INTERNATIONAL CENTER FOR NUMERICAL METHODS IN ENGINEERING

Mechanics of a Continuum Medium
Vol. III

A. Fusco

Monograph, CIMNE Nº 20, October 1993

Mechanics of a Continuum Medium
Vol. III

A. Fusco

Publication CIMNE Nº-20, October 1993

Research made in collaboration with CIMNE, Barcelona, Spain,
FE Modelling s.r.l., Trescore Baln. (BG), Italy, and ISMES, S.p.A., Bergamo,Italy

International Center for Numerical Methods in Engineering
Gran Capitán s/n, 08034 Barcelona, Spain

INTERNATIONAL CENTER FOR NUMERICAL METHODS IN ENGINEERING
Edificio C1, Campus Norte UPC
Gran Capitán s/n
08034 Barcelona, Spain
www.cimne.upc.es

First edtion: January 1993

Mechanics of a Continuum Medium Vol. III

Monograph CIMNE Nº 20

ISBN: 84-87867-34-0

Contents

1 Introduction **1**
1.1 Origin and Motivation 1
1.2 Acknowledgements 3

2 Mathematical Foundations **5**
2.1 Introduction 5
2.2 Notations Convention 5
2.2.1 Elements of matrix algebra 5
2.2.2 Matrix algebraic operations 11
2.2.3 Indicial notations 13
2.3 Scalars and Vectors 16
2.3.1 Addition of vectors 17
2.3.2 Product of vectors 18
2.4 Cartesian Reference System 19
2.4.1 Algebraic vector operations 21
2.4.2 Surface and Volume 23
2.5 Elements of Differential Calculus 24
2.5.1 Differential vector operators 25
2.5.2 Stokes and Gauss Theorems 27
2.6 Curvilinear Coordinate System 30
2.6.1 Functional relationships between a CaORS and CuRS 33
2.6.2 Area and volume in curvilinear coordinate 36
2.6.3 Cylindrical coordinate system 37
2.7 Change of CaORS 39

3 Tensors **43**
3.1 Introduction 43
3.2 2-nd Order Tensors 44

3.2.1 Addition of tensors 45
3.2.2 Tensor products 46
3.3 Symmetric and Skew-symmetric Tensors 47
3.4 Tensor Transformation Rules 49
3.5 Principal Values 50
3.6 Deviatoric tensor 53
3.7 Analytical Solution for Principal Values 56
3.8 Geometrical Meaning of the Tensor Components 58
3.9 Mohr Circles 65
3.9.1 Derivation 66
3.9.2 Remarks 68
3.10 Applications 69

4 Mechanics of a Material Particle 75
4.1 Introduction 75
4.2 Motion 76
4.2.1 Tangential and normal acceleration 78
4.2.2 Circular motion 80
4.3 Equivalent Motion 82
4.4 Inertial Frame of References 84
4.4.1 Relative motion between inertial frame of references . 85
4.4.2 Transformation laws between inertial frame of references 86
4.5 Newton's Laws of Motion 87
4.6 Kinetic and Mechanical Quantities 91
4.7 Conservative Force Fields 94
4.8 Mechanical Principles 97

5 System of Interacting Particles 99
5.1 Introduction 99
5.2 Kinetic and Mechanical Quantities 100
5.3 Center of Mass 102
5.4 Equivalence of Force Fields 106
5.5 System of Interacting Particles 108
5.6 Mechanical Principles 112

6 Motion and Deformation in a Continuum 115
6.1 Introduction 115
6.2 Generality 116
6.3 Material Description: Lagrangian Formulation 118

6.3.1 Lagrangian tensors 120
6.3.2 Measure of Deformation 121
6.3.3 Surface and volume 125
6.3.4 Remarks 127
6.4 Spatial Description: Eulerian Formulation 130
6.4.1 Material derivative operator 132
6.4.2 Eulerian Tensors 133
6.4.3 Measure of Deformation 137
6.4.4 Surface and volume 140
6.4.5 Remarks 142
6.4.6 The velocity field 144
6.4.7 The acceleration field 145
6.5 Relationships Between the Two Formulations 146
6.5.1 Relationships between the Lagrange and the Euler tensors 147
6.5.2 Remarks 149
6.6 Small Deformation Theory 151
6.6.1 Linear Lagrangian tensors 153
6.6.2 Measure of Deformation 155
6.6.3 Surface and volume 158
6.6.4 Remarks 160
6.7 Compatibility Conditions for Strains 162
6.8 Change of Reference System 164
6.9 Strain Invariant Definitions 168
6.10 Strain Vector Definition 169

7 Force and Stress in a Continuum 171
7.1 Introduction 171
7.2 Continuity Equation 172
7.3 Kinetic Quantities 174
7.4 Center of Mass 177
7.5 Mechanical Quantities 178
7.6 Mechanical Principles 180
7.7 Cauchy Stress Tensor 181
7.8 Piola-Kirchhoff Stress Tensors 183
7.9 Co-rotational and Convected Stress Tensors 186
7.10 Cauchy Equations of Motion and Equilibrium 189
7.11 Stress Invariant Definitions 191
7.12 Stress Vector Definition 193

8 Mechanical Energy Principles **195**
8.1 Introduction . . . 195
8.2 Theorem of Stress Mean . . . 196
8.3 The Principle of Virtual Work . . . 198
8.3.1 Deformed configuration . . . 199
8.3.2 Undeformed configuration . . . 200
8.4 The Principle of Virtual Power . . . 202
8.5 Work and Power . . . 204
8.6 Stress Energy and Power . . . 206
8.7 Energy Principles . . . 209

9 Thermodynamic Principles **211**
9.1 Introduction . . . 211
9.2 Thermodynamic Systems . . . 212
9.3 Thermal Energy . . . 214
9.4 The First Principle of Thermodynamics . . . 215
9.5 The Second Principle of Thermodynamics . . . 217
9.6 The Specific Entropy . . . 220
9.7 Thermodynamic Pressure and Temperature . . . 222
9.8 Equations of State . . . 223
9.9 Thermodynamic Processes . . . 224
9.9.1 The Legendre Transformation Technique . . . 226
9.9.2 The Generalized Legendre Transformation Technique . 227
9.10 Potential Thermodynamic Functions . . . 231
9.11 The Clausius-Duhem Inequality . . . 235
9.12 Fully Recoverable Stress Power Processes . . . 236
9.13 Identification of the Thermodynamic Pressure . . . 238
9.14 Non Fully Recoverable Stress Power Processes . . . 240
9.15 Fluids . . . 242

10 Constitutive Equations Theory **245**
10.1 Introduction . . . 245
10.2 Purely Mechanical Theory Postulates . . . 245
10.3 Drucker's Stability Criterion . . . 248
10.4 Finite Elastic Constitutive Equations . . . 250
10.5 Hyperelastic Materials . . . 252
10.6 Isotropic Materials . . . 254
10.7 Energy Function for Isotropic Materials . . . 257
10.8 Linear Elastic Materials . . . 259

10.9 Linear Elasticity Isotropic Materials 260
10.10 Newtonian Fluids 264

11 Hypoelastic Constitutive Equations 269
11.1 Introduction 269
11.2 Triaxial Apparatus 271
11.3 Failure Criteria 274
11.4 A Modified Mohr-Coulomb Surface 279
11.5 Yield Surface 283
11.6 Hypoelastic Theory 284
11.7 Variable Moduli Models 288
11.8 Hyperbolic Models 290
11.9 A Complete Variable Moduli Model 293
11.9.1 Law of variation of the Young modulus 297
11.9.2 Law of variation of the Poisson Ratio 298
11.9.3 Law of variation of the volumetric strain 299
11.9.4 Yield surface 300

12 The Incremental Theory of Plasticity 303
12.1 Introduction 303
12.2 Elasto-Plastic Strain Definition 304
12.3 The Yield Function 305
12.4 The Hardening Parameters 307
12.5 The Plastic Multiplier 308
12.6 Stress Based Elasto-Plastic Criterion 309
12.7 Strain Based Elasto-Plastic Criterion 312
12.8 The Elasto-Plastic Constitutive Equation 314
12.9 Stable Materials 315
12.10 Isotropic Elasto-Plastic Materials 318
12.11 Perfect Plasticity 320
12.12 Work Hardening Materials 321
12.13 Strain Hardening Models 323
12.14 Capped Plastic Models 325
12.15 Cam-Clay Model 326
12.16 ECAM Model 331
12.17 Extension to Visco-Plasticity 332

13 A Generalized Theory of Plasticity 335
13.1 Introduction . . . 335
13.2 The Main Hypotheses . . . 336
13.3 Remarks . . . 343
13.4 Recovery of the Classical Theory . . . 345
13.5 A Generalized Von-Mises Strain Hardening Model . . . 346
13.5.1 The constitutive equation . . . 348
13.5.2 The uniaxial response of the model . . . 353
13.5.3 Experimental determination of the material properties 354
13.5.4 Note on the hysteretic loop . . . 357

14 Field Equations for Solids and Fluids 359
14.1 Introduction . . . 359
14.2 Field Equations for Solids . . . 360
14.2.1 Linearized theory of elasticity . . . 361
14.2.2 Nonlinear solid and small deformation theory . . . 363
14.3 Field equations for Newtonian Fluids . . . 363
14.3.1 Steady Flow . . . 367
14.3.2 Irrotational Flow . . . 368
14.3.3 Barotropic flows . . . 369
14.4 Perfect Fluids . . . 371
14.4.1 Barotropic flows . . . 372
14.4.2 The Bernoulli equation . . . 373
14.4.3 The gas dynamical equation . . . 375
14.4.4 The wave equations . . . 378
14.5 Incompressible Newtonian Fluids . . . 381
14.5.1 The Poisson and Stokes pressure equations . . . 383
14.5.2 Incompressible potential flow . . . 384
14.6 Incompressible Inviscid Fluid . . . 385

15 Coupled Solid-Fluid Problems 389
15.1 Introduction . . . 389
15.2 Preliminary Definitions . . . 390
15.3 Seepage Law . . . 392
15.4 Principle of Effective Stress . . . 397
15.5 Stress-Strain Relationships . . . 398
15.6 Flow Continuity Relation . . . 400
15.7 Steady Seepage Flow . . . 402
15.8 The Physics in a Coupled Soild-Fluid Problem . . . 403

15.9 Undrained Conditions 406
15.10 The Coupled Theory of Consolidation 408
15.11 The Oedometer Problem 410
15.11.1 The mathematical derivation 412
15.11.2 Analytical solution 415
15.12 The Uncoupled Theory of Consolidation 417

16 A Constitutive Equation for Soils **421**
16.1 Introduction 421
16.2 Reference Soil Tests 422
16.2.1 Isotropic drained tests 422
16.2.2 Undrained triaxial tests 425
16.3 The Mathematical Framework 427
16.4 The Bounding, the Yield and the Elastic Surfaces 435
16.5 Remarks on the Mathematical Formulation 440
16.6 The Kinematics of the Bounding Surface $\overline{F}$ 444
16.7 The Elastic Parameters 446
16.8 Functional Relationship for $\overline{p}_y(v^{(p)})$ 447
16.9 Functional Relationship for $\overline{p}_y(p, v)$ 449
16.10 Functional Relationships for $\overline{p}_c$ 452
16.11 Functional Relationships for $\hat{p}_y$ 454
16.12 Functional Relationship for $\overline{p}_d$ 456
16.13 Functional Relationship for $\hat{p}_d$ 456
16.14 The $v^{(p)}$ vs. p Plane 457
16.15 Functional Relationships for p_y 458
16.16 Functional Relationship for $\partial p_y/\partial v^{(p)}$ 460
16.16.1 v^p vs. p in an isotropic test 460
16.16.2 v^p vs. p_y in an isotropic test 460
16.16.3 A mathematical expression for $\partial p_y/\partial v^{(p)}$ 465
16.17 Functional Relationship for $\partial p_y/\partial \epsilon_s^{(p)}$ 468
16.17.1 $\epsilon_s^{(p)}$ vs. q in a triaxial test 469
16.17.2 $\epsilon_s^{(p)}$ vs. p_y in a triaxial test 470
16.17.3 A mathematical expression for $\partial p_y/\partial \epsilon_s^{(p)}$ 473
16.18 Plastic Modulus 475
16.19 The Limit State Surface 476
16.20 Critical State Conditions 479
16.21 Verification of the Physical Hypotheses 480
16.22 Soil Model Predictions 481

16.23 Undrained Triaxial Cyclic Test Simulation 483
16.24 Undrained Shear Cyclic Test Simulation 487
16.25 Conclusions 489

A The Gradient of a Tensor Function 491
A.1 Introduction 491
A.2 The Gradient of a Generic Stress Function 492
A.3 The Gradient of a Stress Invariant Function 494
A.4 Deviatoric Strain and Strain Invariants 499
A.5 The Hessian Matrix 500

List of Tables

2.1 Matrix definitions (I part) 7
2.2 Matrix definitions (II part) 8
2.3 Differential vector operators 28

3.1 2-nd order tensor invariants 65

6.1 Strain tensor transformation under a change of reference frame 165

10.1 Relationship among elastic moduli 262

15.1 Typical values of the permeability coefficients in soils 394

List of Figures

2.1 Addition of vectors 17
2.2 Vector products 19
2.3 Cartesian reference system 20
2.4 Surface and volume in a CaORS 23
2.5 Oriented surface Γ and bounding curve C 29
2.6 Curvilinear reference system 30
2.7 Orthogonal cylindrical coordinate system 38
2.8 Change of Cartesian coordinate system 40

