q, r-SFS)

For any finite set $\mathcal{L}$, a p,q,r-Spherical Fuzzy Set (p,q,r-SFS) [18] $\widetilde{P}$ over an element $\mathfrak{g} \in \mathcal{L}$ can be expressed as:

$$\widetilde{P} = \{\mathfrak{g}, (\mathfrak{k}_{\widetilde{P}}(\mathfrak{g}), \mathfrak{z}_{\widetilde{P}}(\mathfrak{g}), \mathfrak{r}_{\widetilde{P}}(\mathfrak{g})) \mid \mathfrak{g} \in \mathcal{L}\} \tag{5}$$

In this context, $\mathfrak{k}_{\widetilde{P}}(\mathfrak{g}) \in [0,1]$ represents membership grade, $\mathfrak{z}_{\widetilde{P}}(\mathfrak{g}) \in [0,1]$ represents impartial membership grade, while $\mathfrak{r}_{\widetilde{P}}(\mathfrak{g}) \in [0,1]$ represents non-membership grade of a component $\mathfrak{g} \in \mathcal{L}$ satisfying the condition $(\mathfrak{k}_{\widetilde{P}}(\mathfrak{g}))^p + (\mathfrak{r}_{\widetilde{P}}(\mathfrak{g}))^q + (\mathfrak{z}_{\widetilde{P}}(\mathfrak{g}))^r \leq 1$, here p and q are positive integers, and $r = LCM(p,q)$. The grade of rejection is given by:

$$\widetilde{P}(\mathfrak{g}) = \sqrt{1 - (\mathfrak{k}_{\widetilde{P}}(\mathfrak{g}))^p - (\mathfrak{r}_{\widetilde{P}}(\mathfrak{g}))^q - (\mathfrak{z}_{\widetilde{P}}(\mathfrak{g}))^r} \tag{6}$$

2.4 Interval-Valued p, q, r-Spherical Fuzzy Set (IVp, q, r-SFS)

For any finite set $\mathcal{L}$, an interval-valued p,q,r-spherical fuzzy set (IV$p,q,r$-SFS) [24] $\widetilde{Q}$ over an element $\mathfrak{g} \in \mathcal{L}$ can be expressed as:

$$\widetilde{Q} = \{\mathfrak{g}, (\mathfrak{k}_{\widetilde{Q}}(\mathfrak{g}), \mathfrak{z}_{\widetilde{Q}}(\mathfrak{g}), \mathfrak{r}_{\widetilde{Q}}(\mathfrak{g})) \mid \mathfrak{g} \in \mathcal{L}\} \tag{7}$$

Here, $\mathfrak{k}_{\widetilde{Q}}(\mathfrak{g}) = [\mathfrak{k}^l_{\widetilde{Q}}(\mathfrak{g}), \mathfrak{k}^u_{\widetilde{Q}}(\mathfrak{g})] \in [0,1]$ represents membership grade, $\mathfrak{z}_{\widetilde{Q}}(\mathfrak{g}) = [\mathfrak{z}^l_{\widetilde{Q}}(\mathfrak{g}), \mathfrak{z}^u_{\widetilde{Q}}(\mathfrak{g})] \in [0,1]$ represents impartial membership grade, while $\mathfrak{r}_{\widetilde{Q}}(\mathfrak{g}) = [\mathfrak{r}^l_{\widetilde{Q}}(\mathfrak{g}), \mathfrak{r}^u_{\widetilde{Q}}(\mathfrak{g})] \in [0,1]$ represents non-membership grade of a component $\mathfrak{g} \in \mathcal{L}$ satisfying the condition $(\mathfrak{k}_{\widetilde{Q}}(\mathfrak{g}))^p + (\mathfrak{r}_{\widetilde{Q}}(\mathfrak{g}))^q + (\mathfrak{z}_{\widetilde{Q}}(\mathfrak{g}))^r \leq 1$, where p and q are positive integers, and $r = LCM(p,q)$. The grade of rejection is given by:

$$\widetilde{Q}(\mathfrak{g}) = \left[\widetilde{Q}^l(\mathfrak{g}), \widetilde{Q}^u(\mathfrak{g})\right]$$

$$\widetilde{Q}^l(\mathfrak{g}) = \left(1 - \left(\mathfrak{k}^u_{\widetilde{Q}}(\mathfrak{g})\right)^p - \left(\mathfrak{r}^u_{\widetilde{Q}}(\mathfrak{g})\right)^q - \left(\mathfrak{z}^u_{\widetilde{Q}}(\mathfrak{g})\right)^r\right)^{min\{p,q,r\}}$$

$$\widetilde{Q}^u(\mathfrak{g}) = \left(1 - \left(\mathfrak{k}^l_{\widetilde{Q}}(\mathfrak{g})\right)^p - \left(\mathfrak{r}^l_{\widetilde{Q}}(\mathfrak{g})\right)^q - \left(\mathfrak{z}^l_{\widetilde{Q}}(\mathfrak{g})\right)^r\right)^{min\{p,q,r\}}$$

The triplet $(\mathfrak{k}_{\widetilde{Q}}(\mathfrak{g}), \mathfrak{z}_{\widetilde{Q}}(\mathfrak{g}), \mathfrak{r}_{\widetilde{Q}}(\mathfrak{g}))$ is called as IVp, q, r-SFN.

Definition of the score function of a IVp, q, r-SFN $(\mathfrak{k}_{\widetilde{Q}}, \mathfrak{z}_{\widetilde{Q}}, \mathfrak{r}_{\widetilde{Q}})$ is as follows:

$$Sc(\widetilde{Q}(\mathfrak{g})) = \frac{\left((\mathfrak{k}^l_{\widetilde{Q}})^p(1 - (\mathfrak{r}^l_{\widetilde{Q}})^q - (\mathfrak{z}^l_{\widetilde{Q}})^r)\right)\left((\mathfrak{k}^u_{\widetilde{Q}})^p(1 - (\mathfrak{r}^u_{\widetilde{Q}})^q - (\mathfrak{z}^u_{\widetilde{Q}})^r)\right)}{3} \tag{8}$$

In this context, $0 \leq Sc(\widetilde{Q}) \leq 1$.

The accuracy function of a IVp, q, r-SFN $(\mathfrak{k}_{\widetilde{Q}}, \mathfrak{z}_{\widetilde{Q}}, \mathfrak{r}_{\widetilde{Q}})$ is defined as follows:

$$Ac(\widetilde{Q}(\mathfrak{g})) = \frac{\left((\mathfrak{k}^l_{\widetilde{Q}})^p(1 + (\mathfrak{z}^l_{\widetilde{Q}})^q + (\mathfrak{z}^l_{\widetilde{Q}})^r)\right)\left((\mathfrak{k}^u_{\widetilde{Q}})^p(1 + (\mathfrak{r}^u_{\widetilde{Q}})^q + (\mathfrak{z}^u_{\widetilde{Q}})^r)\right)}{3} \tag{9}$$

In this context, $0 \leq Ac(\widetilde{Q}) \leq 1$.

2.5 Confidence Level-Based p, q, r-Spherical Fuzzy Weighted Averaging Operator $C_{p,q,r}$ SFWA

Assume that $\widetilde{Q}_1, \widetilde{Q}_2, \ldots, \widetilde{Q}_n$ is a group of p,q,r-SFNs, with corresponding confidence values ψ_i such that $0 \leq \psi_i \leq 1$. Consider $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be the weights for these p,q,r-SFNs, where each $\xi_j \in [0,1]$, and $\sum \xi_j = 1$ for all $j = 1,2,\ldots,n$. Mapping of confidence based p,q,r-spherical fuzzy

weighted averaging (Cp, q, rSFWA) operator [28] is $C_{p,q,r}SFWA : \triangle^n \to \triangle$ and can be described as:

$$C_{p,q,r}SFWA = C\left(\widetilde{Q}_1, \psi_1\right), C\left(\widetilde{Q}_2, \psi_2\right), \ldots, C\left(\widetilde{Q}_n, \psi_n\right)$$
$$= \xi_1 C\left(\psi_1 \widetilde{Q}_1\right) \oplus \xi_2 C\left(\psi_2 \widetilde{Q}_2\right) \oplus \ldots \oplus \xi_n C\left(\psi_n \widetilde{Q}_n\right)$$

Theorem 1: Suppose that $\widetilde{Q}_i = (\mathfrak{k}_i, \mathfrak{z}_i, \mathfrak{r}_i)(i = 1, 2, \ldots, n)$ be a set of np, q, r-SFNs and ψ_i the their confidence levels, then the aggregated value attained by $C_{p,q,r}$SFWA operator is a p, q, r-SFN and can be expressed as given below:

$$C_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_n\left(\widetilde{Q}_n, \psi_n\right)\right) = \bigoplus_{i=1}^{n} \xi_i(\psi_i \widetilde{Q}_i).$$

$$C_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_n\left(\widetilde{Q}_n, \psi_n\right)\right)$$
$$= \left(\sqrt[p]{1 - \left(1 - \prod_{i=1}^{n} \mathfrak{k}_i^p\right)^{\psi_i \xi_i}}, \prod_{i=1}^{n} (\mathfrak{z}_i)^{\psi_i \xi_i}, \prod_{i=1}^{n} (\mathfrak{r}_i)^{\psi_i \xi_i}\right) \tag{10}$$

where $\xi_i(i = 1, 2, \ldots, n)$ is the weight of p, q, r-SFN $\widetilde{Q}_i$ satisfying the conditions $0 \leq \xi_i \leq 1$ and $\sum(\xi_i) = 1$ for all i.

2.6 Confidence Level-Based p, q, r-Spherical Fuzzy Weighted Geometric Operator $C_{p,q,r}$ SFWG

Consider that a group of p, q, r-SFNs $\widetilde{Q}_1, \widetilde{Q}_2, \ldots, \widetilde{Q}_n$, with regarding confidence values ψ_i such that $0 \leq \psi_i \leq 1$. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be the weights for these p, q, r-SFNs, where each $\xi_j \in [0, 1]$, and $\sum \xi_j = 1$ for all $j = 1, 2, \ldots, n$. Confidence levels-based mapping of p, q, r-spherical fuzzy weighted averaging (Cp, q, rSFWG) operator [28] is $C_{p,q,r}$ SFWG: $\triangle^n \to \triangle$ and can be expressed as given below:

$$C_{p,q,r}SFWG\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_n\left(\widetilde{Q}_n, \psi_n\right)\right) = \bigotimes_{i=1}^{n} (\widetilde{Q}_i^{\psi_i})^{\xi_i}$$

Theorem 2: Suppose that $\widetilde{Q}_i = (\mathfrak{k}_i, \mathfrak{z}_i, \mathfrak{r}_i)(i = 1, 2, \ldots, n)$ be a set np, q, r-SFNs and ψ_i the corresponding confidence levels then the aggregated value obtained by $C_{p,q,r}$-SFWG operator is also p, q, r-SFN and can be expressed as follows:

$$C_{p,q,r}SFWG\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_n\left(\widetilde{Q}_n, \psi_n\right)\right) = \bigotimes_{i=1}^{n} (\widetilde{Q}_i^{\psi_i})^{\xi_i}$$
$$= \left(\prod_{i=1}^{n} (\mathfrak{k}_i^{\psi_i})^{\xi_i}, \prod_{i=1}^{n} (\mathfrak{z}_i^{\psi_i})^{\xi_i}, \sqrt[q]{1 - \left(1 - \prod_{i=1}^{n} \mathfrak{r}_i^q\right)^{\psi_i \xi_i}}\right) \tag{11}$$

3 Proposed Aggregation Operators

In the following section, we introduce aggregation operators that combine the fundamental operational laws outlined below with the concept of confidence levels.