3.1 The vector components of $\mathbf{u} = \mathbf{T}\hat{\mathbf{e}}^{(1)}$ 58
3.2 Octahedral plane in the principal CaORS 60
3.3 Projection of a vector into the octahedral plane 61
3.4 Octahedral vector representation of a tensors 63
3.5 The Mohr circles 67
3.6 Construction of the Mohr circle relative to a plane tensor . . 70
3.7 Graphical procedure on the Mohr circle for determining $\chi = \mathbf{T}\mathbf{n}$ when $\mathbf{T}$ is a plane tensor 73

4.1 Motion in space of a material point P 77
4.2 Tangential and normal acceleration 79
4.3 Circular motion 81
4.4 Motion of a material particle as seen by two reference systems traveling and rotating one with respect to the other. 82
4.5 Force acting on a particle 89
4.6 Mechanical quantities acting on a particle 92
4.7 Action-at-a-distance forces between two material particles . . 96

5.1 System of particles 100
5.2 Center of mass in a system of particles 103

5.3 Force field 107

6.1 Motion in space of a continuous medium 116
6.2 Material description of the deformation process 121
6.3 Deformation process of an infinitesimal portion of material . 125
6.4 Eulerian description of the deformation process 137
6.5 Deformation process of an infinitesimal portion of material . 141
6.6 Material description of a *small* deformation process 155

7.1 Spatial domain at two distinct times 172
7.2 Kinetic quantities associated to a particle in a continuum . . 175
7.3 External forces acting on a free body. 178
7.4 The Cauchy tetrahedron. 182
7.5 Force vectors for Piola-Kirchhoff stress definition 185

11.1 Typical triaxial response of a real solid material 270
11.2 Simple triaxial apparatus 272
11.3 Stress paths in a simple triaxial test 273
11.4 Mohr-Coulomb failure criterion in the σ vs. τ Mohr-plane . 274
11.5 Mohr-Coulomb and Tresca failure surfaces 275
11.6 Drucker-Prager and Von-Mises failure surfaces 277
11.7 A modified Mohr-Coulomb surface 281
11.8 Loading/unloading/reloading criterion 285
11.9 Hyperbolic stress-strain curve 291
11.10 Material response under triaxial conditions 294
11.11 Volumetric strain prediction 299
11.12 Failure and yield surface 301

12.1 A convex and not convex yield surfaces in the stress space . 317
12.2 Triaxial response of a perfectly plastic Mohr-Coulomb material 321
12.3 Strain hardening Drucker-Pragher elasto-plastic model . . . 324
12.4 Cam-Clay model 328
12.5 ECAM model 331

13.1 Material response under uniaxial cyclic loading 347
13.2 The uniaxial material response scaled by the elastic strain . 355

14.1 Interrelationships of variables in the solution of a continuum mechanic problem 360

15.1 Seepage through a porous medium 393
15.2 Total and effective stress in a porous media 397
15.3 Steady seepage flow in a porous solid medium 402
15.4 Confined and unconfined seepage flow in a porous solid medium 404
15.5 Excess pore pressure vs. time factor in the center of a spherical sample loaded uniformly by Q 410
15.6 1D consolidation problem 411

16.1 Material response in a isotropic loading drained test; p_a is the atmospheric pressure. 423
16.2 Response of OCS in a triaxial loading undrained test. 426
16.3 The bounding, the yield and the elastic surfaces in the q vs. p plane 436
16.4 Expansion of $\overline{F}$ in the q vs. p plane 444
16.5 Determination of $\overline{p}_y$ 448
16.6 Plane $v^{(p)}$ vs. p. 457
16.7 Experimental graph $v^{(p)}$ vs. p of an isotropic test 461
16.8 Extrapolated graph $v^{(p)}$ vs. p, of an isotropic test 461
16.9 Extrapolated graph $v^{(p)}$ vs. p_y of an isotropic test 462
16.10 Families of curves $v^{(p)}$ vs. p_y for isotropic drained test. . . . 463
16.11 Experimental graph $(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$ vs. q, in the undrained triaxial test TUDP0 469
16.12 Experimental graph $(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$ vs. p_y, in the undrained triaxial test TUDP0 470
16.13 Family of curves $(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$ vs. p_y for undrained triaxial test TUDP0. 472
16.14 The Limit State Surface, LSS: 478
16.15 Reference experimental tests 481
16.16 Isotropic drained test 483
16.17 Triaxial drained test 484
16.18 Triaxial undrained test 484
16.19 Triaxial cyclic undrained test $(q_M^{(1)} \leq \tilde{q}^{(t)})$ 486
16.20 Triaxial cyclic undrained test $(q_M^{(2)} > \tilde{q}^{(t)})$ 486
16.21 Imposed pure shear strain 487
16.22 Shear cyclic undrained test $(\gamma_{max} = \gamma_M^{(1)})$ 488
16.23 Shear cyclic undrained test $(\gamma_{max} = \gamma_M^{(2)})$ 488
16.24 Variation of G_d and λ with γ_M. 489

Chapter 1

Introduction

1.1 Origin and Motivation

The Continuum Mechanics is a theory which establishes the field equations governing the physical processes of a *continuous medium* or *continuum*. A continuum is any portion of matter (solid, liquid or gas) which can be considered without gaps or empty spaces.

Although much of the formal developments were carried out during the ninenteen century, only a part of this theory is of common knowledge among the engineers. This is probably due to the fact that the complete theory yields to a set of highly nonlinear differential equations which, in general, cannot be solved with the traditional analytical tools. Thus, continuum mechanics books for engineers are usually confined to those problems whose relative field equations may be simplified in a linear form and which, consequently, may be solved analytically.

Nowadays, the solution of a complete set of governing equations is not anymore a major problem. For example, a mathematical procedure known as the Finite Element Method allows to transform the set of nonlinear differential equations into a system of nonlinear equations which can then be solved numerically by means of digital computers.

This book presents the fundamental assumptions and the successive mathematical developments which allow to establish the complete field equations of a continuum. The aim is to be sufficiently general and self-content using, however, mathematical procedures which can be always understood by an engineer with a common background of Linear Algebra and elementary Differential Calculus.

Advanced textbooks in Continuum Mechanics present the theory assuming general curvilinear reference systems; the relative mathematical developments require a special mathematical technique known as *Tensorial Calculus*. One of the main reason of using curvilinear coordinates is that the geometrical shape of a solid body or a fluid container may be such that the relative boundary-value problem results simpler to solve analytically. However, mathematical solution techniques such as the Finite Element Method may solve any type of 3D problems using the field equations relative to the simpler Cartesian orthogonal reference system.

Thus, this book presents all the theory assuming Cartesian orthogonal reference system which allows to consider all the *tensorial operations* as simple linear algebraic operations. The necessary theoretical basis of Linear Algebra can be found in [38].

The first part of this book is concerned with the classical Continuum Mechanics for a *one phase medium*, that is a medium consisting of a solid or a fluid phase only. Then, the theory is generalized to take into account *coupled solid-fluid problems* which actually represent the origin of this research work. The Coupled Solid-Fluid Theory establishes the field equations for the solution of continuum mechanical problems in which the medium is represented by a porous solid fully saturated by a fluid phase. This theory finds important applications for the solution of geotechnical structure in which the medium consists of fully saturated soils.

The most noteworthy recent developments in Continuum Mechanics have been in the Theory of Constitutive Equations. However, instead of reporting just an overview of these modern theories, I preferred to describe only the simple classical ones and then explain in detail the *Incremental Theory of Plasticity*, with its limits and possible generalizations. This presentation is then completed by reporting the mathematical developments and the predicted response of an elasto-plastic model for soils which is the result of our most recent research work.

As mentioned, the solution of a complete set of field equations requires special mathematical methods, such as the Finite Element Method. Hence, the content of this book is bounded to the setup of the complete set of field equations while details for their solution by Finite Element may be found in [40].

1.2 Acknowledgements

While writing this book, I have received valuable suggestions from many unselfish friends and colleagues. To them all, I give my sincere thanks. I especially wish to thank:

- dott. ing. Alberto De Crescenzo from FEModelling s.r.l., Trescore Balneario (BG), Italy, who carried out most of the research work on new constitutive equations and corrected all the contents of this book.
- dotts. Paolo Bonaldi and Alberto Peano from ISMES S.p.A., Bergamo, Italy, for their encouragement and support.
- Professors Miguel Cervera and Eugenio Oñate from the Universitat Politecnica de Catalunya, Barcelona, Spain, for their stimulus for completing this book.

Sincere gratitude I address to the ISMES S.p.A., Bergamo, Italy, which has financed my research studies in the last six years, thus giving me the opportunity to complete the manuscript.

Last, but not least, I thank my wife Lisa for her patience, understanding and assistance in reading the last version of this book.

Chapter 2

Mathematical Foundations

2.1 Introduction

The Continuum Mechanics Theory deals with physical quantities which, by definition, are independent of the position and orientation of the observer. For this reason, the equations of the physical laws are *vector equations* or *tensor equations* which are independent of any reference system. However, for the mathematical description of such quantities, it is necessary to refer to a coordinate system.

In this Chapter, we essentially review the elementary concept of scalars and vectors. Then, we introduce the definition of a reference coordinate system and the mathematical relationships between two arbitrary reference systems.

2.2 Notations Convention

Most of the mathematical developments in Continuum Mechanics Theory may be performed according to the Linear Algebra Theory. Consequently, the notation convention used in this book is fully based on this Linear Algebra Theory, [38].

2.2.1 Elements of matrix algebra

From an elementary point of view, a *vector* consists of a set of ordered numbers, real or complex, arranged in an array; for example

$$(-3, a, -2b, c+1)$$

On the other hand, a *matrix* consists of a set of ordered numbers, real or complex, arranged in a rectangular array; for example

$$\begin{pmatrix} -2, & 3a, & c \\ b, & 1, & d-3 \end{pmatrix}$$

The separate numbers in a given array are called *elements* or *components* of the vector or of the matrix. In general, the elements in the array are completely independent of each other.

Definition 2.2.1 (Matrix.) *A matrix* $\mathbf{A}$ *of order* $m \times n$ *consists of a collection of* $m \times n$ *(read* m *by* n*) numbers arranged in* m *rows and* n *columns,*

$$\mathbf{A} = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & . & A_{1n} \\ A_{21} & A_{22} & . & A_{2n} \\ . & . & . & . \\ A_{m1} & A_{m2} & . & A_{mn} \end{bmatrix} \tag{2.1}$$

The symbol A_{ij} denotes the element in the i-th row and j-th column of the array

$$\begin{array}{c} \\ i\text{-th row} \end{array} \begin{bmatrix} & & \vdots & \\ & & j\text{-th column} & \\ . & . & A_{ij} & \\ & & & \end{bmatrix}$$

As a special case of matrices we have:

- *Real matrix*, a matrix whose elements are all real numbers.
- *Complex matrix*, a matrix whose elements are complex.
- *Null matrix*, written as $\mathbf{O}$, a matrix whose elements are all equal to zero.
- *Square matrix*, a matrix whose numbers of rows and columns are equal.

A square matrix of order $m \times m$ may be simply called as matrix of order m. Special types of square matrices are reported in Tabs. 2.1 and 2.2 along with a brief summary of their major properties.

Table 2.1: Matrix definitions (I part)

List of items	Definitions	Properties and notes
Symmetric matrix	$\mathbf{A} = \begin{bmatrix} A_{11} & . & A_{1j} & . & A_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{1j} & . & A_{ij} & . & A_{im} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{1m} & . & A_{im} & . & A_{mm} \end{bmatrix}$	$A_{ij} = A_{ji}$ $\mathbf{A} = \mathbf{A}^T$
Skew-symmetric matrix	$\mathbf{A} = \begin{bmatrix} A_{11} & . & A_{1j} & . & A_{1m} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -A_{1j} & . & A_{ij} & . & A_{im} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -A_{1m} & . & -A_{im} & . & A_{mm} \end{bmatrix}$	$A_{ij} = -A_{ji}$ $\mathbf{A} = -\mathbf{A}^T$
Diagonal square matrix	$\mathbf{A} = \begin{bmatrix} A_{11} & & & & \\ & \ddots & & & \\ & & A_{ii} & & \\ & & & \ddots & \\ & & & & A_{mm} \end{bmatrix}$	$A_{ij} = A_{ii}\delta_{ij}$ $\mathbf{A} = \mathbf{A}^T$ $\mathbf{A}^{-1} = [\delta_{ij}/A_{ii}]$
Identity square matrix	$\mathbf{I} = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$	$I_{ij} = \delta_{ij}$ $\mathbf{I} = \mathbf{I}^T = \mathbf{I}^{-1}$ $\det \mathbf{I} = 1$
Lower triangular square matrix	$\mathbf{A} = \begin{bmatrix} A_{11} & & & \\ A_{21} & A_{22} & & \\ \vdots & \vdots & \ddots & \\ A_{m1} & A_{m2} & . & A_{mm} \end{bmatrix}$	$\det \mathbf{A} = \prod_{i=1}^{m} A_{ii}$

Table 2.2: Matrix definitions (II part)

List of items	Definitions	Properties and notes
Singular (square) matrix	$\det \mathbf{A} = 0$	The system $\mathbf{Ax} = \mathbf{o}$ admits solutions $\mathbf{x} \neq \mathbf{o}$. In general, the system $\mathbf{Ax} = \mathbf{b}$ has no solution
Orthogonal (square) matrix	$\mathbf{A}^T\mathbf{A} = \mathbf{I}$ $A_{ki}A_{kj} = \delta_{ij}$	$\lvert\det \mathbf{A}\rvert = 1$ $\mathbf{A}^T = \mathbf{A}^{-1}$ $\mathbf{AA}^T = \mathbf{I}$
Positive definite (square) matrix	$r = \mathbf{a}^T\mathbf{Aa} = a_i A_{ij} a_j > 0$ for all choices of non trivial $\mathbf{a}$	$\det \mathbf{A}^{(k)} > 0, \quad \forall\ k$ $\det \mathbf{A} > 0$
Semipositive definite (square) matrix	$r = \mathbf{a}^T\mathbf{Aa} = a_i A_{ij} a_j \geq 0$ for all choices of non trivial $\mathbf{a}$ N.B.: There exists at least one non trivial $\mathbf{a}$ for which $r = 0$	$\det \mathbf{A} = 0$
Strictly diagonally dominant (square) matrix	$\lvert A_{ii}\rvert > \sum_{j=1, j\neq i}^{m} \lvert A_{ij}\rvert$ for all $i = 1, 2, \ldots, m$	$\det \mathbf{A} \neq 0$
Diagonally dominant (square) matrix	$\lvert A_{ii}\rvert \geq \sum_{j=1, j\neq i}^{m} \lvert A_{ij}\rvert$ for all $i = 1, 2, \ldots, m$	

Definition 2.2.2 (Vector.) *A vector* $\mathbf{v}$ *of order* m *is a matrix of order* $m \times 1$

$$\mathbf{v} = \begin{Bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{Bmatrix} = \{v_1, v_2, \ldots, v_m\}^T$$

Definition 2.2.3 (The transpose of a matrix.) *The transpose of a* $m \times n$ *matrix* $\mathbf{A}$*, written as* $\mathbf{A}^T$*, is a* $n \times m$ *matrix obtained by interchanging the rows and columns of* $\mathbf{A}$

$$\mathbf{A}^T = \begin{bmatrix} A_{11} & A_{21} & . & A_{m1} \\ A_{12} & A_{22} & . & A_{m2} \\ . & . & . & . \\ A_{1n} & A_{2n} & . & A_{mn} \end{bmatrix} \tag{2.2}$$

Definition 2.2.4 (Orthogonal matrix) *A square matrix* $\mathbf{A}$ *such that*

$$\mathbf{A}\mathbf{A}^T = \mathbf{A}^T\mathbf{A} = \mathbf{I}$$

is called orthogonal (or orthonormal) matrix.

Definition 2.2.5 (Determinant of a square matrix.) *The determinant of a square matrix* $\mathbf{A}$ *of order* m*, usually denoted as*

$$\det \mathbf{A} = \begin{vmatrix} A_{11} & A_{12} & . & A_{1m} \\ A_{21} & A_{22} & . & A_{2m} \\ . & . & . & . \\ A_{m1} & A_{m2} & . & A_{mm} \end{vmatrix}$$

is a scalar value calculated as

$$\det \mathbf{A} = \sum_{s_1}^{s_m} (-1)^s A_{1,s_1} A_{2,s_2} \ldots A_{m,s_m}$$

where the sum is extended to all $m!$ *permutations* $(s_1, s_2, \ldots, s_m)$*, over the index* $(i = 1, 2, \ldots, m)$ *and* s *is the number of the inversion of the permutation* $(s_1, s_2, \ldots, s_m)$ *with respect to the fundamental permutation* $(1, 2, \ldots, m)$.

Thus for example:

- the determinant of a square matrix of order $m = 1$

$$\mathbf{A} = [A]$$

  is equal to

$$\det \mathbf{A} = A$$

- the determinant of a $2nd$ order square matrix

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

  is equal to

$$\det \mathbf{A} = A_{11}A_{22} - A_{12}A_{21}$$

- the determinant of a $3rd$ order square matrix

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

  is equal to

$$\begin{aligned} \det \mathbf{A} &= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - \\ &\quad - A_{11}A_{23}A_{32} - A_{12}A_{21}A_{33} - A_{13}A_{22}A_{31} \end{aligned}$$

Definition 2.2.6 (Inverse of a square matrix.) *The inverse of a square matrix* $\mathbf{A}$*, commonly denoted as* $\mathbf{A}^{-1}$*, is that matrix which, multiplied by* $\mathbf{A}$*, gives the identity matrix*

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Theorem 2.2.1 (The inverse of A.) *The inverse of* $\mathbf{A}$ *exists if and only if* $\det \mathbf{A} \neq 0$*. In this case* $\mathbf{A}^{-1}$ *is equal to the adjoint matrix of* $\mathbf{A}$*,* $\operatorname{adj} \mathbf{A}$*, divided by the determinant of* $\mathbf{A}$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \operatorname{adj} \mathbf{A}$$

where $\operatorname{adj} \mathbf{A}$ *denotes the adjoint matrix of* $\mathbf{A}$*, Section 3.8 in [38].*

2.2.2 Matrix algebraic operations

Similarly to the ordinary numbers, matrices and vectors can be algebraically manipulated. In this Section we report the basic definition of addition (subtraction) and product for matrices and vectors.

Definition 2.2.7 (Matrix equality.) *Two matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *are equal*

$$\mathbf{A} = \mathbf{B}$$

if and only if they are of the same order and all the corresponding elements are equal, that is

$$A_{ij} = B_{ij}$$

for all $i = 1, 2, \ldots, m$ *and* $j = 1, 2, \ldots, n$

Definition 2.2.8 (Addiction (subtraction)). *Two matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *can be added (subtracted) to obtain*

$$\mathbf{S} = \mathbf{A} + \mathbf{B}$$

if and only if they are of the same order. The addition (subtraction) is performed by adding (subtracting) all the corresponding elements, that is

$$S_{ij} = A_{ij} + B_{ij}$$

The resulting matrix $\mathbf{S}$ *has the same order of* $\mathbf{A}$ *and* $\mathbf{B}$

Theorem 2.2.2 *Let* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *be matrices of the same order* $m \times n$. *According to the above reported definition of addiction (subtraction), it is immediate to verify that*

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A} \\ (\mathbf{A} + \mathbf{B})^T &= \mathbf{A}^T + \mathbf{B}^T = \mathbf{B}^T + \mathbf{A}^T \\ \mathbf{A} + (\mathbf{B} + \mathbf{C}) &= \mathbf{A} + \mathbf{B} + \mathbf{C} \end{aligned}$$

Notice that, the subtraction of a matrix by itself results in a null matrix,

$$\mathbf{A} - \mathbf{A} = \mathbf{O}$$

Definition 2.2.9 (Product of a matrix by a scalar.) *The product of a matrix* $\mathbf{A}$ *of order* $m \times n$ *by a scalar* c *is an operation where all elements of* $\mathbf{A}$ *are multiplied by this scalar* c, *that is*

$$S_{ij} = cA_{ij}$$

The resulting matrix $\mathbf{S} = c\mathbf{A}$ *has the same order of* $\mathbf{A}$.

Theorem 2.2.3 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices of the same order* $m \times n$ *and* c *be a scalar. According to the above reported definition of product, it is immediate to verify that*

$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B}$$

Definition 2.2.10 (Product of two matrices.) *Two matrices* $\mathbf{A}$ *and* $\mathbf{B}$ *can be multiplied to obtain* $\mathbf{S} = \mathbf{AB}$, *if and only if the number of columns in* $\mathbf{A}$ *is equal to the number of rows in* $\mathbf{B}$. *To calculate the* S_{ij} *elements we multiply the elements in the* i*-th row of* $\mathbf{A}$ *by the elements in the* j*-th column of* $\mathbf{B}$ *and add all the individual results. Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two matrices of order* $m \times p$ *and* $p \times n$, *respectively; then, the matrix* $\mathbf{S}$ *is calculated as*

$$S_{ij} = \sum_{k=1}^{p} A_{ik} B_{kj}$$

The order of the matrix $\mathbf{S}$ *is* $m \times n$.

Theorem 2.2.4 *If* $\mathbf{A}, \mathbf{I}, \mathbf{B}$ *and* $\mathbf{C}$ *are matrices of appropriate order so that they can be correctly multiplied, then*

$$\begin{aligned} \mathbf{AI} &= \mathbf{A} \\ \mathbf{AB} &\neq \mathbf{BA}; \qquad \textit{(in general)} \\ \mathbf{A}(\mathbf{B} + \mathbf{C}) &= \mathbf{AB} + \mathbf{AC} \\ (\mathbf{B} + \mathbf{C})\mathbf{A} &= \mathbf{BA} + \mathbf{CA} \\ (\mathbf{ABC}\ldots)^T &= \ldots \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T \end{aligned}$$

Definition 2.2.11 (Power of a square matrix.) *The* n*-th power of a square matrix* $\mathbf{A}$ *is defined as*

$$\mathbf{A}^n = \mathbf{A}^{n-1}\mathbf{A}$$

where n *is any integer positive value, with the convention*

$$\mathbf{A}^0 = \mathbf{I}$$

Theorem 2.2.5 *Let* $\mathbf{A}$ *be a square matrix of order* m. *According to the above reported definition of power of a matrix, it is immediate to verify that*

$$\mathbf{A}^p \mathbf{A}^q = \mathbf{A}^q \mathbf{A}^p = \mathbf{A}^{p+q}$$

Definition 2.2.12 (Product of a matrix by a vector.) *Vectors* $\mathbf{v}$ *and* $\mathbf{v}^T$ *of order* m *are in effect matrices of order* $m \times 1$ *and* $1 \times m$, *respectively. Hence, the rules of the product of matrices applies as well as for the product of a matrix by a vector. Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be two matrices of order* $p \times m$ *and* $m \times p$ *respectively; then*

$$\begin{aligned} s_i &= A_{ik} v_k \\ t_j &= v_k B_{kj} \end{aligned}$$

The order of the vectors $\mathbf{s} = \mathbf{A}\mathbf{v}$ *and* $\mathbf{t} = \mathbf{v}^T\mathbf{B}$ *is* p.

Theorem 2.2.6 *Let* $\mathbf{A}$ *and* $\mathbf{C}$ *be matrices of the same order* $p \times m$ *and let* $\mathbf{v}$ *and* $\mathbf{t}$ *be vectors of the same order* m. *According to the above reported definition of product of a matrix by a vector, it is immediate to verify that*

$$\begin{aligned} \mathbf{s} &= (\mathbf{A} + \mathbf{C})\,\mathbf{v} = \mathbf{A}\mathbf{v} + \mathbf{C}\mathbf{v} \\ \mathbf{s}^T &= [(\mathbf{A} + \mathbf{C})\,\mathbf{v}]^T = (\mathbf{A}\mathbf{v})^T + (\mathbf{C}\mathbf{v})^T = \\ &= \mathbf{v}^T\mathbf{A}^T + \mathbf{v}^T\mathbf{C}^T = \mathbf{v}^T(\mathbf{A} + \mathbf{C})^T \\ a &= \mathbf{v}^T\mathbf{t} = \mathbf{t}^T\mathbf{v} \\ b &= \mathbf{s}^T\mathbf{A}\mathbf{v} = \mathbf{v}^T\mathbf{A}^T\mathbf{s} \\ c &= \mathbf{s}^T(\mathbf{A} + \mathbf{B})\,\mathbf{v} = \mathbf{s}^T\mathbf{A}\mathbf{v} + \mathbf{s}^T\mathbf{B}\mathbf{v} \end{aligned}$$

where $\mathbf{s}$ *is a vector of order* p *while* a, b *and* c *are scalar values.*

2.2.3 Indicial notations

Indicial notations permit to shorten lengthy mathematical expressions and to speed up the relative calculations. In indicial notations, any component of a vector

$$\mathbf{a} = \{a_1, a_2, \ldots, a_m\}^T$$

may be indicated as

$$a_i \quad , \quad a^i$$

where the letter index i, either subscript or superscript, is understood to take on the values

$$i = 1, 2, \ldots, m$$

In this notation, the repetition of an index in a term denotes a summation with respect to that index over the range, namely

$$
\begin{aligned}
a_i a_i &= \sum_{i=1}^{m} a_i a_i = a_1 a_1 + a_2 a_2 + \cdots + a_m a_m \\
a_i b_i &= \sum_{i=1}^{m} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_m b_m
\end{aligned}
$$

Similarly, any component of a matrix

$$
\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & . & A_{1m} \\ A_{21} & A_{22} & . & A_{2m} \\ . & . & . & . \\ A_{m1} & A_{m2} & . & A_{mn} \end{bmatrix}
$$

may be indicated as

$$A_{ij} \ , \ A_i^j$$

where the indices i and j are understood to take on the values

$$
\begin{aligned}
i &= 1, 2, \ldots, m \\
j &= 1, 2, \ldots, n
\end{aligned}
$$

The notation $A_{ij} a_j$ indicates a sum over the index j, that is

$$A_{ij} a_j = \sum_{j=1}^{n} A_{ij} a_j = A_{i1} a_1 + A_{i2} a_2 + \cdots + A_{in} a_n$$

Definition 2.2.13 (The Kronecker Delta operator.) *The Kronecker Delta is an operator defined as*

$$\delta_{ij} = \begin{cases} 0 & \textit{for } i \neq j \\ 1 & \textit{for } i = j \end{cases}$$

where

$$
\begin{aligned}
i &= 1, 2, 3 \\
j &= 1, 2, 3
\end{aligned}
$$

According to the above reported definition of Kronecker Delta operator, it is easy to verify that

$$\begin{aligned}
\delta_{ij} &= \delta_{ji} && (2.3)\\
\delta_{ij}\delta_{ij} &= \delta_{ii} = 3 && (2.4)\\
\delta_{ij}\delta_{jk} &= \delta_{ik} && (2.5)\\
\delta_{ij}\delta_{ik}\delta_{jk} &= 3 && (2.6)
\end{aligned}$$