3.1 *Fundamental Arithmetic Operations*

Addition Operation

Let $\tilde{A}_s$ and $\tilde{B}_s$ be two IV$p,q,r$-SFS. Then addition operator of these two IV$p,q,r$-SFS can be define as follows:

$$\tilde{A}_s \oplus \tilde{B}_s = \{\tilde{\mathfrak{k}}, \tilde{\mathfrak{z}}, \tilde{\mathfrak{r}}\}$$

$$\tilde{\mathfrak{k}} = \left[\sqrt[p]{\left(\mathfrak{k}_A^l\right)^p + \left(\mathfrak{k}_B^l\right)^p - \left(\mathfrak{k}_A^l\right)^p \left(\mathfrak{k}_B^l\right)^p}, \sqrt[p]{\left(\mathfrak{k}_A^u\right)^p + \left(\mathfrak{k}_B^u\right)^p - \left(\mathfrak{k}_A^u\right)^p \left(\mathfrak{k}_B^u\right)^p}\right]$$

$$\tilde{\mathfrak{z}} = \left[\mathfrak{z}_A^l \mathfrak{z}_B^l, \mathfrak{z}_A^u \mathfrak{z}_B^u\right] \qquad \tilde{\mathfrak{r}} = \left[\mathfrak{r}_A^l \mathfrak{r}_B^l, \mathfrak{r}_A^u \mathfrak{r}_B^u\right]$$

Multiplication Operation

$$\tilde{A}_s \otimes \tilde{B}_s = \{\tilde{\mathfrak{k}}, \tilde{\mathfrak{z}}, \tilde{\mathfrak{r}}\}$$

$$\tilde{\mathfrak{k}} = \left[\mathfrak{k}_A^l \mathfrak{k}_B^l, \mathfrak{k}_A^u \mathfrak{k}_B^u\right] \qquad \tilde{\mathfrak{r}} = \left[\mathfrak{r}_A^l \mathfrak{r}_B^l, \mathfrak{r}_A^u \mathfrak{r}_B^u\right]$$

$$\tilde{\mathfrak{r}} = \left[\sqrt[q]{\left(\mathfrak{r}_A^l\right)^q + \left(\mathfrak{r}_B^l\right)^q - \left(\mathfrak{r}_A^l\right)^q \left(\mathfrak{r}_B^l\right)^q}, \sqrt[q]{\left(\mathfrak{r}_A^u\right)^q + \left(\mathfrak{r}_B^u\right)^q - \left(\mathfrak{r}_A^u\right)^q \left(\mathfrak{r}_B^u\right)^q}\right]$$

Multiplication by a Scalar Number $\lambda > 0$

$$\lambda \cdot \tilde{A}_s = \{\tilde{\mathfrak{k}}, \tilde{\mathfrak{z}}, \tilde{\mathfrak{r}}\}$$

$$\{\tilde{\mathfrak{k}} = \left[\sqrt[p]{\left(1 - \left(1 - \left(\mathfrak{k}_A^l\right)^p\right)\right)^\lambda}, \sqrt[p]{\left(1 - \left(1 - \left(\mathfrak{k}_A^u\right)^p\right)\right)^\lambda}\right]$$

$$\tilde{\mathfrak{z}} = \left[\left(\mathfrak{z}_A^l\right)^\lambda, \left(\mathfrak{z}_A^u\right)^\lambda\right] \qquad \left[\left(\mathfrak{r}_A^l\right)^\lambda, \left(\mathfrak{r}_A^u\right)^\lambda\right]$$

Scalar Power $\lambda > 0$

$$\tilde{A}_s^\lambda = \{\tilde{\mathfrak{k}}, \tilde{\mathfrak{z}}, \tilde{\mathfrak{r}}\}$$

$$\tilde{\mathfrak{k}} = \left[\left(\mathfrak{k}_A^l\right)^\lambda, \left(\mathfrak{k}_A^u\right)^\lambda\right] \qquad \tilde{\mathfrak{z}} = \left[\left(\mathfrak{z}_A^l\right)^\lambda, \left(\mathfrak{z}_A^u\right)^\lambda\right]$$

$$\tilde{\mathfrak{r}} = \left[\sqrt[q]{\left(1 - \left(1 - \mathfrak{r}_A^l\right)^q\right)^\lambda}, \sqrt[q]{\left(1 - \left(1 - \mathfrak{r}_A^u\right)^q\right)^\lambda}\right]$$

3.2 *Confidence Level-Based Interval-Valued p, q, r-Spherical Fuzzy Weighted Averaging Operator*

Assume that $\widetilde{Q}_1, \widetilde{Q}_2, \ldots, \widetilde{Q}_n$ is a group of interval-valued p,q,r-SFNs, with regarding confidence values ψ_i such that $0 \leq \psi_i \leq 1$. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be the weights for these interval-valued p,q,r-SFNs, where each $\xi_j \in [0,1]$, and $\sum \xi_j = 1$ for all $j = 1, 2, \ldots, n$. A confidence based mapping of interval-valued p,q,r-spherical fuzzy weighted averaging ($CIV_{p,q,r}$SFWA) operator is $CIV_{p,q,r}SFWA : \triangle^n \to \triangle$ and can be defined as:

$$CIV_{p,q,r}SFWA = C\left(\widetilde{Q}_1, \psi_1\right), C\left(\widetilde{Q}_2, \psi_2\right), \ldots, C\left(\widetilde{Q}_n, \psi_n\right)$$

$$= \xi_1 C\left(\psi_1 \widetilde{Q}_1\right) \oplus \xi_2 C\left(\psi_2 \widetilde{Q}_2\right) \oplus \ldots \oplus \xi_n C\left(\psi_n \widetilde{Q}_n\right)$$

Theorem 3: Suppose that $\widetilde{Q}_i = (\mathfrak{k}_i, \mathfrak{z}_i, \mathfrak{r}_i)(i = 1, 2, \ldots, n)$ be a set of n IV$p, q, r$-SFNs and ψ_i the corresponding confidence levels, then the aggregated value achieved by $CIV_{p,q,r}$SFWA operator is also IV$p, q, r$-SFN and expressed as follows:

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_n\left(\widetilde{Q}_n, \psi_n\right)\right) = \bigoplus_{i=1}^{n} \xi_i(\psi_i \widetilde{Q}_i).$$

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_n\left(\widetilde{Q}_n, \psi_n\right)\right) = \left(\widetilde{\mathcal{U}}, \widetilde{\mathcal{W}}, \widetilde{\mathcal{V}}\right) \tag{12}$$

and

$$\widetilde{\mathcal{U}} = \left[\sqrt[p]{1 - \left(1 - \prod_{i=1}^{n} (\mathfrak{k}_i^l)^p\right)^{\psi_i \xi_i}}, \sqrt[p]{1 - \left(1 - \prod_{i=1}^{n} (\mathfrak{k}_i^u)^p\right)^{\psi_i \xi_i}}\right]$$

$$\widetilde{\mathcal{W}} = \left[\prod_{i=1}^{n} (\mathfrak{z}_i^l)^{\psi_i \xi_i}, \prod_{i=1}^{n} (\mathfrak{z}_i^u)^{\psi_i \xi_i}\right]$$

$$\widetilde{\mathcal{V}} = \left[\prod_{i=1}^{n} (\mathfrak{r}_i)^{\psi_i \xi_i}, \prod_{i=1}^{n} (\mathfrak{r}_i)^{\psi_i \xi_i}\right]$$

where $\xi_i(i = 1, 2, \ldots, n)$ is the weight of IV$p, q, r$-SFN $\widetilde{Q}_i$ satisfying the conditions $0 \leq \xi_i \leq 1$ and $\sum(\xi_i) = 1$ for all i.

Proof: The result above was obtained by the induction method, which relied on the value of n.

Step 1. For $n = 2$, we have

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right)\right) = \xi_1\left(\psi_1 \widetilde{Q}_1\right) \oplus \xi_2\left(\psi_2 \widetilde{Q}_2\right).$$

By definition of scalar multiplication,

$$\xi \widetilde{Q} =$$

$$\left(\left[\sqrt[p]{1 - (1 - (\mathfrak{k}^l)^p)^{\xi}}, \sqrt[p]{1 - (1 - (\mathfrak{k}^u)^p)^{\xi}}\right], \left[(\mathfrak{z}^l)^{\xi}, (\mathfrak{z}^u)^{\xi}\right], \left[(\mathfrak{r}^l)^{\xi}, (\mathfrak{r}^u)^{\xi}\right]\right)$$

Thus, we have

$$\xi_1\left(\psi_1 \widetilde{Q}_1\right) =$$

$$\left(\left[\sqrt[p]{1 - (1 - (\mathfrak{k}_1^l)^p)^{\xi}}, \sqrt[p]{1 - (1 - (\mathfrak{k}_1^u)^p)^{\xi}}\right], \left[(\mathfrak{z}_1^l)^{\xi}, (\mathfrak{z}_1^u)^{\xi}\right], \left[(\mathfrak{r}_1^l)^{\xi}, (\mathfrak{r}_1^u)^{\xi}\right]\right)$$

and

$$\xi_2\left(\psi_2 \widetilde{Q}_2\right) =$$

$$\left(\left[\sqrt[p]{1 - (1 - (\mathfrak{k}_2^l)^p)^{\xi}}, \sqrt[p]{1 - (1 - (\mathfrak{k}_2^u)^p)^{\xi}}\right], \left[(\mathfrak{z}_2^l)^{\xi}, (\mathfrak{z}_2^u)^{\xi}\right], \left[(\mathfrak{r}_2^l)^{\xi}, (\mathfrak{r}_2^u)^{\xi}\right]\right)$$

$$\xi_1\left(\psi_1 \widetilde{Q}_1\right) \oplus \xi_2\left(\psi_2 \widetilde{Q}_2\right)$$

$$= \left(\left[\sqrt[p]{1 - (1 - (\mathfrak{k}_1^l)^p)^{\xi}}, \sqrt[p]{1 - (1 - (\mathfrak{k}_1^u)^p)^{\xi}}\right], \left[(\mathfrak{z}_1^l)^{\xi}, (\mathfrak{z}_1^u)^{\xi}\right], \left[(\mathfrak{r}_1^l)^{\xi}, (\mathfrak{r}_1^u)^{\xi}\right]\right)$$

$$\oplus \left(\left[\sqrt[p]{1 - (1 - (\mathfrak{k}_2^l)^p)^{\xi}}, \sqrt[p]{1 - (1 - (\mathfrak{k}_2^u)^p)^{\xi}}\right], \left[(\mathfrak{z}_2^l)^{\xi}, (\mathfrak{z}_2^u)^{\xi}\right], \left[(\mathfrak{r}_2^l)^{\xi}, (\mathfrak{r}_2^u)^{\xi}\right]\right)$$

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1,\psi_1\right),\xi_2\left(\widetilde{Q}_2,\psi_2\right)\right)=\left(\widetilde{\mathcal{M}},\widetilde{\mathcal{N}},\widetilde{\mathcal{O}}\right)$$

and

$$\widetilde{\mathcal{M}}=\left[\sqrt[p]{1-\left(1-\prod_{i=1}^{2}(\mathfrak{k}_i^l)^p\right)^{\psi_i\xi_i}},\sqrt[p]{1-\left(1-\prod_{i=1}^{2}(\mathfrak{k}_i^u)^p\right)^{\psi_i\xi_i}}\right]$$

$$\widetilde{\mathcal{N}}=\left[\prod_{i=1}^{2}(\mathfrak{z}_i^l)^{\psi_i\xi_i},\prod_{i=1}^{2}(\mathfrak{z}_i^u)^{\psi_i\xi_i}\right]$$

$$\widetilde{\mathcal{O}}=\left[\prod_{i=1}^{2}(\mathfrak{r}_i)^{\psi_i\xi_i},\prod_{i=1}^{2}(\mathfrak{r}_i)^{\psi_i\xi_i}\right]$$

Step 2. We assume that Eq. (12) is true for $n=k$, i.e.,

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1,\psi_1\right),\xi_2\left(\widetilde{Q}_2,\psi_2\right),\ldots,\xi_k\left(\widetilde{Q}_k,\psi_k\right)\right)=\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}},\widetilde{\mathfrak{g}}\right) \tag{13}$$

and

$$\widetilde{\mathcal{E}}=\left[\sqrt[p]{1-\left(1-\prod_{i=1}^{k}(\mathfrak{k}_i^l)^p\right)^{\psi_i\xi_i}},\sqrt[p]{1-\left(1-\prod_{i=1}^{k}(\mathfrak{k}_i^u)^p\right)^{\psi_i\xi_i}}\right]$$

$$\widetilde{\mathcal{F}}=\left[\prod_{i=1}^{k}(\mathfrak{z}_i^l)^{\psi_i\xi_i},\prod_{i=1}^{k}(\mathfrak{z}_i^u)^{\psi_i\xi_i}\right]$$

$$\widetilde{\mathfrak{g}}=\left[\prod_{i=1}^{k}(\mathfrak{r}_i)^{\psi_i\xi_i},\prod_{i=1}^{k}(\mathfrak{r}_i)^{\psi_i\xi_i}\right]$$

Step 3. In this step, we assume that Eq. (12) is valid for $n=k+1$, i.e., as we have an expression for $n=k$ in Eq. (13), and by using Eq. (13), we have

$$\xi_{k+1}(\psi_{k+1}\widetilde{Q}_{k+1})=[Z,F]$$

$$Z=\left(\sqrt[p]{1-(1-(\mathfrak{k}_{k+1}^l)^p)^{\xi}},(\mathfrak{z}_{k+1}^l)^{\xi},(\mathfrak{r}_{k+1}^l)^{\xi}\right)$$

$$F=\left(\sqrt[p]{1-(1-(\mathfrak{k}_{k+1}^u)^p)^{\xi}},(\mathfrak{z}_{k+1}^u)^{\xi},(\mathfrak{r}_{k+1}^u)^{\xi}\right)$$

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1,\psi_1\right),\xi_2\left(\widetilde{Q}_2,\psi_2\right),\ldots,\left(\widetilde{Q}_k,\psi_k\right)\right)\oplus\xi_{k+1}(\psi_{k+1}\widetilde{Q}_{k+1})$$

$$=\left(\widetilde{\mathcal{E}},\widetilde{\mathcal{F}},\widetilde{\mathfrak{g}}\right)\oplus\left(\sqrt[p]{1-(1-\mathfrak{k}_{k+1}^p)^{\xi}_{k+1}},\mathfrak{z}_{k+1}^{\xi_{k+1}},\mathfrak{r}_{k+1}^{\xi_{k+1}}\right)$$

$$=\left(\widetilde{\mathcal{E}}',\widetilde{\mathcal{F}}',\widetilde{\mathfrak{g}}'\right)$$

$$\widetilde{\mathcal{E}}'=\left[\sqrt[p]{1-\left(1-\prod_{i=1}^{k+1}(\mathfrak{k}_i^l)^p\right)^{\psi_i\xi_i}},\sqrt[p]{1-\left(1-\prod_{i=1}^{k+1}(\mathfrak{k}_i^u)^p\right)^{\psi_i\xi_i}}\right]$$

$$\widetilde{\mathcal{F}}'=\left[\prod_{i=1}^{k+1}(\mathfrak{z}_i^l)^{\psi_i\xi_i},\prod_{i=1}^{k+1}(\mathfrak{z}_i^u)^{\psi_i\xi_i}\right]$$

$$\widetilde{\mathfrak{g}}'=\left[\prod_{i=1}^{k+1}(\mathfrak{r}_i)^{\psi_i\xi_i},\prod_{i=1}^{k+1}(\mathfrak{r}_i)^{\psi_i\xi_i}\right]$$

Thus Eq. (13) is valid for $n=k+1$. □

Property 1: If $\widetilde{Q}_i = \widetilde{Q} = ((\mathfrak{k}, \mathfrak{z}, \mathfrak{r}), \psi)$ for all $i = 1, 2, \ldots, n$, i.e., $\mathfrak{k}_i = \mathfrak{k}$, $\mathfrak{z}_i = \mathfrak{z}$, $\mathfrak{r}_i = \mathfrak{r}$, and $\psi_i = \psi$, then

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_n\left(\widetilde{Q}_n, \psi_n\right)\right) = \psi\widetilde{Q}$$

Property 2: Assume that

$$\widetilde{Q}^- = \left(\min_i \xi_i\{\psi_i\{\mathfrak{k}_i^l, \mathfrak{k}_i^u\}\}, \min_i \xi_i\{\psi_i\{\mathfrak{z}_i^l, \mathfrak{z}_i^u\}\}, \max_i \xi_i\{\psi_i\{\mathfrak{r}_i^l, \mathfrak{r}_i^u\}\}\right)$$

and

$$\widetilde{Q}^+ = \left(\max_i \xi_i\{\psi_i\{\mathfrak{k}_i^l, \mathfrak{k}_i^u\}\}, \max_i \xi_i\{\psi_i\{\mathfrak{z}_i^l, \mathfrak{z}_i^u\}\}, \min_i \xi_i\{\psi_i\{\mathfrak{r}_i^l, \mathfrak{r}_i^u\}\}\right)$$

then

$$\widetilde{Q}^- \leq CIV_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_n\left(\widetilde{Q}_n, \psi_n\right)\right) \leq \widetilde{Q}^+.$$

Property 3: Suppose that $\widetilde{Q}$ and a are two different sets of IV$p, q, r$-SFNs and $\widetilde{Q}_i \leq a_i$ for any i then

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_n\left(\widetilde{Q}_n, \psi_n\right)\right)$$
$$\leq CIV_{p,q,r}SFWA\left(\xi_1\left(a_1, \psi_1\right), \xi_2\left(a_2, \psi_2\right), \ldots, \xi_n\left(a_n, \psi_n\right)\right)$$

3.3 Example

Suppose

$$\mathfrak{E} = ([0.7, 0.8], [0.4, 0.5], [0.5, 0.6])$$

and

$$\mathfrak{F} = ([0.6, 0.7], [0.3, 0.4], [0.5, 0.6])$$

be two IVSFN and $\psi_1 = 0.8$ and $\psi_2 = 0.6$, the weights of these two IVSFN are $\xi_1 = 0.58$ and $\xi_2 = 0.42$, respectively. Here p = 4, q = 4, and r = LCM(4, 4) = 4. $CIV_{p,q,r}SFWA$ operator calculation is as follows:

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\mathfrak{E}, \psi_1\right), \xi_2\left(\mathfrak{F}, \psi_2\right)\right) = \left(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{O}}\right)$$

and

$$\widetilde{\mathcal{M}} = \left[\sqrt[p]{1 - \left(1 - \prod_{i=1}^{2}(\mathfrak{k}_i^l)^p\right)^{\psi_i\xi_i}}, \sqrt[p]{1 - \left(1 - \prod_{i=1}^{2}(\mathfrak{k}_i^u)^p\right)^{\psi_i\xi_i}}\right] = [0.7451, 0.8241]$$

$$\widetilde{\mathcal{N}} = \left[\prod_{i=1}^{2}(\mathfrak{z}_i^l)^{\psi_i\xi_i}, \prod_{i=1}^{2}(\mathfrak{z}_i^u)^{\psi_i\xi_i}\right] = [0.4826, 0.5755]$$

$$\widetilde{\mathcal{O}} = \left[\prod_{i=1}^{2}(\mathfrak{r}_i)^{\psi_i\xi_i}, \prod_{i=1}^{2}(\mathfrak{r}_i)^{\psi_i\xi_i}\right] = [0.6088, 0.6937]$$

$$CIV_{p,q,r}SFWA\left(\xi_1\left(\mathfrak{E}, \psi_1\right), \xi_2\left(\mathfrak{F}, \psi_2\right)\right) = \left(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{O}}\right)$$
$$= ([0.7451, 0.8241], [0.4826, 0.5755], [0.6088, 0.6937])$$

3.4 Confidence Level-Based Interval-Valued p, q, r - Spherical Fuzzy Weighted Geometric Operator

Assume that a group of interval-valued p, q, r-SFNs $\widetilde{Q}_1, \widetilde{Q}_2, \ldots, \widetilde{Q}_n$, with corresponding confidence levels ψ_i such that $0 \leq \psi_i \leq 1$. Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be the weights for these IV$p, q, r$-SFNs, where each $\xi_j \in [0, 1]$, and $\sum \xi_j = 1$ for all $j = 1, 2, \ldots, n$. A confidence levels-based interval-valued p, q, r-spherical fuzzy weighted averaging ($CIV_{p,q,r}SFWG$) operator is a mapping $CIV_{p,q,r}SFWG : \triangle^n \to \triangle$

and can be expressed as:

$$CIV_{p,q,r}SFWG\left(\xi_1\left(\widetilde{Q}_1,\psi_1\right),\xi_2\left(\widetilde{Q}_2,\psi_2\right),\ldots,\xi_n\left(\widetilde{Q}_n,\psi_n\right)\right)=\bigotimes_{i=1}^{n}(\widetilde{Q}_i^{\psi_i})^{\xi_i}$$

Theorem 4: Suppose that $\widetilde{Q}_i = (\mathfrak{k}_i,\mathfrak{z}_i,\mathfrak{r}_i)(i = 1,2,\ldots,n)$ be a set n IV$p,q,r$-SFNs and ψ_i the corresponding confidence levels then the aggregated value obtained by $CIV_{p,q,r}$-SFWG operator is also IV$p,q,r$-SFN and can be expressed as follows:

$$CIV_{p,q,r}SFWG\left(\xi_1\left(\widetilde{Q}_1,\psi_1\right),\xi_2\left(\widetilde{Q}_2,\psi_2\right),\ldots,\xi_n\left(\widetilde{Q}_n,\psi_n\right)\right)=\bigotimes_{i=1}^{n}(\widetilde{Q}_i^{\psi_i})^{\xi_i}=(\mathfrak{B},\mathfrak{C},\mathfrak{D}) \tag{14}$$

$$\mathfrak{B}=\left[\prod_{i=1}^{n}(\mathfrak{k}_i^l)^{\psi_i\xi_i},\prod_{i=1}^{n}(\mathfrak{k}_i^u)^{\psi_i\xi_i}\right]$$

$$\mathfrak{C}=\left[\prod_{i=1}^{n}(\mathfrak{z}_i^l)^{\psi_i\xi_i},\prod_{i=1}^{n}(\mathfrak{z}_i^u)^{\psi_i\xi_i}\right]$$

$$\mathfrak{D}=\left[\sqrt[q]{1-\left(1-\prod_{i=1}^{n}(\mathfrak{r}_i^l)^q\right)^{\psi_i\xi_i}},\sqrt[q]{1-\left(1-\prod_{i=1}^{n}(\mathfrak{r}_i^u)^q\right)^{\psi_i\xi_i}}\right]$$

Proof: Based on the value of n, this result has been determined by using the induction method.

Step 1. For $n = 2$, we have

$$CIV_{p,q,r}SFWG\left(\xi_1(\widetilde{Q}_1,\psi_1),(\xi_2(\widetilde{Q}_2,\psi_2)\right)=(\widetilde{Q}_1^{\psi_1})^{\xi_1}\otimes(\widetilde{Q}_2^{\psi_2})^{\xi_2}$$

By Definition of scalar multiplication, we get

$$(\widetilde{Q}_1^{\psi_1})^{\xi_1}=(\mathfrak{A}_1,\mathfrak{B}_1,\mathfrak{C}_1)$$

$$\mathfrak{A}_1=\left[\left(\mathfrak{k}_1^l\right)^{\psi_1\xi_1},\left(\mathfrak{k}_1^u\right)^{\psi_1\xi_1}\right]\quad\mathfrak{B}_1=\left[\left(\mathfrak{z}_1^l\right)^{\psi_1\xi_1},\left(\mathfrak{z}_1^u\right)^{\psi_1\xi_1}\right]$$

$$\mathfrak{C}_1=\left[\sqrt[q]{\left(1-(1-(\mathfrak{r}_1^l)^q)\right)^{\psi_1\xi_1}},\sqrt[q]{(1-(1-(\mathfrak{r}_1^u)^q))^{\psi_1\xi_1}}\right]$$

and

$$(\widetilde{Q}_2^{\psi_2})^{\xi_2}=(\mathfrak{A}_2,\mathfrak{B}_2,\mathfrak{C}_2)$$

$$\mathfrak{A}_2=\left[\left(\mathfrak{k}_2^l\right)^{\psi_2\xi_2},\left(\mathfrak{k}_2^u\right)^{\psi_2\xi_2}\right]\quad\mathfrak{B}_2=\left[\left(\mathfrak{z}_2^l\right)^{\psi_2\xi_2},\left(\mathfrak{z}_2^u\right)^{\psi_2\xi_2}\right]$$

$$\mathfrak{C}_2=\left[\sqrt[q]{\left(1-(1-(\mathfrak{r}_2^l)^q)\right)^{\psi_2\xi_2}},\sqrt[q]{(1-(1-(\mathfrak{r}_2^u)^q))^{\psi_2\xi_2}}\right]$$

Therefore,

$$(\widetilde{Q}_1^{\psi_1})^{\xi_1} \otimes (\widetilde{Q}_2^{\psi_2})^{\xi_2}$$
$$= (\mathfrak{A}_1, \mathfrak{B}_1, \mathfrak{C}_1) \otimes (\mathfrak{A}_2, \mathfrak{B}_2, \mathfrak{C}_2)$$
$$= (\mathfrak{E}, \mathfrak{F}, \mathfrak{G})$$
$$\mathfrak{E} = \left[\prod_{i=1}^{2} (\mathfrak{k}_i^l)^{\psi_i \xi_i}, \prod_{i=1}^{2} (\mathfrak{k}_i^u)^{\psi_i \xi_i}\right]$$
$$\mathfrak{F} = \left[\prod_{i=1}^{2} (\mathfrak{z}_i^l)^{\psi_i \xi_i}, \prod_{i=1}^{2} (\mathfrak{z}_i^u)^{\psi_i \xi_i}\right]$$
$$\mathfrak{G} = \left[\sqrt[q]{1 - \left(1 - \prod_{i=1}^{2} (\mathfrak{r}_i^l)^q\right)^{\psi_i \xi_i}}, \sqrt[q]{1 - \left(1 - \prod_{i=1}^{2} (\mathfrak{r}_i^u)^q\right)^{\psi_i \xi_i}}\right]$$