Definition 2.2.14 (The permutation symbol or alternating tensor.) *The permutation symbol is an operator defined as*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if the values of i,j,k are an even permutation of 1,2,3} \\ & \text{(i.e. if they appear in sequence as in the arrangement 1 2 3 1 2)} \\ -1 & \text{if the values of i,j,k are an odd permutation of 1,2,3} \\ & \text{(i.e. if they appear in sequence as in the arrangement 3 2 1 3 2)} \\ 0 & \text{if the values of i,j,k are not a permutation of 1,2,3} \\ & \text{(i.e. if two or more of the indices have the same value)} \end{cases}$$

According to the above reported definition of permutation symbol, it is easy to verify that

$$\begin{aligned}
\epsilon_{ijk} &= -\epsilon_{kji} && (2.7)\\
\delta_{ij}\epsilon_{ijk} &= 0 && (2.8)
\end{aligned}$$

Moreover, it is possible to prove that, Theorems 3.3.1 and 3.4.3 in [38],

$$\det \mathbf{A} = \det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{23} & A_{33} \end{bmatrix} = \epsilon_{ijk} A_{1i} A_{2j} A_{3k} \qquad (2.9)$$

and

$$\det \begin{bmatrix} A_{mp} & A_{mq} & A_{ms} \\ A_{np} & A_{nq} & A_{ns} \\ A_{rp} & A_{rq} & A_{rs} \end{bmatrix} = \epsilon_{mnr}\epsilon_{pqs} \det \mathbf{A} \qquad (2.10)$$

Hence, setting

$$\mathbf{A} = \mathbf{I} = [\delta_{ij}]$$

we can prove that the product between two permutation symbols results to be equal to

$$\epsilon_{ijk}\epsilon_{npq} = \det \begin{bmatrix} \delta_{ni} & \delta_{nj} & \delta_{nk} \\ \delta_{pi} & \delta_{pj} & \delta_{pk} \\ \delta_{qi} & \delta_{qj} & \delta_{qk} \end{bmatrix} \qquad (2.11)$$

from which we can establish, for example,

$$\epsilon_{ijk}\epsilon_{kpq} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \qquad (2.12)$$
$$\epsilon_{ijk}\epsilon_{kij} = 6 \qquad (2.13)$$

As an example of the use of the above defined operators, let define

$$\begin{aligned}
\mathbf{a} &= \{a_1, a_2, a_3\}^T \\
\mathbf{b} &= \{b_1, b_2, b_3\}^T \\
\mathbf{c} &= \{c_1, c_2, c_3\}^T \\
\mathbf{A} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}
\end{aligned}$$

The Kronecker delta and the permutation symbol may be used to indicate the following algebraic operations on them:

$$\begin{aligned}
\delta_{ij}a_ib_j &= a_ib_i = a_jb_j = a_1b_1 + a_2b_2 + a_3b_3 \\
\delta_{ij}A_{ij} &= A_{ii} = A_{jj} = A_{11} + A_{22} + A_{33} \\
\epsilon_{ijk}a_ib_jc_k &= (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)
\end{aligned}$$

2.3 Scalars and Vectors

In a three dimensional Euclidean space, *vectors* are entities possessing both *magnitude* and *direction* and obeying certain laws; conventionally, a vector is represented by an arrow pointing in the appropriate direction and having the length proportional to the magnitude of the vector. On the other hand, *scalars* are entities possessing only magnitude and do not have any direction associated with them.

According to *Gibbs notation*, vectors are designated by bold-faced letters such as $\mathbf{a}, \mathbf{b}, \ldots$, while scalars are denoted by italic letters such as $a, b, \ldots$. The magnitude of a vector $\mathbf{a}$ is denoted by a (the same letter in italic), or by $|\mathbf{a}|$, and is always a positive number or zero. By definition:

- Two vectors are *equal* if they have the same direction and the same magnitude.

- The *negative* of a vector **a** is that vector −**a** having the same magnitude but opposite direction.
- A *null* or *zero* vector **o** is one having zero length and an unspecified direction.
- A *unit* vector is one having unit length; unit vector are usually distinguished by a caret placed over the bold-faced vector, for example, $\hat{\mathbf{a}}, \hat{\mathbf{b}}, \ldots$.

2.3.1 Addition of vectors

Any two vectors of the same kind may be *summed* according to the *parallelogram law*, Fig. 2.1a, which defines the vector sum of two vectors as the diagonal of a parallelogram having the component vectors as adjacent sides. This law for addition is equivalent to the *triangular rule* which defines the

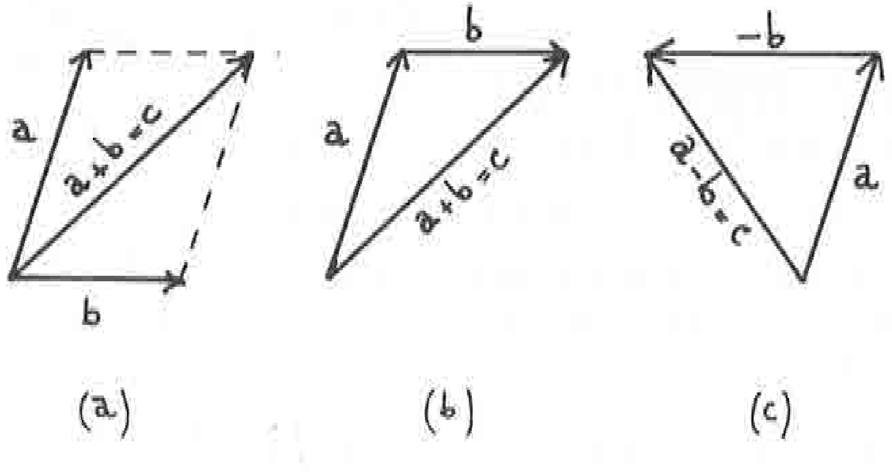


Figure 2.1: Addition of vectors

sum of two vectors as the vector extending from the tail of the first to the head of the second, Fig. 2.1b.

Algebraically, the addition is expressed by the vector equation

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$

Vector addition obeys the *commutative and associative laws*, so that

$$\begin{aligned} \mathbf{a}+\mathbf{b} &= \mathbf{b}+\mathbf{a} \\ (\mathbf{a}+\mathbf{b})+\mathbf{c} &= \mathbf{a}+(\mathbf{b}+\mathbf{c}) \end{aligned}$$

Vector subtraction is accomplished by addition of the negative vector as shown, for example, in Fig. 2.1c.

A vector $\mathbf{a}$ may be *multiplied by a scalar* m to yield a new vector $m\mathbf{a}$ of magnitude $|ma|$. If m is negative, $m\mathbf{a}$ has the direction of $-\mathbf{a}$; if m is zero, $m\mathbf{a}$ is a null vector. Multiplication of a vector by a scalar obeys the *associative and distributive laws*, so that

$$\begin{aligned} m(n\mathbf{a}) &= (mn)\mathbf{a} = n(m\mathbf{a}) \\ (m+n)\mathbf{a} &= (n+m)\mathbf{a} = n\mathbf{a}+m\mathbf{a} \\ m(\mathbf{a}+\mathbf{b}) &= m(\mathbf{b}+\mathbf{a}) = m\mathbf{a}+m\mathbf{b} \end{aligned}$$

We remark that according to the definition, a unit vector $\hat{\mathbf{a}}$ can be expressed as

$$\hat{\mathbf{a}} = \mathbf{a}/a$$

2.3.2 Product of vectors

The *dot* or *scalar product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the scalar

$$c = \mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = ab\cos\theta$$

where θ is the smaller anlge between the vectors as shown in Fig. 2.2a. The dot product $\mathbf{a}\cdot\hat{\mathbf{b}}$ is the projection of $\mathbf{a}$ in the direction of $\hat{\mathbf{b}}$; it is immediate to verify that

$$\mathbf{a}\cdot\mathbf{a} = a^2.$$

The *cross* or *vector product* of two vectors $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{c} = \mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} = (ab\sin\theta)\hat{\mathbf{e}}$$

where

- θ is the smaller angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, i.e. $\theta \leq 180°$;
- $\hat{\mathbf{e}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ in the sense that $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right-handed system.

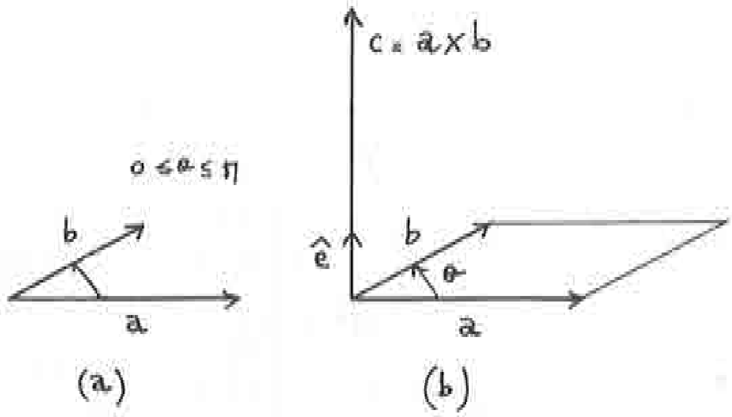


Figure 2.2: Vector products

- the magnitude c of $\mathbf{c}$ is equal to the area of the parallelogram defined by the vectors $\mathbf{a}$ and $\mathbf{b}$ as shown in Fig. 2.2b.

The cross product is not commutative; while, we have

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

It is easy to verify that

$$\begin{aligned} \mathbf{a} \times \mathbf{a} &= \mathbf{o} \\ m(\mathbf{a} \times \mathbf{b}) &= m\mathbf{a} \times \mathbf{b} = \mathbf{a} \times m\mathbf{b} \\ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \end{aligned}$$

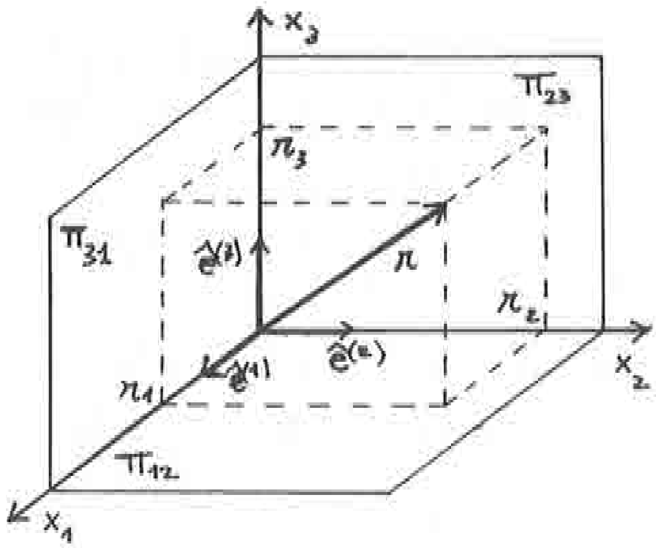
The *scalar triple product* is a dot product of two vectors, one of which is a cross product, namely

$$\lambda = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{c}$$

Geometrically, the absolute value of λ is the volume of the parallelepiped formed with $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ as three edges.

2.4 Cartesian Reference System

In the 3D Euclidean space of elementary geometry, a *Cartesian Reference System*, CaRS, is defined by three oriented (straight) lines crossing a unique point O in the space, Fig. 2.3. These three oriented lines and the crossing



x_i = Cartesian coordinate reference axes;

π_{ij} = Cartesian coordinate reference plane associated to $x_i x_j$;

r_i = Cartesian coordinates;

$\widehat{e}^{(i)}$ = unit vectors associated to x_i (length of $\widehat{e}^{(i)} = 1$).

Figure 2.3: Cartesian reference system

point are called the *Cartesian Coordinate Reference Axes*, CaRA, and the *origin* of the CaRS, respectively. The three planes identified by the CaRA are the *Cartesian Coordinate Reference Planes*, CaRP.

In this space, a *unit measure* is selected for segments so that there is a unique correspondence between the domain of the real numbers and the different lengths of segments on the CaRA measured from the origin. The positive versus of the CaRA correspond to positive real values.

The unit vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ in in Fig. 2.3, oriented in the positive direction of the CaRA, are the *unit base vectors* of the CaRS. Consequently, any vector $\mathbf{a}$ in this space may be uniquely identified as

$$\mathbf{a} = a_1\hat{\mathbf{e}}^{(1)} + a_2\hat{\mathbf{e}}^{(2)} + a_3\hat{\mathbf{e}}^{(3)} \tag{2.14}$$

where a_1, a_2, a_3 are the vector components (projection) of $\mathbf{a}$ with respect to the CaRA. When the CaRA form right angles, the CaRS is said to be a *Cartesian Orthogonal Coordinate Reference System*, CaORS.