Step 2. Let Eq. (14) be true for $n = k$, i.e.,

$$CIV_{p,q,r}SFWG\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_k\left(\widetilde{Q}_k, \psi_k\right)\right) = \bigotimes_{i=1}^{k} (\widetilde{Q}_i^{\psi_i})^{\xi_i} = (\mathbb{J}, \mathbb{K}, \mathbb{L}) \tag{15}$$

$$\mathbb{J} = \left[\prod_{i=1}^{k} (\mathfrak{k}_i^l)^{\psi_i \xi_i}, \prod_{i=1}^{k} (\mathfrak{k}_i^u)^{\psi_i \xi_i}\right]$$
$$\mathbb{K} = \left[\prod_{i=1}^{k} (\mathfrak{z}_i^l)^{\psi_i \xi_i}, \prod_{i=1}^{k} (\mathfrak{z}_i^u)^{\psi_i \xi_i}\right]$$
$$\mathbb{L} = \left[\sqrt[q]{1 - \left(1 - \prod_{i=1}^{k} (\mathfrak{r}_i^l)^q\right)^{\psi_i \xi_i}}, \sqrt[q]{1 - \left(1 - \prod_{i=1}^{k} (\mathfrak{r}_i^u)^q\right)^{\psi_i \xi_i}}\right]$$

Step 3. In this stage, we are assuming that Eq. (14) is correct for $n = k + 1$. By using Eq. (15), we get

$$CIV_{p,q,r}SFWG\left(\xi_1\left(\widetilde{Q}_1, \psi_1\right), \xi_2\left(\widetilde{Q}_2, \psi_2\right), \ldots, \xi_k\left(\widetilde{Q}_k, \psi_k\right)\right) = \bigotimes_{i=1}^{k} (\widetilde{Q}_i^{\psi_i})^{\xi_i}$$
$$= (\mathbb{J}, \mathbb{K}, \mathbb{L})$$
$$\mathbb{J} = \left[\prod_{i=1}^{k} (\mathfrak{k}_i^l)^{\psi_i \xi_i}, \prod_{i=1}^{k} (\mathfrak{k}_i^u)^{\psi_i \xi_i}\right]$$
$$\mathbb{K} = \left[\prod_{i=1}^{k} (\mathfrak{z}_i^l)^{\psi_i \xi_i}, \prod_{i=1}^{k} (\mathfrak{z}_i^u)^{\psi_i \xi_i}\right]$$
$$\mathbb{L} = \left[\sqrt[q]{1 - \left(1 - \prod_{i=1}^{k} (\mathfrak{r}_i^l)^q\right)^{\psi_i \xi_i}}, \sqrt[q]{1 - \left(1 - \prod_{i=1}^{k} (\mathfrak{r}_i^u)^q\right)^{\psi_i \xi_i}}\right]$$

and by using Eq. (14), we have

$$(\widetilde{Q}_{k+1}^{\psi_{k+1}})^{\xi_{k+1}} = (D, E, F)$$

$$D = \left[\left(\mathfrak{k}_{k+1}^{l}\right)^{\psi_{k+1}\xi_{k+1}}, \left(\mathfrak{k}_{k+1}^{u}\right)^{\psi_{k+1}\xi_{k+1}}\right]$$

$$E = \left[\left(\mathfrak{z}_{k+1}^{l}\right)^{\psi_{k+1}\xi_{k+1}}, \left(\mathfrak{z}_{k+1}^{u}\right)^{\psi_{k+1}\xi_{k+1}}\right]$$

$$F = \sqrt[q]{\left(1-\left(1-\left(\mathfrak{r}_{k+1}^{l}\right)^{q}\right)\right)^{\psi_{k+1}\xi_{k+1}}}, \sqrt[q]{\left(1-\left(1-\left(\mathfrak{r}_{k+1}^{u}\right)^{q}\right)\right)^{\psi_{k+1}\xi_{k+1}}}$$

This implies that

$$CIV_{p,q,r}SFWG\left(\xi_1\left(\widetilde{Q}_1,\psi_1\right),\xi_2\left(\widetilde{Q}_2,\psi_2\right),\ldots,\xi_k\left(\widetilde{Q}_k,\psi_k\right)\right)\otimes(\widetilde{Q}_{k+1}^{\psi_{k+1}})^{\xi_{k+1}}$$
$$= (\mathbb{J}', \mathbb{K}', \mathbb{L}')$$

$$\mathbb{J}' = \left[\prod_{i=1}^{k+1}(\mathfrak{k}_i^{l})^{\psi_i\xi_i}, \prod_{i=1}^{k+1}(\mathfrak{k}_i^{u})^{\psi_i\xi_i}\right]$$

$$\mathbb{K}' = \left[\prod_{i=1}^{k+1}(\mathfrak{z}_i^{l})^{\psi_i\xi_i}, \prod_{i=1}^{k+1}(\mathfrak{z}_i^{u})^{\psi_i\xi_i}\right]$$

$$\mathbb{L}' = \left[\sqrt[q]{1-\left(1-\prod_{i=1}^{k+1}(\mathfrak{r}_i^{l})^{q}\right)^{\psi_i\xi_i}}, \sqrt[q]{1-\left(1-\prod_{i=1}^{k+1}(\mathfrak{r}_i^{u})^{q}\right)^{\psi_i\xi_i}}\right]$$

Thus Eq. (15) is hold true for n = k + 1. □

Property 4: Assume that $\widetilde{Q}_i = \widetilde{Q} = ((\mathfrak{k}, \mathfrak{z}, \mathfrak{r}), \psi)$ for all $i = 1, 2, \ldots, n$, i.e., $\mathfrak{k}_i = \mathfrak{k}$, $\mathfrak{z}_i = \mathfrak{z}$, $\mathfrak{r}_i = \mathfrak{r}$, and $\psi_i = \psi$ then

$$CIV_{p,q,r}SFWG\left(\xi_1(\widetilde{Q}_1,\psi_1),(\xi_2(\widetilde{Q}_2,\psi_2),\ldots,(\xi_n(\widetilde{Q}_n,\psi_n)\right) = \widetilde{Q}^{\psi}.$$

Property 5: Suppose that

$$\widetilde{Q}^{-} = \left(min_i\xi_i\{\psi_i[\mathfrak{k}_i^{l},\mathfrak{k}_i^{u}]\}, min_i\xi_i\{\psi_i[\mathfrak{z}_i^{l},\mathfrak{z}_i^{u}]\}, max_i\xi_i\{\psi_i[\mathfrak{r}_i^{l},\mathfrak{r}_i^{u}]\}\right)$$

and

$$\widetilde{Q}^{+} = \left(max_i\xi_i\{\psi_i[\mathfrak{k}_i^{l},\mathfrak{k}_i^{u}]\}, max_i\xi_i\{\psi_i[\mathfrak{z}_i^{l},\mathfrak{z}_i^{u}]\}, min_i\xi_i\{\psi_i[\mathfrak{r}_i^{l},\mathfrak{r}_i^{u}]\}\right)$$

then

$$\widetilde{Q}^{-} \leq CIV_{p,q,r}SFWG\left(\xi_1(\widetilde{Q}_1,\psi_1),\xi_2(\widetilde{Q}_2,\psi_2),\ldots,\xi_n(\widetilde{Q}_n,\psi_n)\right) \leq \widetilde{Q}^{+}$$

Property 6: For any two IV$p, q, r$-SFNs $\widetilde{Q}_i$ and a_i, if $\widetilde{Q}_i \leq a_i$ for any $i = 1, 2, \ldots, n$, then

$$CIV_{p,q,r}SFWG\left(\xi_1(\widetilde{Q}_1,\psi_1),\xi_2(\widetilde{Q}_2,\psi_2),\ldots,\xi_n(\widetilde{Q}_n,\psi_n)\right)$$
$$\leq CIV_{p,q,r}SFWG\left(\xi_1(a_1,\psi_1),\xi_2(a_2,\psi_2),\ldots,\xi_n(a_n,\psi_n)\right)$$

3.5 Some Special Casses of the Proposed Operators

1. If we let $\mathfrak{k}^{l} = \mathfrak{k}^{u}$, $\mathfrak{z}^{l} = \mathfrak{z}^{u}$, and $\mathfrak{r}^{l} = \mathfrak{r}^{u}$, then the suggested operators become aggregation operators for confidence level based $(p, q, r - SFS)$ [28].

2. If we let $\mathfrak{k}^{l} = \mathfrak{k}^{u}$, $\mathfrak{z}^{l} = \mathfrak{z}^{u}$, $\mathfrak{r}^{l} = \mathfrak{r}^{u}$, and $\psi = 1$ then the suggested operators become aggregation operators for $(p, q, r - SFS)$ [18].

3. If we let $p = q = r$ and $\psi = 1$ then the suggested operators become aggregation operators for IVt-SFS [23].

4. If we let $p = q = r$, $\mathfrak{k}^l = \mathfrak{k}^u$, $\mathfrak{z}^l = \mathfrak{z}^u$, $\mathfrak{r}^l = \mathfrak{r}^u$, and $\psi = 1$ then the suggested operators become aggregation operators for t-SFS [16].

5. If we let $p = q = r = 2$, $\mathfrak{k}^l = \mathfrak{k}^u$, $\mathfrak{z}^l = \mathfrak{z}^u$, $\mathfrak{r}^l = \mathfrak{r}^u$, and $\psi = 1$ then the suggested operators become aggregation operators for SFS [15].

6. If we let $p = q = r$, $\mathfrak{z}^l = 0$ and $\mathfrak{x}^l = \mathfrak{z}^u$, then the suggested operators become aggregation operators for IV q-ROSFS [22].

From the discussion above, it becomes apparent that the proposed operators are highly generalized and free from limitations.

3.6 Example

Suppose $\mathfrak{E} = ([0.8, 0.9], [0.4, 0.5], [0.6, 0.7])$ and

$\mathfrak{F} = ([0.6, 0.7], [0.3, 0.4], [0.4, 0.6])$

be two IVSFN and $\psi_1 = 0.8$ and $\psi_2 = 0.6$, the weights of these two IVSFN are $\xi_1 = 0.58$ and $\xi_2 = 0.42$, respectively. Here p = 4, q = 4, and r = LCM(4, 4) = 4. $CIV_{p,q,r}SFWG$ operator calculation is as follows:

$$CIV_{p,q,r}SFWG\left(\xi_1\left(\mathfrak{E}, \psi_1\right), \xi_2\left(\mathfrak{F}, \psi_2\right)\right) = \left(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{O}}\right)$$

and

$$\widetilde{\mathcal{M}} = \left[\prod_{i=1}^{2}\left(\mathfrak{k}_i^l\right)^{\psi_i\xi_i}, \prod_{i=1}^{2}\left(\mathfrak{k}_i^u\right)^{\psi_i\xi_i}\right] = [0.7927, 0.8704]$$

$$\widetilde{\mathcal{N}} = \left[\prod_{i=1}^{2}\left(\mathfrak{z}_i^l\right)^{\psi_i\xi_i}, \prod_{i=1}^{2}\left(\mathfrak{z}_i^u\right)^{\psi_i\xi_i}\right] = [0.4826, 0.5755]$$

$$\widetilde{\mathcal{O}} = \left[\sqrt[q]{1 - \left(1 - \prod_{i=1}^{2}\left(\mathfrak{r}_i^l\right)^q\right)^{\psi_i\xi_i}}, \sqrt[q]{1 - \left(1 - \prod_{i=1}^{2}\left(\mathfrak{r}_i^u\right)^q\right)^{\psi_i\xi_i}}\right] = [0.6263, 0.7451]$$

$$CIV_{p,q,r}SFWG\left(\xi_1\left(\mathfrak{E}, \psi_1\right), \xi_2\left(\mathfrak{F}, \psi_2\right)\right) = \left(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}, \widetilde{\mathcal{O}}\right)$$

$$= ([0.7927, 0.8704], [0.4826, 0.5755], [0.6263, 0.7451])$$

3.7 Score Function and Accuracy Function

Definition of the score function of a confidence level based interval-valued p, q, r fuzzy number ($CIV_{p,q,r}SFN$), $\widetilde{Q} = (\mathfrak{k}_{\widetilde{Q}}, \mathfrak{z}_{\widetilde{Q}}, \mathfrak{r}_{\widetilde{Q}})$ is as follows:

$$Sc(\widetilde{Q}(\mathfrak{g})) = \frac{\left((\mathfrak{k}_{\widetilde{Q}}^l)^p(1 - (\mathfrak{z}_{\widetilde{Q}}^l)^r - (\mathfrak{r}_{\widetilde{Q}}^l)^q)\right)\left((\mathfrak{k}_{\widetilde{Q}}^u)^p(1 - (\mathfrak{z}_{\widetilde{Q}}^u)^r - (\mathfrak{r}_{\widetilde{Q}}^u)^q)\right)}{3} \tag{16}$$

In this context, $0 \leq Sc(\widetilde{Q}) \leq 1$.

The accuracy function of a $CIV_{p,q,r}SFN\widetilde{Q} = (\mathfrak{k}_{\widetilde{Q}}, \mathfrak{z}_{\widetilde{Q}}, \mathfrak{r}_{\widetilde{Q}})$ is defined as follows:

$$Ac(\widetilde{Q}(\mathfrak{g})) = \frac{\left((\mathfrak{k}_{\widetilde{Q}}^l)^p(1 + (\mathfrak{z}_{\widetilde{Q}}^l)^r + (\mathfrak{r}_{\widetilde{Q}}^l)^q)\right)\left((\mathfrak{k}_{\widetilde{Q}}^u)^p(1 + (\mathfrak{z}_{\widetilde{Q}}^u)^r + (\mathfrak{r}_{\widetilde{Q}}^u)^q)\right)}{3} \tag{17}$$

In this context, $0 \leq Ac(\widetilde{Q}) \leq 1$.

4 Proposed MCDM for Interval-Valued *p, q, r*-Spherical Fuzzy Information

As a robust approach to addressing complex issues, Multi-Criteria Decision-Making (MCDM) is an excellent way to take into account multiple factors at once. By combining qualitative and quantitative criteria, this method facilitates the evaluation of options comprehensively, allowing the most optimal course of action to be determined. MCDM can be applied in numerous domains, including economics, engineering, and environmental management, as a result of its adaptability. In MCDM, intricate decision scenarios are transformed into well-informed strategic decisions by leveraging mathematical models, optimization techniques, and decision analysis. An important feature of MCDM is that it offers an effective strategy for managing decision complexities, ensuring optimal outcomes whenever a variety of factors are at play. Modern decision challenges can be more effectively addressed with MCDM's innovative methodologies integrating proposed aggregation techniques.

Let's consider a decision-making scenario involving m distinct alternatives denoted as $\mathfrak{A} = \{\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_m\}$ and n criteria labeled as

$\beth = \{\beth_1, \beth_2, \ldots, \beth_n\}$. The criteria weight vector is given by: $\xi = \{\xi_1, \xi_2, \ldots, \xi_n\}$ where each $0 \leq \xi_j \leq 1$, and $\sum_{j=1}^{n}(\xi_j) = 1$. Consider a group of decision-makers $\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_k$ having the weight vector $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_k)$ where $0 \leq \zeta_e \leq 1$ and $\sum_{e=1}^{k}(\zeta_e)$, provide the rating values of alternatives using interval-valued p, q, r-spherical fuzzy numbers with confidence levels ψ_j such that $0 \leq \psi_j \leq 1$. The subsequent procedures are employed to ascertain the optimal choice from a set of feasible options.

Step 1. The decision-maker constructs a decision matrix $\widetilde{D}_{ij}^{t}$ using the evidence provided by the IVp,q,r-SFNs in order to assess the alternatives based on the provided criteria that they have provided.

$$\widetilde{D}_{ij}^{t} = \begin{bmatrix} \mathfrak{d}_{11} & \mathfrak{d}_{12} & \mathfrak{d}_{13} & \ldots & \mathfrak{d}_{1n} \\ \mathfrak{d}_{21} & \mathfrak{d}_{22} & \mathfrak{d}_{23} & \ldots & \mathfrak{d}_{2n} \\ \vdots & & & & \\ \mathfrak{d}_{m1} & \mathfrak{d}_{m2} & \mathfrak{d}_{m3} & \ldots & \mathfrak{d}_{mn} \end{bmatrix}$$

In this matrix values of $\mathfrak{d}_{ij}$ are given below and $i = (1, 2, 3, \ldots, m)$ $j = (1, 2, 3, \ldots, n)$.

$$\mathfrak{d}_{ij} = \left(\left\langle\left[(\mathfrak{k}_{ij}^{l})^{e}, (\mathfrak{k}_{ij}^{u})^{e}\right], \left[(\mathfrak{z}_{ij}^{l})^{e}, (\mathfrak{z}_{ij}^{u})^{e}\right], \left[(\mathfrak{r}_{ij}^{l})^{e}, (\mathfrak{r}_{ij}^{u})^{e}\right]\right\rangle \psi_{ij}^{e}\right)$$

Step 2. A decision matrix is established in Step 1 and each row element represents a IVp,q,r-SFN in the matrix. By using the $CIV_{p,q,r}SFWA$ or $CIV_{p,q,r}SFWG$ operators, row elements are aggregated.

Step 3. By using the aggregate operator values for each alternative, the score values will be calculated for each alternative.

Step 4. Based on the scores given to all the alternatives, rank them in order of importance.

Step 5. Choose the alternative that has the highest score value out of all the alternatives.

There is a flowchart shown in Fig. 1 that demonstrating the steps involved in the MCDM method.

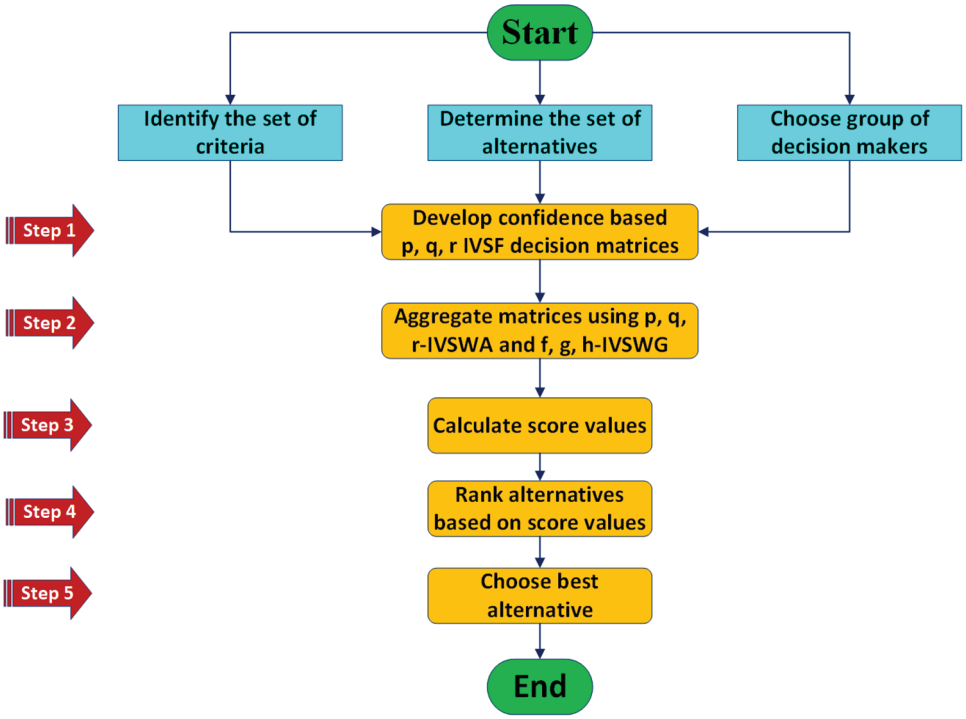

Figure 1: Decision-making flowchart of proposed method

Example

The following steps illustrate how to apply the model step-by-step, along with example calculations:

The first step is to construct the decision matrix: When we have three alternatives and three criteria (m = 3, n = 3), this decision matrix could look something like this:

$$\widetilde{D}^{t} = \begin{bmatrix} \mathfrak{d}_{11} & \mathfrak{d}_{12} & \mathfrak{d}_{13} \\ \mathfrak{d}_{21} & \mathfrak{d}_{22} & \mathfrak{d}_{23} \\ \mathfrak{d}_{31} & \mathfrak{d}_{32} & \mathfrak{d}_{33} \end{bmatrix}$$

For example, let's define:

$$\mathfrak{d}_{11} = (\langle [0.4, 0.6], [0.3, 0.4], [0.4, 0.5] \rangle 0.8)\,,$$

$$\mathfrak{d}_{12} = (\langle [0.6, 0.8], [0.5, 0.6], [0.3, 0.4] \rangle 0.6)\,,$$

$$\mathfrak{d}_{13} = (\langle [0.5, 0.6], [0.3, 0.4], [0.3, 0.5] \rangle 0.7)\,,$$

$$\mathfrak{d}_{21} = (\langle [0.6, 0.8], [0.2, 0.4], [0.3, 0.4] \rangle 0.8)\,,$$

$$\mathfrak{d}_{22} = (\langle [0.4, 0.6], [0.3, 0.4], [0.4, 0.5] \rangle 0.6)\,,$$

$$\mathfrak{d}_{23} = (\langle [0.6, 0.8], [0.2, 0.4], [0.3, 0.4] \rangle 0.7)\,,$$

$$\mathfrak{d}_{31} = (\langle [0.6, 0.8], [0.5, 0.6], [0.3, 0.4] \rangle 0.8)\,,$$

$$\mathfrak{d}_{32} = (\langle [0.5, 0.6], [0.3, 0.4], [0.3, 0.5] \rangle 0.6)\,,$$

$\mathfrak{d}_{33} = (\langle [0.4, 0.6], [0.3, 0.4], [0.4, 0.5] \rangle 0.7)$.

The second step is to aggregate the row elements by using $CIV_{p,q,r}SFWA$ operator (**Theorem 3**) or $CIV_{p,q,r}SFWG$ operator (**Theorem 4**).

Here the row elements are aggregated using the $CIV_{p,q,r}SFWA$ operator (**Theorem 3**):

$$\mathfrak{A} = CIV_{p,q,r}SFWA\,(\mathfrak{d}_{11}, \mathfrak{d}_{12}, \mathfrak{d}_{13})$$
$$= ([0.5861, 0.7295], [0.4624, 0.5575], [0.4653, 0.5880])$$

$$\mathfrak{H} = CIV_{p,q,r}SFWA\,(\mathfrak{d}_{21}, \mathfrak{d}_{22}, \mathfrak{d}_{23})$$
$$= ([0.6493, 0.8128], [0.3403, 0.5208], [0.4453, 0.5407])$$

$$\mathfrak{Y} = CIV_{p,q,r}SFWA\,(\mathfrak{d}_{31}, \mathfrak{d}_{32}, \mathfrak{d}_{33})$$
$$= ([0.6156, 0.7621], [0.4997, 0.5929], [0.4526, 0.5684])$$

The third step is to calculate the score function values by using 3.7:

$$Sc(\mathfrak{A}) = 0.1097$$
$$Sc(\mathfrak{H}) = 0.1785$$
$$Sc(\mathfrak{Y}) = 0.1297$$

Ranking the alternatives according to score function values:

$$Sc(\mathfrak{H}) > Sc(\mathfrak{Y}) > Sc(\mathfrak{A})$$

The alternative with the highest score should be chosen. Based on the provided criteria and scores, $Sc(\mathfrak{H})$ is the best choice.

5 Application of IV *p,q,r–SFS* Operators to Industrial Waste Management

Pakistan Steel Mills Corporation (PkMC), commonly recognized as Pak Steels, is the country's leading manufacturer of long-rolled steel and heavy metal products. With a production capability ranging from 1.1 to 5.0 million tonnes annually, PkMC stands as Pakistan's biggest industrial conglomerate. Established in the 1970s with major technical and financial support from the Soviet Union, the facility was equipped with approximately 330,000 t of industrial-grade machinery, structural steel, and electrical systems. Given its industrial scale and environmental footprint, the Environmental Protection Agency (EPA), under the Ministry of Environment, enforces strict compliance with environmental regulations. A dedicated system oversees occupational safety and the maintenance of an appropriate work setting. Beyond its manufacturing operations, PkMC has also launched several initiatives aimed at improving local climatic conditions.