2.4.1 Algebraic vector operations

With reference to a generic CaRS, the basic vector operations defined in the previous Section can be explicited as follows:

$$\begin{aligned}
\mathbf{a} \pm \mathbf{b} &= (a_1 \pm b_1)\hat{\mathbf{e}}^{(1)} + (a_2 \pm b_2)\hat{\mathbf{e}}^{(2)} + (a_3 \pm b_3)\hat{\mathbf{e}}^{(3)} \\
\mathbf{a} \cdot \mathbf{b} &= a_1b_1(\hat{\mathbf{e}}^{(1)} \cdot \hat{\mathbf{e}}^{(1)}) + a_1b_2(\hat{\mathbf{e}}^{(1)} \cdot \hat{\mathbf{e}}^{(2)}) + a_1b_3(\hat{\mathbf{e}}^{(1)} \cdot \hat{\mathbf{e}}^{(3)}) + \\
&+ a_2b_1(\hat{\mathbf{e}}^{(2)} \cdot \hat{\mathbf{e}}^{(1)}) + a_2b_2(\hat{\mathbf{e}}^{(2)} \cdot \hat{\mathbf{e}}^{(2)}) + a_2b_3(\hat{\mathbf{e}}^{(2)} \cdot \hat{\mathbf{e}}^{(3)}) + \\
&+ a_3b_1(\hat{\mathbf{e}}^{(3)} \cdot \hat{\mathbf{e}}^{(1)}) + a_3b_2(\hat{\mathbf{e}}^{(3)} \cdot \hat{\mathbf{e}}^{(2)}) + a_3b_3(\hat{\mathbf{e}}^{(3)} \cdot \hat{\mathbf{e}}^{(3)}) \\
\mathbf{a} \times \mathbf{b} &= a_1b_1(\hat{\mathbf{e}}^{(1)} \times \hat{\mathbf{e}}^{(1)}) + a_1b_2(\hat{\mathbf{e}}^{(1)} \times \hat{\mathbf{e}}^{(2)}) + a_1b_3(\hat{\mathbf{e}}^{(1)} \times \hat{\mathbf{e}}^{(3)}) + \\
&+ a_2b_1(\hat{\mathbf{e}}^{(2)} \times \hat{\mathbf{e}}^{(1)}) + a_2b_2(\hat{\mathbf{e}}^{(2)} \times \hat{\mathbf{e}}^{(2)}) + a_2b_3(\hat{\mathbf{e}}^{(2)} \times \hat{\mathbf{e}}^{(3)}) + \\
&+ a_3b_1(\hat{\mathbf{e}}^{(3)} \times \hat{\mathbf{e}}^{(1)}) + a_3b_2(\hat{\mathbf{e}}^{(3)} \times \hat{\mathbf{e}}^{(2)}) + a_3b_3(\hat{\mathbf{e}}^{(3)} \times \hat{\mathbf{e}}^{(3)})
\end{aligned}$$

where

$$\begin{aligned}
\hat{\mathbf{e}}^{(1)} \cdot \hat{\mathbf{e}}^{(1)} = \hat{\mathbf{e}}^{(2)} \cdot \hat{\mathbf{e}}^{(2)} = \hat{\mathbf{e}}^{(3)} \cdot \hat{\mathbf{e}}^{(3)} &= 1 \\
\hat{\mathbf{e}}^{(1)} \times \hat{\mathbf{e}}^{(1)} = \hat{\mathbf{e}}^{(2)} \times \hat{\mathbf{e}}^{(2)} = \hat{\mathbf{e}}^{(3)} \times \hat{\mathbf{e}}^{(3)} &= \mathbf{o}
\end{aligned}$$

In a CaORS, since $(\hat{\mathbf{e}}^{(1)}, \hat{\mathbf{e}}^{(2)}, \hat{\mathbf{e}}^{(3)})$ form a right-handed system, we have

$$\hat{\mathbf{e}}^{(1)} \cdot \hat{\mathbf{e}}^{(2)} = \hat{\mathbf{e}}^{(2)} \cdot \hat{\mathbf{e}}^{(3)} = \hat{\mathbf{e}}^{(3)} \cdot \hat{\mathbf{e}}^{(1)} = 0$$

and

$$\begin{array}{lll}\hat{\mathbf{e}}^{(1)}\times\hat{\mathbf{e}}^{(2)}=\hat{\mathbf{e}}^{(3)}; & \hat{\mathbf{e}}^{(2)}\times\hat{\mathbf{e}}^{(3)}=\hat{\mathbf{e}}^{(1)}; & \hat{\mathbf{e}}^{(3)}\times\hat{\mathbf{e}}^{(1)}=\hat{\mathbf{e}}^{(2)};\\ \hat{\mathbf{e}}^{(2)}\times\hat{\mathbf{e}}^{(1)}=-\hat{\mathbf{e}}^{(3)}; & \hat{\mathbf{e}}^{(3)}\times\hat{\mathbf{e}}^{(2)}=-\hat{\mathbf{e}}^{(1)}; & \hat{\mathbf{e}}^{(1)}\times\hat{\mathbf{e}}^{(3)}=-\hat{\mathbf{e}}^{(2)};\end{array}$$

Hence, in a CaORS, the scalar and the vector products between two vectors semplify into

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= a_1b_1+a_2b_2+a_3b_3\\ \mathbf{a}\times\mathbf{b} &= (a_2b_3-a_3b_2)\hat{\mathbf{e}}^{(1)}+(a_3b_1-a_1b_3)\hat{\mathbf{e}}^{(2)}+(a_1b_2-a_2b_1)\hat{\mathbf{e}}^{(3)}\end{aligned}$$

and, consequently, they can be simply indicated as, Section 2.2.3,

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^T\mathbf{b}=a_ib_i=\delta_{ij}a_ib_j \tag{2.15}$$

$$\mathbf{a}\times\mathbf{b} = \epsilon_{ijk}a_jb_k\hat{\mathbf{e}}^{(i)} \tag{2.16}$$

It is interesting to notice that the vector product and the scalar triple product can be respectively indicated as

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\hat{\mathbf{e}}^{(1)} & \hat{\mathbf{e}}^{(2)} & \hat{\mathbf{e}}^{(3)}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} \tag{2.17}$$

$$\mathbf{a}\times\mathbf{b}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix} \tag{2.18}$$

and the following identities hold for the triple vector product

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{c})\mathbf{b}=\mathbf{v} \tag{2.19}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}=\mathbf{w} \tag{2.20}$$

In fact, according to Eqs. 2.16 and 2.12,

$$\begin{aligned}v_k &= \epsilon_{kji}a_j(\epsilon_{irs}b_rc_s)=\epsilon_{kji}\epsilon_{irs}a_jb_rc_s=\\ &= (\delta_{kr}\delta_{js}-\delta_{ks}\delta_{jr})a_jb_rc_s=\\ &= (\delta_{js}a_jc_s)\delta_{kr}b_r-(\delta_{jr}a_jb_r)\delta_{ks}c_s=\\ &= (a_jc_j)b_k-(a_jb_j)c_k\end{aligned}$$

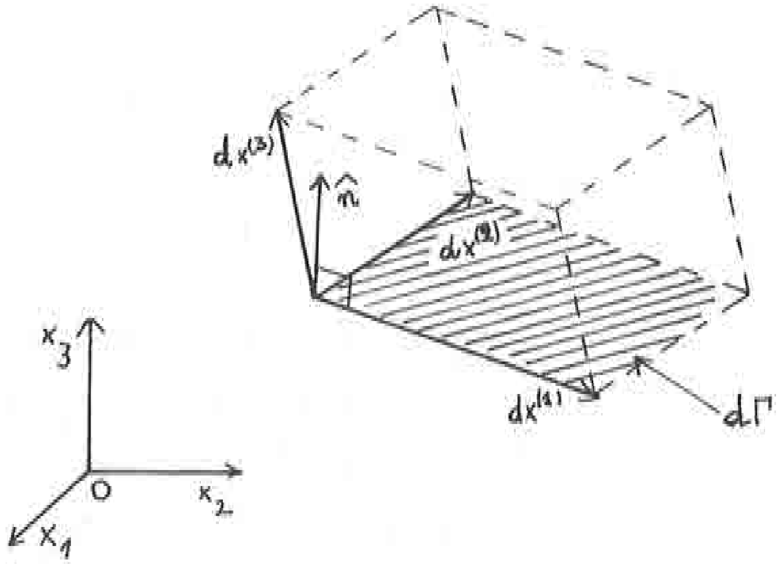


Figure 2.4: Surface and volume in a CaORS

2.4.2 Surface and Volume

Let $d\mathbf{x}^{(i)}$, for $i = 1, 2, 3$, be the three (infinitesimal) vectors defined in the CaORS shown in Fig. 2.4. According to the basic definitions reported in Section 2.3, we have that

- The modulus of any vector $d\mathbf{x}^{(i)}$ can be calculated as

$$dx^{(i)} = |d\mathbf{x}^{(i)}| = (d\mathbf{x}^{(i)T} d\mathbf{x}^{(i)})^{\frac{1}{2}} \tag{2.21}$$

  where

$$d\mathbf{x}^{(i)T} d\mathbf{x}^{(i)} = dx_1^{(i)2} + dx_2^{(i)2} + dx_3^{(i)2}$$

- The angle θ between two vectors $d\mathbf{x}^{(i)}$ and $d\mathbf{x}^{(j)}$ can be calculated as

$$\cos\theta = \frac{d\mathbf{x}^{(i)T} d\mathbf{x}^{(j)}}{|d\mathbf{x}^{(i)}||d\mathbf{x}^{(j)}|} \tag{2.22}$$

  where

$$d\mathbf{x}^{(i)T} d\mathbf{x}^{(j)} = dx_1^{(i)} dx_1^{(j)} + dx_2^{(i)} dx_2^{(j)} + dx_3^{(i)} dx_3^{(j)}$$

- The unit vector $\hat{\mathbf{n}}$ orthogonal to the plane defined by $d\mathbf{x}^{(i)}$ and $d\mathbf{x}^{(j)}$ can be calculated as

$$\hat{\mathbf{n}} = \frac{d\mathbf{x}^{(i)} \times d\mathbf{x}^{(j)}}{|d\mathbf{x}^{(i)} \times d\mathbf{x}^{(j)}|} \tag{2.23}$$

where

$$d\mathbf{x}^{(i)} \times d\mathbf{x}^{(j)} = \begin{vmatrix} \hat{\mathbf{e}}^{(1)} & \hat{\mathbf{e}}^{(2)} & \hat{\mathbf{e}}^{(3)} \\ dx_1^{(i)} & dx_2^{(i)} & dx_3^{(i)} \\ dx_1^{(j)} & dx_2^{(j)} & dx_3^{(j)} \end{vmatrix} = \epsilon_{rst}\hat{\mathbf{e}}^{(r)} dx_s^{(i)} dx_t^{(j)}$$

- The (infinitesimal) area $d\Gamma$ identified by $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ is defined as

$$\begin{aligned} \hat{\mathbf{n}} d\Gamma &= d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} = \begin{vmatrix} \hat{\mathbf{e}}^{(1)} & \hat{\mathbf{e}}^{(2)} & \hat{\mathbf{e}}^{(3)} \\ dx_1^{(1)} & dx_2^{(1)} & dx_3^{(1)} \\ dx_1^{(2)} & dx_2^{(2)} & dx_3^{(2)} \end{vmatrix} = \\ &= \epsilon_{rst}\hat{\mathbf{e}}^{(r)} dx_s^{(1)} dx_t^{(2)} \end{aligned} \tag{2.24}$$

where $\hat{\mathbf{n}}$ in the unit vector orthogonal to the plane $(d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)})$.

- The (infinitesimal) volume $d\Omega$ identified by $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$ and $d\mathbf{x}^{(3)}$ is defined as

$$\begin{aligned} d\Omega &= d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} \cdot d\mathbf{x}^{(3)} = \begin{vmatrix} \hat{\mathbf{e}}^{(1)} & \hat{\mathbf{e}}^{(2)} & \hat{\mathbf{e}}^{(3)} \\ dx_1^{(1)} & dx_2^{(1)} & dx_3^{(1)} \\ dx_1^{(2)} & dx_2^{(2)} & dx_3^{(2)} \\ dx_1^{(3)} & dx_2^{(3)} & dx_3^{(3)} \end{vmatrix} = \\ &= \epsilon_{rst} dx_r^{(1)} dx_s^{(2)} dx_t^{(3)} \end{aligned} \tag{2.25}$$

2.5 Elements of Differential Calculus

We know from the elementary Differential Calculus Theory that, if a multiple variable scalar function

$$F = F(a_1, a_2, \dots, a_n) \tag{2.26}$$

is continuous and differentiable over a_i, the total differential of F with respect to a_i is given by

$$dF = \frac{\partial F}{\partial a_1} da_1 + \frac{\partial F}{\partial a_2} da_2 + \cdots + \frac{\partial F}{\partial a_n} da_n = \frac{\partial F}{\partial a_j} da_j \tag{2.27}$$

where, each term

$$\frac{\partial F}{\partial a_j} da_j = F_{,j} da_j$$

no sum on j, represents the variation of F for an infinitesimal variation of a_j when the other variables are kept fixed.

In Continuum Mechanics, the variables of a scalar function are the spatial position $\mathbf{x} = \{x_1, x_2, x_3\}^T$ of the material point and the time t. Thus, let $F = F(\mathbf{x}, t)$ be a scalar function continuous in both $\mathbf{x}$ and t. The relative total differential can be indicated as

$$dF = \delta F + \frac{\partial F}{\partial t} dt \tag{2.28}$$

where

- the first term on the right

  $$\delta F = \frac{\partial F}{\partial x_i} dx_i$$

  represents the variation of F in space when t is kept constant.