At PkMC, steel is manufactured using iron ore via blast furnaces (BF) and basic oxygen furnaces (BOF), or from recycled scrap through electric arc furnaces (EAF). In BFs, coke reduces iron ore to molten iron; in BOFs, oxygen decarburizes this molten iron to produce steel. Fluxes are added during melting to bind impurities, forming slag that's removed and air-cooled into a crystalline form. Refining occurs in ladles, where alloying elements are introduced, gases reduced, and impurities eliminated to improve steel quality. The refined molten steel is adjusted to the target composition and temperature,

then cast into billets, beam blanks, or other forms. These are finally shaped into finished products in rolling mills. Steel production is energy-intensive and leads to significant environmental issues through emissions and industrial waste. Despite being the most recyclable material, steel manufacturing still generates unsustainable material flows and requires effective waste management strategies. Main waste sources include coke ovens, sinter plants, blast furnaces, BOFs, melting shops, and rolling mills. Typical solid wastes are slag, flue dust, mill scale, sludge, and damaged refractory materials. At PkMC, these processes generate large quantities of solid waste.

Open dumping of industrial waste on land leads to environmental issues like dust and leachate contamination. With rapid population growth, land availability for waste disposal is shrinking, highlighting the need for effective waste management strategies. PkMC generates significantly more waste, about one tonne per tonne of steel, compared to the industry average of 0.55 t, largely due to low-quality raw materials. Given rising material and energy shortages, strict environmental regulations, and disposal costs, industrial waste should be viewed as a potential resource within the steel industry. Various economically viable procedures are employed to manage industrial waste effectively. However, due to the complexity, uncertainty, and vagueness of decision-making, selecting the optimal industrial waste management strategy (IWMS) remains a challenge for PkMC. The proposed framework is implemented to identify the most suitable IWMS for PkMC solid waste.

We utilized artificial data for our illustrated example. We selected three experts who evaluated alternatives by considering four criteria. Environmental scientists investigate the sources and impacts of industrial waste and develop strategies to minimize its environmental footprint, including sustainable waste treatment and recycling solutions. Finance professionals are vital to mobilizing resources for waste management initiatives. They design funding mechanisms and financial models to support investments in cleaner production technologies and efficient waste handling systems. Engineers contribute by designing and implementing advanced technologies for waste reduction, recycling, and safe disposal, all of which are essential for effective industrial waste management.

5.1 Description of Criteria

Industrial Waste Management Strategies (IWMS) are evaluated based on several aspects. To rank these strategies, we need to consider several key aspects.

5.1.1 Technology ($\mathfrak{c}_1$)

The technical aspects play a vital role in evaluating various waste treatment methods. This includes factors such as the location of treatment facilities, the availability of skilled personnel, and the operational efficiency of the technologies used. Management is also responsible for assessing the capacity and long-term economic viability of the selected industrial waste management approach.

5.1.2 Cost ($\mathfrak{c}_2$)

Cost is a critical aspect of industrial waste management, encompassing all the expenditures involved in planning and implementing waste management strategies. In developing nations such as Pakistan, financial constraints often make cost the top priority in decision-making. Moreover, the effectiveness of any waste management approach also depends on the public's willingness and capacity to support and adopt the chosen methods.

5.1.3 *Environmental Criteria ($\mathfrak{c}_3$)*

Ineffective waste management leads to material shortages, pollution, environmental destruction, and health problems for both animals and the public. Poor waste management significantly harms our ecological system.

5.1.4 *Time ($\mathfrak{c}_4$)*

This measurement focuses on quantifying the time required to accomplish an industrial waste management activity and the processing duration for waste. It helps estimate the volume of waste to be stored and the necessary transit time during waste handling.

5.2 *Description of Strategies*

Below is a description of waste management strategies.

5.2.1 *Bio-Reprocessing ($\mathfrak{A}$)*

This methodology involves the decomposition of organic waste, such as food scraps, plant matter, and paper materials. It is possible to recover organic materials using composting and digestion methods. Biological reprocessing accelerates the spontaneous breakdown of these materials. Additionally, the methane gas generated during the process can be captured and converted into heat and energy.

5.2.2 *Recycling ($\mathfrak{H}$)*

Recycling is a resource recovery method that involves the collection and reuse of waste materials. In this process, materials that would typically be disposed of are broken down, processed, and repurposed for new uses. This reduces the need for raw material extraction and minimizing waste.

5.2.3 *Energy Recovery ($\mathfrak{Y}$)*

This approach involves converting non-recyclable waste substances into useful heat, electric power, or energy, a process commonly referred to as waste-to-energy. Energy recovery from waste is an essential component of non-hazardous waste management hierarchy.

5.2.4 *Pyrolysis ($\mathfrak{V}$)*

Pyrolysis is a process that transforms various residential and industrial waste materials into valuable recovered products. By utilizing different waste inputs, pyrolysis can generate sustainable alternatives to fossil fuels. During slow pyrolysis, gases and solid charcoal are generated as products.

5.3 *Computation of Proposed Method*

To resolve this industrial waste management MCDM problem, we define the alternatives $\mathfrak{A}, \mathfrak{H}, \mathfrak{YV}$ and criteria $\mathfrak{c}_j$ (j = 1, 2, 3, 4) are expressed using IVp,q,r-SFN, we choose p = 4, q = 4, and r = 4 (p, q, r $\geq$ 1). As choosing p = q = r = 1, for Table 4, elements of row **H** against $\mathfrak{c}_2$ would not meet requirements of Karaaslan and Karamaz study [24] $((\mathbf{k}^u)^p + (\mathfrak{z}^u)^q + (\mathfrak{r}^u)^r \leq 1)$. $(0.6)^1 + (0.4)^1 + (0.7)^1 = 1.7 \nleq 1$ and even $(0.6)^2 + (0.4)^2 + (0.7)^2 = 1.01 \nleq 1$ not correct. So, we choose p = q = r = 4 to meet the requirements and help us to avoid errors in calculations.

According to professionals, the rating values of alternatives for each criterion, as provided by them, are assumed to be described by IV$p,q,r$-SFN.

Step 1. The information provided by these four professionals about the alternatives with respect to the criteria is listed in Tables 4–6.

Step 2. The selection of the best industrial waste management alternative is done using the $CIV_{p,q,r}$SFWA and $CIV_{p,q,r}$SFWG operators.

Step 3. By applying the $CIV_{p,q,r}$SFWA operator, we get Table 7. By applying the $CIV_{p,q,r}$SFWG operator, we get Table 8.

Step 4. Ranking of waste management strategies based on the score values is given below and a graphical presentation can be seen in Fig. 2.

$$CIV_{p,q,r}SFWA = Sc(\mathfrak{Y}) > Sc(\mathfrak{A}) > Sc(\mathfrak{V}) > Sc(\mathfrak{H})$$

$$CIV_{p,q,r}SFWG = Sc(\mathfrak{Y}) > Sc(\mathfrak{V}) > Sc(\mathfrak{A}) > Sc(\mathfrak{H})$$

Step 5. Energy recovery ($\mathfrak{Y}$) is the best option in both cases.

Table 4: DM_1's pairwise comparison matrix

	$\mathfrak{c}_1$							$\mathfrak{c}_2$						
	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ
$\mathfrak{A}$	0.30	0.40	0.30	0.50	0.20	0.30	0.8	0.40	0.50	0.30	0.50	0.40	0.60	0.8
$\mathfrak{H}$	0.20	0.30	0.40	0.50	0.40	0.60	0.6	0.40	0.60	0.30	0.40	0.60	0.70	0.6
$\mathfrak{Y}$	0.50	0.60	0.30	0.40	0.30	0.50	0.8	0.30	0.40	0.40	0.60	0.20	0.30	0.8
$\mathfrak{V}$	0.40	0.60	0.30	0.40	0.60	0.70	0.7	0.50	0.60	0.30	0.50	0.40	0.50	0.7

	$\mathfrak{c}_3$							$\mathfrak{c}_4$						
	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ
$\mathfrak{A}$	0.30	0.40	0.40	0.50	0.20	0.30	0.8	0.60	0.70	0.30	0.40	0.60	0.70	0.8
$\mathfrak{H}$	0.40	0.60	0.30	0.40	0.60	0.70	0.6	0.40	0.60	0.30	0.40	0.60	0.70	0.6
$\mathfrak{Y}$	0.50	0.60	0.40	0.50	0.40	0.60	0.8	0.50	0.60	0.40	0.50	0.40	0.60	0.8
$\mathfrak{V}$	0.50	0.60	0.30	0.40	0.30	0.50	0.7	0.40	0.60	0.30	0.40	0.60	0.70	0.7

Table 5: DM_2's pairwise comparison matrix

	$\mathfrak{c}_1$							$\mathfrak{c}_2$						
	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ
$\mathfrak{A}$	0.30	0.40	0.40	0.50	0.30	0.40	0.8	0.60	0.70	0.40	0.50	0.20	0.30	0.8
$\mathfrak{H}$	0.40	0.60	0.30	0.40	0.60	0.70	0.6	0.60	0.70	0.30	0.40	0.60	0.70	0.6
$\mathfrak{Y}$	0.50	0.60	0.40	0.50	0.30	0.40	0.8	0.50	0.60	0.50	0.60	0.40	0.60	0.8
$\mathfrak{V}$	0.50	0.60	0.30	0.40	0.30	0.50	0.7	0.40	0.60	0.30	0.40	0.60	0.70	0.7

	$\mathfrak{c}_3$							$\mathfrak{c}_4$						
	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ
$\mathfrak{A}$	0.50	0.70	0.30	0.50	0.40	0.60	0.8	0.30	0.40	0.40	0.50	0.20	0.30	0.8
$\mathfrak{H}$	0.40	0.50	0.20	0.30	0.30	0.40	0.6	0.40	0.60	0.30	0.40	0.60	0.70	0.6
$\mathfrak{Y}$	0.50	0.60	0.30	0.40	0.20	0.30	0.8	0.60	0.70	0.30	0.40	0.60	0.70	0.8
$\mathfrak{V}$	0.50	0.60	0.30	0.40	0.30	0.50	0.7	0.50	0.60	0.30	0.50	0.40	0.50	0.7

Table 6: DM_3's pairwise comparison matrix

	$\mathfrak{c}_1$							$\mathfrak{c}_2$						
	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ
$\mathfrak{A}$	0.50	0.60	0.40	0.60	0.20	0.30	0.8	0.40	0.50	0.30	0.50	0.40	0.60	0.8
$\mathfrak{H}$	0.40	0.60	0.30	0.50	0.40	0.50	0.6	0.40	0.60	0.30	0.40	0.60	0.70	0.6
$\mathfrak{Y}$	0.60	0.70	0.30	0.60	0.30	0.40	0.8	0.30	0.40	0.40	0.50	0.20	0.30	0.8
$\mathfrak{V}$	0.50	0.60	0.30	0.40	0.30	0.50	0.7	0.40	0.60	0.30	0.40	0.60	0.70	0.7

	$\mathfrak{c}_3$							$\mathfrak{c}_4$						
	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	ψ
$\mathfrak{A}$	0.60	0.70	0.30	0.50	0.40	0.60	0.8	0.60	0.70	0.30	0.50	0.40	0.60	0.8
$\mathfrak{H}$	0.50	0.60	0.30	0.40	0.20	0.30	0.6	0.40	0.60	0.30	0.40	0.60	0.70	0.6
$\mathfrak{Y}$	0.70	0.80	0.40	0.50	0.30	0.40	0.7	0.50	0.60	0.40	0.50	0.40	0.60	0.7
$\mathfrak{V}$	0.50	0.60	0.30	0.50	0.40	0.50	0.7	0.50	0.60	0.30	0.40	0.30	0.50	0.7

Table 7: $CIV_{p,q,r}$SFWA operator score values and ranking

Alternatives	MB		NMB		ID				
	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	Score	N. Score	Rank
$\mathfrak{A}$	0.5973	0.6988	0.0506	0.1429	0.0386	0.1035	0.1219	0.2508	2
$\mathfrak{H}$	0.5551	0.6787	0.0343	0.0809	0.1509	0.2596	0.1021	0.2100	4
$\mathfrak{Y}$	0.6354	0.7188	0.0638	0.1400	0.0535	0.1341	0.1433	0.2949	1
$\mathfrak{V}$	0.5875	0.6979	0.0337	0.0899	0.0836	0.1953	0.1187	0.2443	3

Table 8: $CIV_{p,q,r}$SFWG operator score values and ranking

Alternatives	MB		NMB		ID				
	$\mathfrak{k}^l$	$\mathfrak{k}^u$	$\mathfrak{z}^l$	$\mathfrak{z}^u$	$\mathfrak{r}^l$	$\mathfrak{r}^u$	Score	N. Score	Rank
$\mathfrak{A}$	0.1228	0.2250	0.0506	0.1429	0.4432	0.5672	0.0008	0.2323	3
$\mathfrak{H}$	0.0872	0.2122	0.0343	0.0809	0.6233	0.7138	0.0005	0.14 32	4
$\mathfrak{Y}$	0.1526	0.2524	0.0638	0.1400	0.4809	0.6051	0.0013	0.3720	1
$\mathfrak{V}$	0.1191	0.2373	0.0337	0.0899	0.5378	0.6647	0.0009	0.2526	2

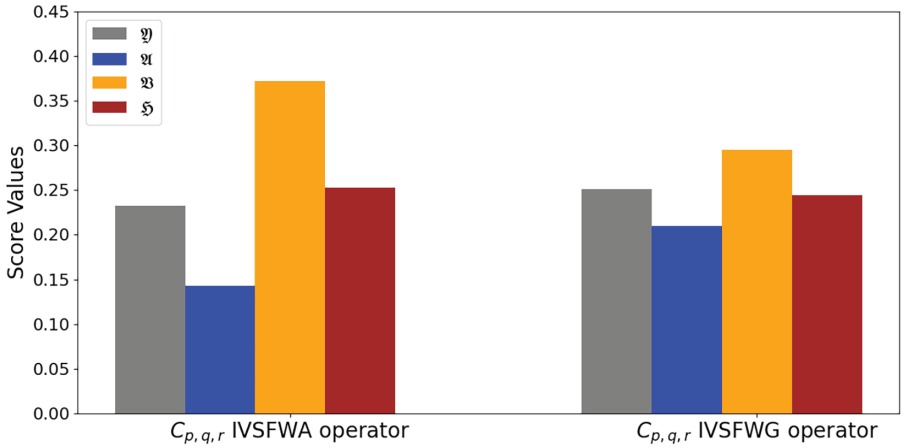

Figure 2: Ranking of industrial waste management strategies

5.4 Sensitivity Analysis

The investigation assessed the impact of each parameter (p, q, and r) on the ultimate ranking of the alternatives. Table 9 presents the resulting score values and the corresponding ranking order. As we discussed in Section 5.3, the choice of p = q = r = 1, for Table 4, elements of row $\mathfrak{H}$ against $\mathfrak{c}_2$ would not meet requirements of Karaaslan and Karamaz study [24] $((\mathfrak{k}^u)^p + (\mathfrak{z}^u)^q + (\mathfrak{r}^u)^r \leq 1)$. $(0.6)^1 + (0.4)^1 + (0.7)^1 = 1.7 \nleq 1$ and even $(0.6)^2 + (0.4)^2 + (0.7)^2 = 1.01 \nleq 1$ are not correct. When we set a new arrangement of parameters, we got $(0.6)^1 + (0.4)^4 + (0.7)^4 = 0.8657 \leq 1$ that is correct. In this scenario, the $CIV_{p,q,r}$ SFWA operator was utilized, with the parameter set at q = 4, while varying

p from 1 to 10, and r = LCM(p, q). Throughout the results, an optimal choice is energy recovery $\mathfrak{Y}$ and it shows the authenticity and reliability of the developed model. The graphical presentation of the alternative ranking is shown in Fig. 3. Similarly, Table 10 presents the resulting score values and the corresponding ranking order. In this scenario, the operator $CIV_{p,q,r}$ SFWG was utilized, with the parameter set at q = 4, while varying p from 1 to 10, and r = LCM(p, q). Throughout the results, an optimal choice is energy recovery $\mathfrak{Y}$ and it shows the authenticity and reliability of the developed model. The graphical presentation of alternative ranking is shown in Fig. 4.

Table 9: Sensitivity analysis based on $CIV_{p,q,r}$ SFWA operator

Parameters			Score values				
p	q	r	$\mathfrak{A}$	$\mathfrak{H}$	$\mathfrak{Y}$	$\mathfrak{V}$	Optimal choice
1	4	4	0.2509	0.2383	0.2621	0.2487	$\mathfrak{Y}$
2	4	4	0.2512	0.2279	0.2736	0.2473	$\mathfrak{Y}$
3	4	12	0.2509	0.2190	0.2843	0.2457	$\mathfrak{Y}$
4	4	4	0.2508	0.2100	0.2949	0.2443	$\mathfrak{Y}$
5	4	20	0.2500	0.2028	0.3044	0.2428	$\mathfrak{Y}$
6	4	12	0.2492	0.1956	0.3138	0.2413	$\mathfrak{Y}$
7	4	28	0.2483	0.1891	0.3226	0.2400	$\mathfrak{Y}$
8	4	8	0.2473	0.1830	0.3311	0.2386	$\mathfrak{Y}$
9	4	36	0.2462	0.1772	0.3393	0.2373	$\mathfrak{Y}$
10	4	20	0.2450	0.1718	0.3472	0.2360	$\mathfrak{Y}$

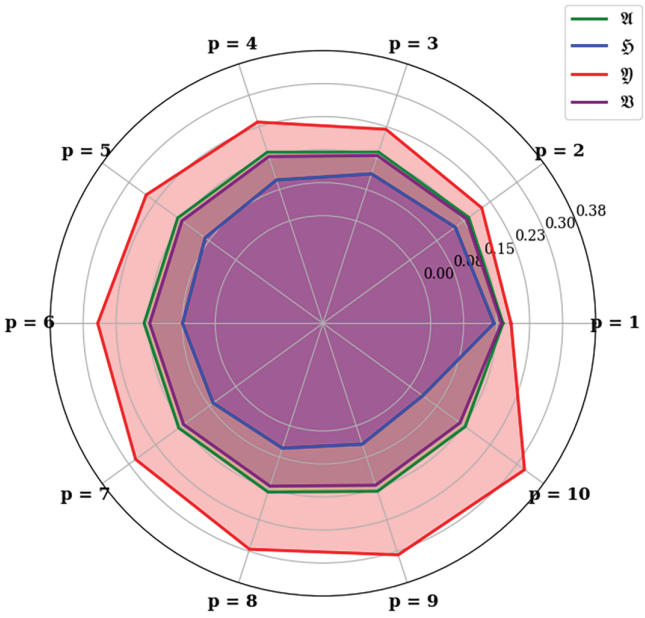

Figure 3: Sensitivity graph of $CIV_{p,q,r}$ SFWA operator

Table 10: Sensitivity analysis based on $CIV_{p,q,r}$ SFWG operator

Parameters			Score values				
p	q	r	$\mathfrak{A}$	$\mathfrak{H}$	$\mathfrak{Y}$	$\mathfrak{V}$	Optimal choice
1	4	4	0.2634	0.1904	0.2990	0.2472	$\mathfrak{Y}$
2	4	4	0.2546	0.1692	0.3282	0.2480	$\mathfrak{Y}$
3	4	12	0.2287	0.1738	0.3389	0.2586	$\mathfrak{Y}$
4	4	4	0.2323	0.1432	0.3720	0.2526	$\mathfrak{Y}$
5	4	20	0.2068	0.1488	0.3789	0.2655	$\mathfrak{Y}$
6	4	12	0.1973	0.1338	0.4017	0.2671	$\mathfrak{Y}$
7	4	28	0.1862	0.1222	0.4230	0.2686	$\mathfrak{Y}$
8	4	8	0.1790	0.1050	0.4507	0.2654	$\mathfrak{Y}$
9	4	36	0.1657	0.0975	0.4699	0.2669	$\mathfrak{Y}$
10	4	20	0.1555	0.0864	0.4937	0.2643	$\mathfrak{Y}$

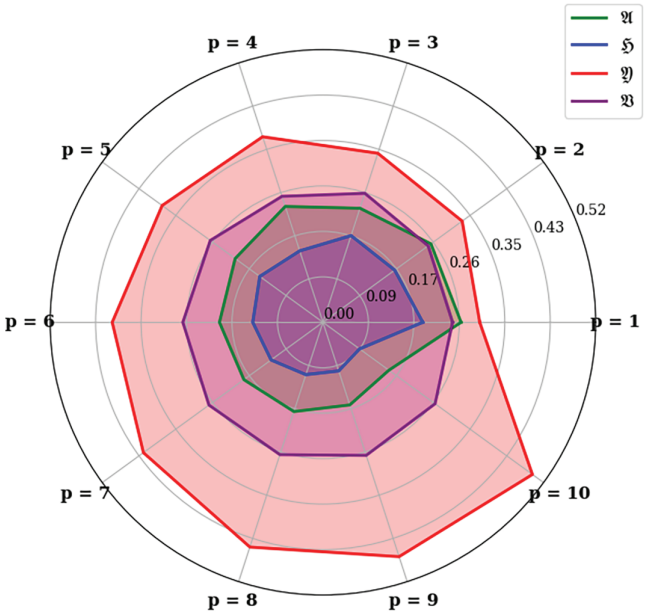

Figure 4: Sensitivity graph of $CIV_{p,q,r}$ SFWG operator

The suggested aggregation operators offer a high level of flexibility by accommodating multiple values for the parameters p, q, and r. Also, facilitate decision makers to express their opinions in an interval-valued context with confidence levels so that they can express their opinions more flexibly. This adaptability empowers decision-makers to scalability the aggregation methodology to diverse complexities and requirements, rendering the model suitable for a diverse variety of decision-making scenarios. The capability to independently adjust criteria for membership, impartial MB, and NMB degrees enhances the overall versatility of the approach. In contrast to previous methodologies, the proposed approach directly incorporates and evaluates decision-makers' confidence levels. A more

realistic depiction of uncertainty is achieved by empowering decision-makers to demonstrate different thresholds of confidence in their judgments. This flexibility allows the model to be more accurate and complex in contexts where it is difficult to obtain complete certainty, thereby enhancing its real-world relevance. Several aggregation operators are indicated that are designed to deal with complex data scenarios, such as those that differ from traditional picFS and SFS requirements. They demonstrate resilience when it comes to managing data that may not meet stringent standards by mitigating constraints and offering a comprehensive framework.

6 Comparative Analysis

To determine whether or not the proposed aggregation operators outperform, we compared our results with the t-SFS aggregation operators [16], existing operators are described by the p, q, r-SF aggregation operators [18], and t-IVSF aggregation operators [23]. The most suitable alternative among them was selected after a comprehensive evaluation. The results of the analysis can be seen in Table 11, which summarizes the scores and the rankings of the various options. It is evident from the results-that energy recovery $\mathfrak{Y}$ is the most suitable option. A graphical presentation of the ranking is shown in Fig. 5. Using the technique used by [16,18,23], we can determine that the ranking of alternatives suggested by our method (Table 11) is identical to the ranking determined through the approach of them. All provide consistent results regardless of uncertain conditions, which is a major advantage of (IVp,q,r-SFWA) and (IVp,q,r-SFWG). A parameter based on the actual decisions made by the decision maker in an interval context is also included in the proposed approach to allow for modification to the aggregate value based on the actual decisions made. In this way, the proposed operators will be more adaptable and versatile as a result. Furthermore, the study shows that novel methods differ from existing methods under different conditions in terms of the computation process. Since the proposed method in this paper considers confidence levels with interval properties, the results obtained are more realistic for decision-making. Making choices based on preferences allows decision makers to make the most informed decisions.

Table 11: Score values and ranking comparison with existing approaches

	Alternatives				
Approaches	$\mathfrak{A}$	$\mathfrak{H}$	$\mathfrak{Y}$	$\mathfrak{V}$	Optimal choice
($CIV_{p,q,r}$SFWA)	0.2508	0.2100	0.2949	0.2443	$\mathfrak{Y}$
($CIV_{p,q,r}$SFWG)	0.2323	0.1432	0.3720	0.2526	$\mathfrak{Y}$
(IVt-SFS)	0.2522	0.2175	0.2798	0.2505	$\mathfrak{Y}$
(t-SFS)	0.2418	0.2101	0.2952	0.2529	$\mathfrak{Y}$
(p,q,r-SFS)	0.2489	0.2442	0.2565	0.2504	$\mathfrak{Y}$

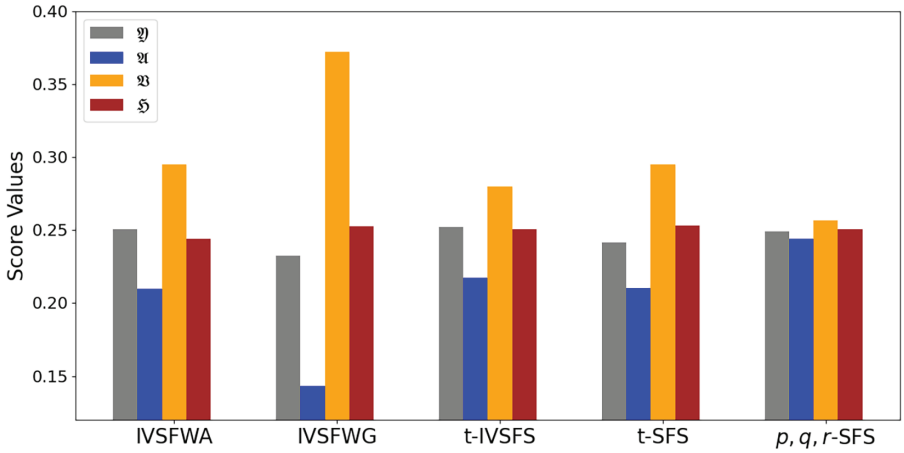

Figure 5: Existing models comparison with proposed model for alternatives ranking

6.1 Advantages

Traditional IV$p, q, r$-SFS still assigns 100% confidence to membership degrees, giving limited control over the importance of the accuracy of the information provided. The $CIV_{p,q,r}$-SFS framework allows you to adjust and regulate how decision makers' confidence levels affect memberships. As ξ increases or decreases, the outcome of memberships provides more accurate results according to real data at a more confident level, which makes it easier for decision-makers to judge real time information. It lets decision-makers customize the impact of membership degrees based on the specific context. Members of the decision-making team can adjust the impact of memberships based on factors like data reliability, information source confidence, or the decision problem nature by assigning confidence levels that are better suited to the decision context. By using $CIV_{p,q,r}$-SFS, decision-makers can balance certainty and uncertainty. In real time application data and information, uncertainty and ambiguity are inherent, so decision-makers can account for the influence of memberships based on their confidence level, which ensures that decisions are not overly based on 100% confidence, but rather consider a broad range of possibilities according to real time situations. In real-world decision-making contexts where data imperfections are common, the proposed approach is more practical because of its adaptability.