- the second term on the right

  $$\frac{\partial F}{\partial t} dt$$

  represents the variation of F in time at a fixed material point, i.e at $\mathbf{x} = const.$

2.5.1 Differential vector operators

With respect to a CaORS, the total differential of a continuous function $F = F(\mathbf{x}, t)$ may be indicated as

$$dF = \delta F + \frac{\partial F}{\partial t} dt \tag{2.29}$$

where the variation in space is given by

$$\delta F = \nabla F^T d\mathbf{x} \tag{2.30}$$

in which

$$\begin{aligned}\nabla &= \left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}^T = \{\partial_i\}^T \\ \nabla F &= \operatorname{grad} F = \left\{\frac{\partial F}{\partial x_i}\right\}^T = \{F_{,i}\}^T\end{aligned}$$

are known as the *differential vector operator* and the *gradient vector* of F, respectively. It is immediate to verify that ∇F is a vector orthogonal to the surface

$$F(\mathbf{x}) = const \tag{2.31}$$

In fact,

$$\delta F = \nabla F^T d\mathbf{x} = 0$$

The scalar product of the differential vector operator ∇ by the gradient of function F defines a scalar ΔF known as the *Laplacian* of F, namely

$$\Delta F = \nabla^2 F = \nabla^T \nabla F = F_{,ii}$$

where the symbol Δ, or the equivalent ∇^2, is known as the *Laplacian differential operator.*

In Continuum Mechanics, vectors may also represent physical quantities varying continuosly in space and time with respect to a given CaORS. Consequently, each component of such type of vectors is actually a continuous function of the position vector $\mathbf{x}$ and time t. Thus, let

$$\mathbf{b}(\mathbf{x}, t) = \{b_1(\mathbf{x}, t), b_2(\mathbf{x}, t), b_3(\mathbf{x}, t)\}^T \tag{2.32}$$

be a vector representing a physical quantity in a given CaORS. The total variation of $\mathbf{b}$ is given by

$$d\mathbf{b} = \delta\mathbf{b} + \frac{\partial \mathbf{b}}{\partial t} dt \tag{2.33}$$

where

$$\begin{aligned}\mathbf{b} &= \mathbf{B} d\mathbf{x} \\ \mathbf{B} &= \left[\frac{\partial b_i}{\partial x_j}\right] = [b_{i,j}]\end{aligned}$$

The first term on the right $\delta\mathbf{b}$ represents the variation of $\mathbf{b}$ in space when t is kept constant. Instead, the second term in the right

$$\frac{\partial \mathbf{b}}{\partial t} dt$$

represents the variation of **b** in time at a fixed material point.

The matrix **B**, also known as the *gradient matrix* of **b**, is usually indicated as

$$\mathbf{B} = \nabla \mathbf{b} = \operatorname{grad} \mathbf{b} \tag{2.34}$$

The trace of this matrix is known as the *divergence* of **b** and it is indicated as

$$\operatorname{tr} \mathbf{B} = \operatorname{div} \mathbf{b} \tag{2.35}$$

Note that the divergence of a vector can be seen as the scalar product of the differential vector operator ∇ by the vector itself, namely

$$\operatorname{div} \mathbf{b} = \nabla^T \mathbf{b} = b_{i,i} \tag{2.36}$$

The vector product of the differential vector operator ∇ by the vector, say **b**, defines a vector known as the *curl* of **b**, namely

$$\operatorname{curl} \mathbf{b} = \nabla \times \mathbf{b} = \{\epsilon_{ijk} b_{k,j}\} \tag{2.37}$$

It is easy to verify that

$$\operatorname{curl} \operatorname{grad} F = \nabla \times \nabla F = \mathbf{o}$$

Finally, the scalar product of the differential vector operator ∇ by the gradient vector, say **b**, defines a scalar

$$\Delta \mathbf{b} = \nabla^2 \mathbf{b} = \nabla^T \nabla \mathbf{b} = b_{k,jj}$$

known as the *Laplacian* differential of **b**.

2.5.2 Stokes and Gauss Theorems

Let Γ be an oriented surface bounded by a closed curve C, Fig. 2.5. If $\hat{\mathbf{n}}$ is the normal to Γ defining its orientation, then the positive sense on C is defined according to the right-hand screw rule as illustrated by the unit tangent $\hat{\mathbf{t}}$.

According to *Stokes Theorem*, see for example [19], the integral of the tangent component of a vector **v** over a closed reducible curve , Fig. 2.5, is equal to the integral of the normal component of the curl of that vector over the surface bounded by the same curve, that is

$$\oint_S \hat{\mathbf{t}} \cdot \mathbf{v} ds = \int_\Gamma \hat{\mathbf{n}} \cdot (\nabla \times \mathbf{v}) d\Gamma \tag{2.38}$$

Table 2.3: Differential vector operators

	Vector notation	Index notation	Extended notation
Differential vector operator	∇	$\{\partial_i\}$	$\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \right\}^T$
Scalar gradient	$\text{grad}\, F = \nabla F$	$(F_{,i})^T$	$\left\{ \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3} \right\}^T$
Vector gradient	$\text{grad}\, \mathbf{b} = \nabla \mathbf{b}$	$[b_{i,j}]$	$\begin{bmatrix} \frac{\partial b_1}{\partial x_1} & \frac{\partial b_1}{\partial x_2} & \frac{\partial b_1}{\partial x_3} \\ \frac{\partial b_2}{\partial x_1} & \frac{\partial b_2}{\partial x_2} & \frac{\partial b_2}{\partial x_3} \\ \frac{\partial b_3}{\partial x_1} & \frac{\partial b_3}{\partial x_2} & \frac{\partial b_3}{\partial x_3} \end{bmatrix}$
Divergence of a vector	$\text{div}\, \mathbf{b} = \nabla^T \mathbf{b}$	$b_{i,i}$	$\left(\frac{\partial b_1}{\partial x_1} + \frac{\partial b_2}{\partial x_2} + \frac{\partial b_3}{\partial x_3} \right)$
Curl of a vector	$\text{curl}\, \mathbf{b} = \nabla \times \mathbf{b}$	$\{\epsilon_{ijk} b_{k,j}\}$	$\left\{ \begin{matrix} \left(\frac{\partial b_3}{\partial x_2} - \frac{\partial b_2}{\partial x_3} \right) \\ \left(\frac{\partial b_1}{\partial x_3} - \frac{\partial b_3}{\partial x_1} \right) \\ \left(\frac{\partial b_2}{\partial x_1} - \frac{\partial b_1}{\partial x_2} \right) \end{matrix} \right\}$
Laplacian of a vector	$\Delta \mathbf{b} = \nabla^T \text{grad}\, \mathbf{b}$	$b_{j,ii}$	$\left\{ \begin{matrix} \left(\frac{\partial^2 b_1}{\partial x_1{}^2} + \frac{\partial^2 b_1}{\partial x_2{}^2} + \frac{\partial^2 b_1}{\partial x_3{}^2} \right) \\ \left(\frac{\partial^2 b_2}{\partial x_1{}^2} + \frac{\partial^2 b_2}{\partial x_2{}^2} + \frac{\partial^2 b_2}{\partial x_3{}^2} \right) \\ \left(\frac{\partial^2 b_3}{\partial x_1{}^2} + \frac{\partial^2 b_3}{\partial x_2{}^2} + \frac{\partial^2 b_3}{\partial x_3{}^2} \right) \end{matrix} \right\}$

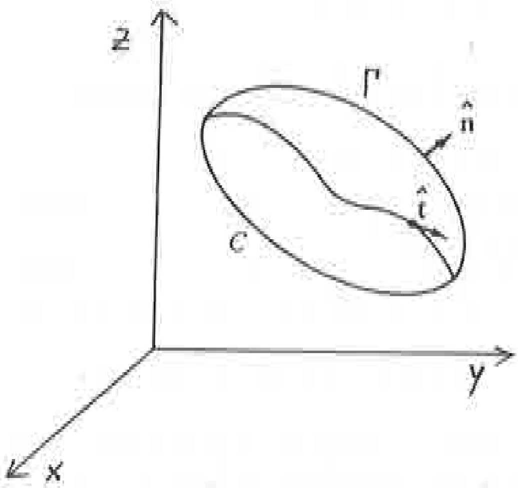


Figure 2.5: Oriented surface Γ and bounding curve C

The generalized form of this theorem takes the form

$$\oint_{s} \hat{\mathbf{t}} * \mathcal{A} ds = \int_{\Gamma} (\hat{\mathbf{n}} \times \nabla) * \mathcal{A} \, d\Gamma \tag{2.39}$$

where $\mathcal{A}$ represents a continuous tensor field of any order (scalar, vector or tensor) with continuous partial derivative in Γ and $*$ represents either the scalar (dot) or vector (cross) product. Note that for $\mathcal{A} \equiv \mathbf{v}$ and when $*$ indicates the scalar product,

$$(\hat{\mathbf{n}} \times \nabla) \cdot \mathbf{v} = \hat{\mathbf{n}} \cdot (\nabla \times \mathbf{v})$$

According to the *Gauss Theorem*, also known as the *Divergence Theorem*, the integral of the outer normal components of a vector $\mathbf{v}$ over a closed surface is equal to the integral of the divergence of that vector over the volume bounded by the same surface, that is

$$\int_{\Gamma} \hat{\mathbf{n}} \cdot \mathbf{v} \, d\Gamma = \int_{\Omega} \nabla \cdot \mathbf{v} \, d\Omega \tag{2.40}$$

The generalized form of this theorem is given by

$$\int_{\Gamma} \hat{\mathbf{n}} * \mathcal{A} \, d\Gamma = \int_{\Omega} \nabla * \mathcal{A} \, d\Omega \tag{2.41}$$

where $\mathcal{A}$ represents a continuous tensor field of any order (scalar, vector or tensor) with continuous partial derivative in Ω and $*$ represents either the scalar (dot) or vector (cross) product.

2.6 Curvilinear Coordinate System

In the 3D Euclidean space of elementary geometry, we can define a system of curvilinear coordinates by specifying three one-to-one functions of a reference system of Cartesian coordinates.

Thus, let P be a point in the 3D Euclidean space which, in a Cartesian reference system $F = Ox_1x_2x_3$, Fig. 2.6, is identified by the vector

$$\mathbf{x} = x_1\hat{\mathbf{e}}^{(1)} + x_2\hat{\mathbf{e}}^{(2)} + x_3\hat{\mathbf{e}}^{(3)} \tag{2.42}$$

where x_i are the Cartesian coordinates of $\mathbf{x}$ and $\hat{\mathbf{e}}^{(i)}$ are the unit vectors of F. With reference to a curvilinear system, CuRS, $F' = O'\xi_1\xi_2\xi_3$ the

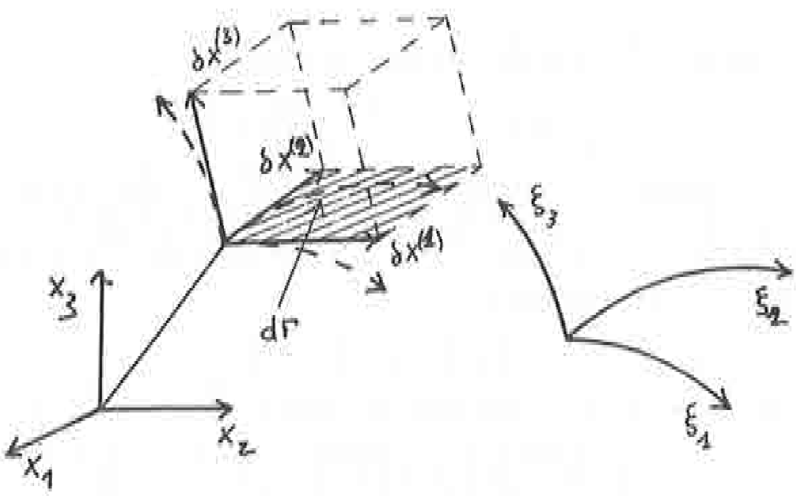


Figure 2.6: Curvilinear reference system

curvilinear coordinates of the same point may by identified as

$$\xi = \xi(\mathbf{x}) \tag{2.43}$$

where

$$\xi_i = \xi_i(x_1, x_2, x_3)$$

are the three one-to-one functions of the reference system of Cartesian coordinates. The one-to-one correspondence guaranties that the inverse relationships exist and they can be formally indicated as

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}) \tag{2.44}$$

that is

$$x_i = x_i(\xi_1, \xi_2, \xi_3)$$

If ξ_1 is held constant, the three Eqs. 2.44 define parametrically a surface, giving its rectangular Cartesian coordinates as function of the two parameters ξ_2 and ξ_3; this surface is a ξ_1-coordinate surface. Similarly, we can define the ξ_2 and ξ_3-coordinate surfaces helding constant the parameters ξ_2 and ξ_3 in turns.

The three coordinate surfaces intersect by pairs in three coordinate curves, on each of which only one curvilinear coordinate varies. The point of intersection of all three coordinate surfaces (and of all coordinate curves) is the point with curvilinear coordinates (ξ_1, ξ_2, ξ_3).

The mapping of the origin O and of the three Cartesian coordinate reference axes (x_1, x_2, x_3) identify a point O' and three curves (ξ_1, ξ_2, ξ_3) which may be used for labeling the curvilinear coordinate system $F' = O'\xi_1\xi_2\xi_3$, Fig. 2.5.

Let $d\mathbf{x}$ be the vector indicating the infinitesimal distance of P from a neighboring point Q. The Cartesian components of this vector are defined by

$$d\mathbf{x} = dx_1\hat{\mathbf{e}}^{(1)} + dx_2\hat{\mathbf{e}}^{(2)} + dx_3\hat{\mathbf{e}}^{(3)} \tag{2.45}$$

According to Eq. 2.43, the curvilinear components of the same vector may be determined as

$$d\xi_i = \frac{\partial \xi_i}{\partial x_j} dx_j$$

that is

$$d\boldsymbol{\xi} = \mathbf{J}\, d\mathbf{x} \tag{2.46}$$

where

$$\mathbf{J} = \left[\frac{\partial \xi_i}{\partial x_j}\right]$$

The condition of one-to-one correspondence imposed on $\boldsymbol{\xi}(\mathbf{x})$ requires the invertibility of $\mathbf{J}$, so that to the curvilinear components $d\xi_i$ there is unique correspondence dx_j; namely

$$d\mathbf{x} = \mathbf{J}^{-1} d\boldsymbol{\xi} \tag{2.47}$$

A necessary and sufficient condition for the inverse to exist is that the determinant J does not vanish for any point in the 3D Euclidian space, namely

$$J = \det \mathbf{J} \neq 0 \tag{2.48}$$

On the other hand, according to Eq. 2.44, $d\mathbf{x}$ may be calculated as

$$d\mathbf{x} = \mathbf{J}' d\boldsymbol{\xi} \tag{2.49}$$

where

$$\mathbf{J}' = \left[\frac{\partial x_i}{\partial \xi_j}\right]$$

Hence, the respect of the condition in Eq. 2.47 requires that

$$\mathbf{J}' = \mathbf{J}^{-1} \tag{2.50}$$

and, consequently,

$$J = \det \mathbf{J} = J'^{-1} \tag{2.51}$$

The matrix $\mathbf{J}$ and its determinant J are known as the *Jacobian matrix* and *Jacobian (determinant)* of the transformation $\mathbf{x} \to \boldsymbol{\xi}$, respectively. Conversely, the matrix $\mathbf{J}'$ and its determinant J' are known as the *Jacobian matrix* and *Jacobian (determinant)* of the transformation $\boldsymbol{\xi} \to \mathbf{x}$, respectively.