6.2 Managerial Implications

This study offers valuable insights for waste management authorities, sustainability managers, and policymakers, particularly those involved in industrial sectors such as steel manufacturing and large-scale industrial operations. The proposed methodology has several managerial implications:

- The proposed confidence-level-based interval-valued p, q, r-spherical fuzzy MCDM methodology provides decision-makers with a more realistic and flexible tool for evaluating sustainable waste management strategies. Unlike traditional fuzzy systems that assume full confidence in input values, this approach captures uncertainty and variability in expert opinions, enhancing decision reliability.
- The use of the $CIV_{p,q,r}$ SFWA and $CIV_{p,q,r}$ SFWG aggregation operators allows for a comprehensive analysis of multiple conflicting criteria, such as environmental impact, cost, feasibility, and

long-term sustainability. This helps managers prioritize waste management strategies aligned with operational goals and environmental policies.

- The case study findings, where "energy recovery" was identified as the most suitable strategy, underline the practical value of converting non-recyclable waste into usable energy. This strategy can significantly reduce landfill use, lower the environmental burden, and optimize resource efficiency.
- Sensitivity analysis confirms the robustness of the proposed model, ensuring it remains effective under varying levels of input data. This is particularly beneficial for long-term planning where uncertainty is high, enabling adaptive and evidence-based policy formulation.
- Policymakers and environmental agencies can adopt this model to guide waste management practices in industries like Pakistan Steel Mills or similar manufacturing environments.
- By incorporating both subjective expert evaluations and objective aggregation mechanisms, the model enhances decision-making transparency and accountability. This strengthens stakeholder confidence in the selected strategies and aligns with national environmental goals and international sustainability frameworks.

7 Conclusion

This paper introduces confidence level-based interval-valued p, q, r-spherical fuzzy sets and fundamental arithmetic operations. Based on these operations, some results have been obtained. Using the induction method, two aggregation operators have been developed for $CIV_{p,q,r}$ SFS, weighted averaging operator $CIV_{p,q,r}$ SFWA and a weighted geometric operator $CIV_{p,q,r}$ SFWG. The proposed $CIV_{p,q,r}$ SFWA and $CIV_{p,q,r}$ SFWG operators, along with a score function, are used to develop an MCDM methodology with a step-by-step algorithm. To illustrate how the proposed MCDM method works, an example of industrial waste management is provided to highlight the reliability and flexibility of the proposed model. Model results indicate that "energy recovery" is the most suitable alternative. Energy recovery emerges as the most suitable option because it transforms non-recyclable waste into valuable energy, thereby reducing landfill dependency while simultaneously maximizing resource efficiency. To prove the validity and effectiveness of our proposed model, a comparison with existing theories was conducted. In addition, we conducted a sensitivity analysis to check the authenticity and reliability of the developed operators. This approach incorporates varying levels of confidence rather than assuming 100% confidence in inputs, resulting in a more realistic assessment of alternatives in decision-making. As a result of this article, we hope that decision makers will find it useful as a reference point for their research.

Some limitations are associated with this study. A small pool of experts is used for the case analysis, which could limit its precision and robustness. Parametric interval-valued spherical fuzzy numbers have a higher mathematical difficulty than crisp or basic fuzzy systems, which adds to their analytical burden. Furthermore, the current model is limited in its application to other domains and limited in its potential for further validation.

Future research can integrate these operators with complex fuzzy sets, complex soft sets, etc. Diverse expert panels can increase the accuracy and robustness of evaluations. In addition, automated computational tools for processing fuzzy numbers could significantly reduce analyst workloads. Other challenges, such as agricultural modernization and the evolution of the digital industry, can be addressed equally with the proposed model.

Acknowledgement: The authors acknowledge Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R308), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia, for supporting this research work.

Funding Statement: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R308), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Author Contributions: The authors confirm contribution to the paper as follows: Study conception and wrote the original draft: Muhammad Amad Sarwar, Umme Habiba and Amel Ali Alhussan; Methodology: Muhammad Amad Sarwar; Review and editing: Umme Habiba and Amel Ali Alhussan. All authors reviewed the results and approved the final version of the manuscript.

Availability of Data and Materials: The data used in this study is included in the article. Further inquiries are directed to the corresponding author.

Ethics Approval: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest to report regarding the present study.

References

1. Pluskal J, Šomplák R, Hrabec D, Nevrlý V, Hvattum LM. Optimal location and operation of waste-to-energy plants when future waste composition is uncertain. Oper Res. 2022;22(5):5765–90. doi:10.1007/s12351-022-00718-w.
2. Gomes K, Caucci S, Morris J, Guenther E, Miggelbrink J. Sustainability transformation in the textile industry—the case of wastewater management. *Bus Strategy Dev*. 2024;7(1):e324. doi:10.1002/bsd2.324.
3. Ahmed F, Hasan S, Rana MS, Sharmin N. A conceptual framework for zero waste management in Bangladesh. Int J Environ Sci Technol. 2023;20(2):1887–1904. doi:10.1007/s13762-022-04127-6.
4. Rahman IU, Ashraf S, Batool B, Gokul KC. Optimal industrial waste management strategy for pakistan steel mill corporation using complex spherical fuzzy einstein aggregation information. Adv Fuzzy Syst. 2025;2025(1):8808643. doi:10.1155/adfs/8808643.
5. Akram M, Ramzan N, Deveci M. Linguistic Pythagorean fuzzy CRITIC-EDAS method for multiple-attribute group decision analysis. Eng Appl Artif Intell. 2023;119:105777. doi:10.1016/j.engappai.2022.105777.
6. Habiba U, Zhang R, Sarwar MA, Gong Y. A robust fractional fuzzy decision support framework for sustainable energy planning. Sci Rep. 2025;15(1):31503. doi:10.1038/s41598-025-16874-w.
7. Sarwar MA, Gong Y, Alzakari SA, Alhussan AA. Quantum-driven spherical fuzzy model for best gate security systems. Comput Model Eng Sci. 2025;143(3):3523–55. doi:10.32604/cmes.2025.066356.
8. Sarwar MA, Gong Y, Alzakari SA, Alhussan AA. A novel Fractional fuzzy approach for multi-criteria decision-making in medical waste management. Sci Rep. 2025;15(1):23999. doi:10.1038/s41598-025-06184-6.
9. Zadeh LA. Fuzzy sets. Inf Control. 1965;8(3):338–53. doi:10.1016/s0019-9958(65)90241-x.
10. Zarei E, Yazdi M, Moradi R, BahooToroody A. Expert judgment and uncertainty in sociotechnical systems analysis. In: Safety causation analysis in sociotechnical systems: advanced models and techniques. Cham, Switzerland: Springer Nature; 2024. p. 487–530 doi:10.1007/978-3-031-62470-4_18.
11. Atanassov KT. More on intuitionistic fuzzy sets. Fuzzy Sets Syst. 1989;33(1):37–45. doi:10.1016/0165-0114(89)90215-7.

12. Yager RR. Pythagorean fuzzy subsets. In: Proceedings of the 2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS); 2013 Jun 24–28; Edmonton, AB, Canada. p. 57–61.
13. Yager RR. Generalized orthopair fuzzy sets. IEEE Trans Fuzzy Syst. 2016;25(5):1222–30. doi:10.1109/tfuzz.2016.2604005.
14. Cuong BC, Kreinovich V. Picture fuzzy sets—a new concept for computational intelligence problems. In: Proceedings of the 2013 Third World Congress on Information and Communication Technologies (WICT 2013); 2013 Dec 15–18; Hanoi, Vietnam. p. 1–6.
15. Al-Shamiri MMA, Ali G, Abidin MZU, Adeel A. Multi-attribute group decision-making method under spherical fuzzy bipolar soft expert framework with its application. Comput Model Eng Sci. 2023;137(2):1891–936. doi:10.32604/cmes.2023.027844.
16. Khan MR, Ullah K, Khan Q. Multi-attribute decision-making using Archimedean aggregation operator in T-spherical fuzzy environment. Rep Mech Eng. 2023;4(1):18–38. doi:10.31181/rme20031012023k.
17. Dhumras H, Kumar M, Bajaj RK. Effectiveness of debris flow mitigation measures through T-spherical fuzzy soft dombi aggregation operators with EDAS-based multi-criteria decision making in mountainous regions. J Ind Inf Integr. 2025;48(6):100968. doi:10.1016/j.jii.2025.100968.
18. Karamaz F, Karaaslan F. Distance measures of r, s, t-spherical fuzzy sets and their applications in MCGDM based on TOPSIS. J Supercomput. 2025;81(1):173. doi:10.1007/s11227-024-06560-5.
19. Gorzałczany MB. A method of inference in approximate reasoning based on interval-valued fuzzy sets. Fuzzy Sets Syst. 1987;21(1):1–17. doi:10.1016/0165-0114(87)90148-5.
20. Atanossov K, Gargor G. Interval-valued intuitionistic fuzzy set. Fuzzy Sets Syst. 1989;31(3):343–9. doi:10.1016/0165-0114(89)90205-4.
21. Alhamzi G, Javaid S, Shuaib U, Razaq A, Garg H, Razzaque A. Enhancing interval-valued Pythagorean fuzzy decision-making through Dombi-based aggregation operators. Symmetry. 2023;15(3):765. doi:10.3390/sym15030765.
22. Joshi BP, Singh A, Bhatt PK, Vaisla KS. Interval valued q-rung orthopair fuzzy sets and their properties. J Intell Fuzzy Syst. 2018;35(5):5225–30. doi:10.3233/jifs-169806.
23. Sarfraz M. Application of Interval-valued T-spherical fuzzy Dombi Hamy mean operators in the antiviral mask selection against COVID-19. J Deci Analyt Intell Comput. 2024;4(1):67–98. doi:10.31181/jdaic10030042024s.
24. Karaaslan F, Karamaz F. Interval-valued (p, q, r)-spherical fuzzy sets and their applications in MCGDM and MCDM based on TOPSIS method and aggregation operators. Expert Syst Appl. 2024;255:124575. doi:10.1016/j.eswa.2024.124575.
25. Yu D. Intuitionistic fuzzy information aggregation under confidence levels. Appl Soft Comput. 2014;19:147–60. doi:10.1016/j.asoc.2014.02.001.
26. Khan F, Abdullah S, Mahmood T, Shakeel M, Rahim M, ul Amin N. Pythagorean cubic fuzzy aggregation information based on confidence levels and its application to multi-criteria decision making process. J Intell Fuzzy Syst. 2019;36(6):5669–83. doi:10.3233/jifs-181516.
27. Garg H, Rahim M, Amin F, Jafari S, Hezam M. Confidence levels-based cubic Fermatean fuzzy aggregation operators and their application to MCDM problems. Symmetry. 2023;15(2):260. doi:10.3390/sym15020260.
28. Mandal U, Seikh MR. Confidence level-driven Dombi aggregation operators within the p, q-quasirung orthopair fuzzy environment for sustainable supplier evaluation in automotive industry. Decis Mak Adv. 2025;3(1):285–309. doi:10.31181/dma312025101.