Expressing the components dx_i in Eq. 2.49 in terms of Eqs. 2.44 and 2.42, we recognize that the vector $d\mathbf{x}$ may be viewed as the resultant of the sum of the following three vectors

$$d\mathbf{x} = \delta\mathbf{x}^{(1)} + \delta\mathbf{x}^{(2)} + \delta\mathbf{x}^{(3)} \tag{2.52}$$

where, for example,

$$\delta\mathbf{x}^{(1)} = \frac{\partial \mathbf{x}}{\partial \xi_1} d\xi_1 = \left(\frac{\partial x_1}{\partial \xi_1}\hat{\mathbf{e}}^{(1)} + \frac{\partial x_2}{\partial \xi_1}\hat{\mathbf{e}}^{(2)} + \frac{\partial x_3}{\partial \xi_1}\hat{\mathbf{e}}^{(3)}\right) d\xi_1$$

We can verify that each of these vectors $\delta\mathbf{x}^{(i)}$, and thus $\partial\mathbf{x}/\partial\xi_i$, has direction tangent to the ξ_i-direction.

The vectors

$$\mathbf{g}^{(i)} = J'_{ji}\hat{\mathbf{e}}^{(j)} \tag{2.53}$$

that is

$$\mathbf{g}^{(i)} = \frac{\partial \mathbf{x}}{\partial \xi_i} = \frac{\partial x_1}{\partial \xi_i}\hat{\mathbf{e}}^{(1)} + \frac{\partial x_2}{\partial \xi_i}\hat{\mathbf{e}}^{(2)} + \frac{\partial x_3}{\partial \xi_i}\hat{\mathbf{e}}^{(3)}$$

are called the *natural basis* of the curvilinear system or *covariant base vectors.* In general, the natural basis $\mathbf{g}^{(i)}$ are neither unit vectors nor orthogonal one to the other.

The *unit base vectors* $\hat{\mathbf{e}}'^{(i)}$ of a curvilinear coordinate system at a point are defined as

$$\hat{\mathbf{e}}'^{(i)} = \frac{\mathbf{g}^{(i)}}{|\mathbf{g}^{(i)}|} \tag{2.54}$$

According to Eq. 2.53, they can be related to the unit base of the CaORS as

$$\hat{\mathbf{e}}'^{(i)} = A_{ji}\hat{\mathbf{e}}^{(j)} \tag{2.55}$$

where

$$A_{ji} = \frac{J'_{ji}}{|\mathbf{g}^{(i)}|}$$

The matrix

$$\mathbf{A} = [A_{ij}] \tag{2.56}$$

is sometime called as the *(physical) tensor transformation matrix.*

2.6.1 Functional relationships between a CaORS and CuRS

Let $F = Ox_1x_2x_3$ and $F' = O'\xi_1\xi_2\xi_3$ be a CaORS and a CuRS, respectively, related by

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}) \tag{2.57}$$

that is

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x_1(\xi_1, \xi_2, \xi_3) \\ x_2(\xi_1, \xi_2, \xi_3) \\ x_3(\xi_1, \xi_2, \xi_3) \end{Bmatrix}$$

where the Jacobian matrix of the transformation $\boldsymbol{\xi} \rightarrow \mathbf{x}$

$$\mathbf{J}' = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix}$$

is nonsingular matrix.

According to the derivations presented in the previous Section, we have that:

- The relationship between the Cartesian components $d\mathbf{x}$ and the curvilinear components $d\boldsymbol{\xi}$ of an infinitesimal vector is given by

$$d\mathbf{x} = \mathbf{J}' d\boldsymbol{\xi} \tag{2.58}$$

  that is

$$\begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial \xi_1} & \frac{\partial x_1}{\partial \xi_2} & \frac{\partial x_1}{\partial \xi_3} \\ \frac{\partial x_2}{\partial \xi_1} & \frac{\partial x_2}{\partial \xi_2} & \frac{\partial x_2}{\partial \xi_3} \\ \frac{\partial x_3}{\partial \xi_1} & \frac{\partial x_3}{\partial \xi_2} & \frac{\partial x_3}{\partial \xi_3} \end{bmatrix} \begin{Bmatrix} d\xi_1 \\ d\xi_2 \\ d\xi_3 \end{Bmatrix}$$

- The vector decomposition of an infinitesimal vector $d\mathbf{x}$ in the three curvilinear ξ_i-directions is given by

$$\begin{aligned} d\mathbf{x} &= \delta\mathbf{x}^{(1)} + \delta\mathbf{x}^{(2)} + \delta\mathbf{x}^{(3)} = \\ &= \mathbf{g}^{(1)} d\xi_1 + \mathbf{g}^{(2)} d\xi_2 + \mathbf{g}^{(3)} d\xi_3 \end{aligned} \tag{2.59}$$

  where

$$\mathbf{g}^{(i)} = \frac{\partial \mathbf{x}}{\partial \xi_i}$$

that is

$$\left\{ \begin{array}{c} \mathbf{g}^{(1)} \\ \mathbf{g}^{(2)} \\ \mathbf{g}^{(3)} \end{array} \right\} = \left[\begin{array}{ccc} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_1} \\ \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_2} \\ \dfrac{\partial x_1}{\partial \xi_3} & \dfrac{\partial x_2}{\partial \xi_3} & \dfrac{\partial x_3}{\partial \xi_3} \end{array} \right] \left\{ \begin{array}{c} \hat{\mathbf{e}}^{(1)} \\ \hat{\mathbf{e}}^{(2)} \\ \hat{\mathbf{e}}^{(3)} \end{array} \right\}$$

- The relationships between the unit vectors $\hat{\mathbf{e}}^{(i)}$ of the cartesian reference system and the local unit vectors $\hat{\mathbf{e}}'^{(i)}$ in the curvilinear system are given by

$$\hat{\mathbf{e}}'^{(i)} = A_{ji}\hat{\mathbf{e}}^{(j)} \tag{2.60}$$

that is

$$\left\{ \begin{array}{c} \hat{\mathbf{e}}'^{(1)} \\ \hat{\mathbf{e}}'^{(2)} \\ \hat{\mathbf{e}}'^{(3)} \end{array} \right\} = \left[\begin{array}{ccc} \dfrac{\partial x_1}{\partial \xi_1} \Big/ \left|\dfrac{\partial \mathbf{x}}{\partial \xi_1}\right| & \dfrac{\partial x_2}{\partial \xi_1} \Big/ \left|\dfrac{\partial \mathbf{x}}{\partial \xi_1}\right| & \dfrac{\partial x_3}{\partial \xi_1} \Big/ \left|\dfrac{\partial \mathbf{x}}{\partial \xi_1}\right| \\ \dfrac{\partial x_1}{\partial \xi_2} \Big/ \left|\dfrac{\partial \mathbf{x}}{\partial \xi_2}\right| & \dfrac{\partial x_2}{\partial \xi_2} \Big/ \left|\dfrac{\partial \mathbf{x}}{\partial \xi_2}\right| & \dfrac{\partial x_3}{\partial \xi_2} \Big/ \left|\dfrac{\partial \mathbf{x}}{\partial \xi_2}\right| \\ \dfrac{\partial x_1}{\partial \xi_3} \Big/ \left|\dfrac{\partial \mathbf{x}}{\partial \xi_3}\right| & \dfrac{\partial x_2}{\partial \xi_3} \Big/ \left|\dfrac{\partial \mathbf{x}}{\partial \xi_3}\right| & \dfrac{\partial x_3}{\partial \xi_3} \Big/ \left|\dfrac{\partial \mathbf{x}}{\partial \xi_3}\right| \end{array} \right] \left\{ \begin{array}{c} \hat{\mathbf{e}}^{(1)} \\ \hat{\mathbf{e}}^{(2)} \\ \hat{\mathbf{e}}^{(3)} \end{array} \right\}$$

where

$$\left|\frac{\partial \mathbf{x}}{\partial \xi_i}\right| = \left[\left(\frac{\partial x_1}{\partial \xi_i}\right)^2 + \left(\frac{\partial x_2}{\partial \xi_i}\right)^2 + \left(\frac{\partial x_3}{\partial \xi_i}\right)^2\right]^{\frac{1}{2}}$$

In general, the matrix $\mathbf{A}$ is not an orthogonal matrix. However, if $\hat{\mathbf{e}}'^{(i)}$ are orthogonal one to other the $\mathbf{A}$ becomes an orthogonal matrix, that is

$$\mathbf{A}^{-1} = \mathbf{A}^T \tag{2.61}$$

In fact, according to the definition of scalar product, the cosine of the angle between $\hat{\mathbf{e}}'^{(i)}$ can be determined as

$$\hat{\mathbf{e}}'^{(i)} \cdot \hat{\mathbf{e}}'^{(j)} = \left(A_{si}\hat{\mathbf{e}}^{(s)}\right) \cdot \left(A_{tj}\hat{\mathbf{e}}^{(t)}\right)$$

and, since $\hat{\mathbf{e}}^{(t)}$ form a right-handed triad,

$$\left(A_{si}\hat{\mathbf{e}}^{(s)}\right) \cdot \left(A_{tj}\hat{\mathbf{e}}^{(t)}\right) = \left(A_{sj}A_{ti}\right)\hat{\mathbf{e}}^{(s)} \cdot \hat{\mathbf{e}}^{(t)} = \left(A_{si}A_{tj}\right)\delta_{st} = A_{si}A_{sj}$$

it follows

$$\hat{\mathbf{e}}'^{(i)} \cdot \hat{\mathbf{e}}'^{(j)} = A_{si} A_{sj}$$

If $\hat{\mathbf{e}}^{(i)}$ form a right-handed orthogonal triad, then

$$\hat{\mathbf{e}}'^{(i)} \cdot \hat{\mathbf{e}}'^{(j)} = \delta_{ij}$$

which implies that,

$$A_{si} A_{sj} = \delta_{ij}$$

that is

$$\mathbf{A}^T \mathbf{A} = \mathbf{I}$$

2.6.2 Area and volume in curvilinear coordinate

Let $F = Ox_1x_2x_3$ and $F' = O'\xi_1\xi_2\xi_3$ be a CaORS and a CuRS, respectively, related by

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}) \tag{2.62}$$

that is

$$\left\{ \begin{array}{c} x_1 \\ \\ x_2 \\ \\ x_3 \end{array} \right\} = \left\{ \begin{array}{c} x_1(\xi_1, \xi_2, \xi_3) \\ \\ x_2(\xi_1, \xi_2, \xi_3) \\ \\ x_3(\xi_1, \xi_2, \xi_3) \end{array} \right\}$$

where the Jacobian matrix $\mathbf{J}'$ of the transformation $\boldsymbol{\xi} \to \mathbf{x}$, that is

$$\mathbf{J}' = \begin{bmatrix} \dfrac{\partial x_1}{\partial \xi_1} & \dfrac{\partial x_1}{\partial \xi_2} & \dfrac{\partial x_1}{\partial \xi_3} \\ \dfrac{\partial x_2}{\partial \xi_1} & \dfrac{\partial x_2}{\partial \xi_2} & \dfrac{\partial x_2}{\partial \xi_3} \\ \dfrac{\partial x_3}{\partial \xi_1} & \dfrac{\partial x_3}{\partial \xi_2} & \dfrac{\partial x_3}{\partial \xi_3} \end{bmatrix}$$

is a nonsingular matrix.

According to the relationship reported in Sections 2.4.2 and 2.6.1, it results that areas and volumes can be calculated in terms of curvilinear coordinates as follows:

- The value of the infinitesimal area $d\Gamma$ on the surface Γ at $\xi_3 = const$, defined as

$$\hat{\mathbf{n}}\, d\Gamma = \delta\mathbf{x}^{(1)} \times \delta\mathbf{x}^{(2)}$$

can be calculated in terms of curvilinear coordinates as

$$d\Gamma = \left| \frac{\partial \mathbf{x}}{\partial \xi_1} \times \frac{\partial \mathbf{x}}{\partial \xi_2} \right| d\xi d\eta = \ell d\xi d\eta \tag{2.63}$$

where

$$\begin{aligned} \ell &= \left[\left(\frac{\partial x_2}{\partial \xi_1}\frac{\partial x_3}{\partial \xi_2} - \frac{\partial x_2}{\partial \xi_2}\frac{\partial x_3}{\partial \xi_1} \right)^2 + \left(\frac{\partial x_1}{\partial \xi_2}\frac{\partial x_3}{\partial \xi_1} - \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_3}{\partial \xi_2} \right)^2 + \right. \\ &\quad \left. + \left(\frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_1}{\partial \xi_2}\frac{\partial x_2}{\partial \xi_1} \right)^2 \right]^{\frac{1}{2}} \end{aligned}$$

- The value of the infinitesimal volume defined as

$$d\Omega = \left(\delta\mathbf{x}^{(1)} \times \delta\mathbf{x}^{(2)} \right) \cdot \delta\mathbf{x}^{(3)}$$

can be calculated as

$$d\Omega = \left[\left(\frac{\partial \mathbf{x}}{\partial \xi_1} \times \frac{\partial \mathbf{x}}{\partial \xi_2} \right)^T \frac{\partial \mathbf{x}}{\partial \xi_3} \right] d\xi_1 d\xi_2 d\xi_3 = \det \mathbf{J}' d\xi_1 d\xi_2 d\xi_3 \tag{2.64}$$

2.6.3 Cylindrical coordinate system

One of the classical examples of curvilinear coordinate system is the so-called orthogonal cylindrical coordinate system, Fig. 2.7. The functional relationships between the Cartesian coordinates (x_1, x_2, x_3) and the cylindrical coordinates $(\xi_1 \equiv \varrho, \xi_2 \equiv \theta, \xi_3 \equiv z)$ are of the type

$$\begin{aligned} x_1 &= \varrho \cos\theta \\ x_2 &= \varrho \sin\theta \\ x_3 &\equiv z \end{aligned}$$

It is easy to verify that from these relationships it follows that

- The inverse relationships result to be equal to

$$\begin{aligned} \theta &= \tan^{-1}\frac{x_2}{x_1} \\ \varrho &= (x_1^2 + x_2^2)^{\frac{1}{2}} \\ z &= x_3 \end{aligned}$$

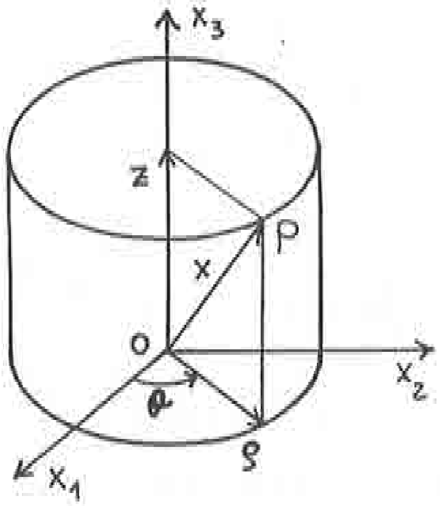


Figure 2.7: Orthogonal cylindrical coordinate system

- The Jacobian Matrix $\mathbf{J}'$ of the transformation $\boldsymbol{\xi} \rightarrow \mathbf{x}$ is given by

$$\mathbf{J}' = \begin{bmatrix} \cos\theta & -\varrho\sin\theta & 0 \\ \sin\theta & \varrho\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- The transformation matrix is given by

$$\mathbf{A} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is important to notice that $\mathbf{A}$ is an orthogonal matrix so that

$$\mathbf{A}^T = \mathbf{A}^{-1}$$

- The partial derivative operators result to be defined as

$$\begin{aligned}
\frac{\partial}{\partial x_1} &= \frac{\partial \varrho}{\partial x_1}\frac{\partial}{\partial \varrho} + \frac{\partial \theta}{\partial x_1}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial \varrho} - \frac{\sin\theta}{\varrho}\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial x_2} &= \frac{\partial \varrho}{\partial x_2}\frac{\partial}{\partial \varrho} + \frac{\partial \theta}{\partial x_2}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial \varrho} - \frac{\cos\theta}{\varrho}\frac{\partial}{\partial \theta} \\
\frac{\partial}{\partial x_3} &= \frac{\partial}{\partial z}
\end{aligned}$$

since, according to the above reported inverse relationships,

$$\begin{aligned}
\frac{\partial \varrho}{\partial x_1} &= \frac{x_1}{\varrho} = \cos\theta \\
\frac{\partial \varrho}{\partial x_2} &= \frac{x_2}{\varrho} = \sin\theta \\
\frac{\partial \theta}{\partial x} &= -\frac{x_2^2}{\varrho^2} = -\frac{\sin\theta}{\varrho} \\
\frac{\partial \theta}{\partial x_2} &= \frac{x^2}{\varrho^2} = -\frac{\cos\theta}{\varrho}
\end{aligned}$$

2.7 Change of CaORS

Let $F = Ox_1x_2x_3$ and $F' = O'\xi_1\xi_2\xi_3$ be two CaORS defined in the Euclidian space of unit base vectors $(\hat{\mathbf{e}}^{(1)}, \hat{\mathbf{e}}^{(2)}, \hat{\mathbf{e}}^{(3)})$ and $(\hat{\mathbf{e}}'^{(1)}, \hat{\mathbf{e}}'^{(2)}, \hat{\mathbf{e}}'^{(3)})$, respectively, Fig. 2.8a. Let indicate the cartesian components of $\hat{\mathbf{e}}'^{(i)}$ in F as

$$\begin{aligned}
\hat{\mathbf{e}}'^{(1)} &= \{l_1, l_2, l_3\}^T \\
\hat{\mathbf{e}}'^{(2)} &= \{m_1, m_2, m_3\}^T \\
\hat{\mathbf{e}}'^{(3)} &= \{n_1, n_2, n_3\}^T
\end{aligned}$$

Then, the Cartesian coordinates $\mathbf{x}$ in F and the Cartesian coordinates $\boldsymbol{\xi}$ in F' of any point P in the space can related as

$$\boldsymbol{\xi} = \mathbf{c} + \mathbf{R}\mathbf{x} \tag{2.65}$$

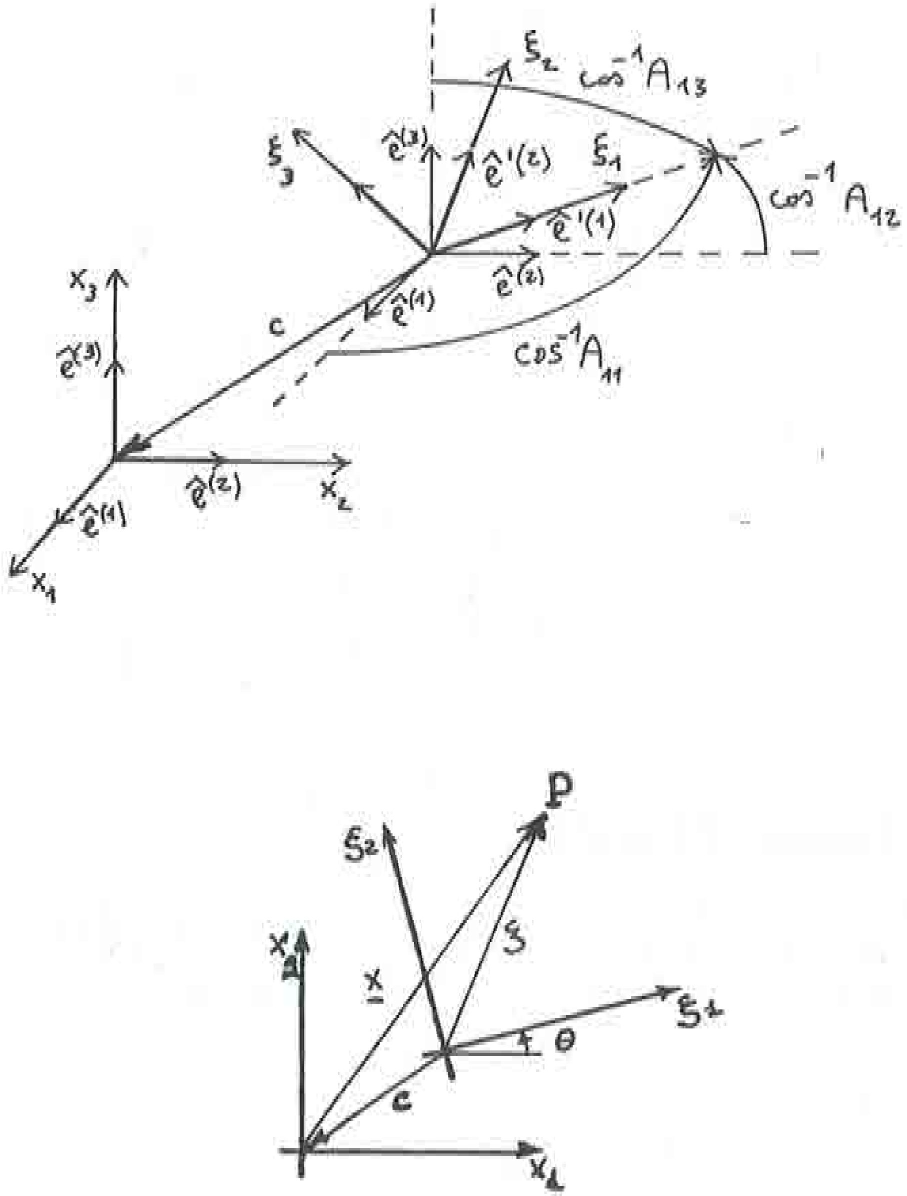


Figure 2.8: Change of Cartesian coordinate system

where $\mathbf{c}$ is the distance of the origin O form O' while

$$\mathbf{R} = \left[\hat{\mathbf{e}}'^{(1)}, \hat{\mathbf{e}}'^{(2)}, \hat{\mathbf{e}}'^{(3)}\right]^T = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \tag{2.66}$$

is an orthogonal constant matrix known as the *rotation matrix*. The inverse relationship is then given by

$$\mathbf{x} = \mathbf{R}^T(\boldsymbol{\xi} - \mathbf{c}) \tag{2.67}$$

In this case, the Jacobian matrix $\mathbf{J}$ and the matrix $\mathbf{A}$ of the transformation $\mathbf{x} \to \boldsymbol{\xi}$ are constant in space and are related to $\mathbf{R}$ as

$$\mathbf{R} \equiv \mathbf{J}^T \equiv \mathbf{A}^T \tag{2.68}$$

and

$$\begin{aligned} \mathbf{J}^{-1} &= \mathbf{J}' = \mathbf{R} \\ J &= J' = 1 \end{aligned}$$

It is useful to note that in case $\hat{\mathbf{e}}'^{(3)} \equiv \hat{\mathbf{e}}^{(3)}$, the rotation matrix takes the following simple form

$$\mathbf{R} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.69}$$

In fact, we first remark that, since F' is an CaORS, $\mathbf{A}$ is an orthogonal matrix. Let

$$\mathbf{x} = x_j \hat{\mathbf{e}}^{(j)} \tag{2.70}$$

be the position vector of a point P, Fig. 2.8b, where x_j are its cartesian component with respect to F. This same vector can be also indicated as

$$\mathbf{x}' = x_i' \hat{\mathbf{e}}'^{(i)} \tag{2.71}$$

where x_j' are the cartesian component of $\mathbf{x}$ with respect to F'. According to Eq. 2.61,

$$\mathbf{x}' = x_i' \hat{\mathbf{e}}'^{(i)} = x_i'(A_{ji} \hat{\mathbf{e}}^{(i)}) = (A_{ji} x_i') \hat{\mathbf{e}}^{(i)}$$

which equalized to Eq. 2.70 yields to enstablish that

$$x_j = A_{ji} x'_i$$

and hence

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \mathbf{R}^T\mathbf{x}$$

Finally, since $\mathbf{x}'$ can be expressed by the vectorial difference

$$\mathbf{x}' = \boldsymbol{\xi} - \mathbf{c}$$

we can obtain the relationship in Eq. 2.65.

Chapter 3

Tensors

3.1 Introduction

The Continuum Mechanics Theory deals with physical quantities which are independent of any particular coordinate system. Such quantities may be mathematically represented by *tensors*. A tensor field assigns a tensor $\mathbf{T}(\mathbf{x}, t)$ to every pair $(\mathbf{x}, t)$, where the position vector $\mathbf{x}$ and time variable t may vary over the spatial and the time domains, respectively.

Tensors may be classified by *rank*, or *order*, and this order reflects the number of components that a given tensor $\mathbf{T}$ possesses in a n-dimensional space. In the ordinary 3D Euclidian space, the number of components of a tensor is 3^N where N is the order of the tensor. Accordingly:

- A tensor of *order zero* is a tensor of *one* component so that it recalls the familiar concept of *scalar*. Scalars, i.e. 0-th order tensors, are used to represent physical quantities having magnitude only.

- A tensor of *order one* is a tensor of *three* components so that it recalls the familiar concept of *vector*. Vectors, i.e. 1-th order tensors, are used to represent physical quantities having both magnitude and direction.

- A tensor of *order two* is a tensor of *nine* components. For the purposes of this book, it is sufficient to consider a 2-nd order tensor as a *linear vector function*.

The mathematical laws obeying scalar and vector functions have been reported in the previous Chapter. The definition of the mathematical laws obeying 2-nd order tensors and their physical meaning are the main topics of this Chapter. Although tensors can be mathematically described with reference to any coordinate system, we will almost exclusively refer to the Cartesian Orthogonal Refence System, CaORS.

3.2 2-nd Order Tensors

A 2-nd order tensor is a tensor of nine components which, according to *Gibbs notation*, may be denoted by a capital bold-face. For brevity, we will often use the word *tensor* meaning 2-nd order tensor. In a CaORS, a tensor is represented by a square matrix of order 3

$$\mathbf{A} = [A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{23} \end{bmatrix}$$

By definition:

- Two tensors **A** and **B** are *equal* if all the corresponding nine components have the same value, that is

$$A_{ij} = B_{ij}$$

- A *null* or *zero* tensor **O** is one having all nine components value equal to zero.

- A *unit* or *identity* tensor **I** is one satisfying the requirement

$$\mathbf{I} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{I} = \mathbf{A}$$

  where the symbol $\cdot$ represent the tensor product, Section 3.2.2. In practice,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As mentioned before, for the purposes of this book, we define a tensor as a linear vector function. By definition, a vector function $\mathcal{F}$ is a *linear vector function* provided that

$$\begin{aligned} \mathcal{F}(\mathbf{a} + \mathbf{b}) &= \mathcal{F}(\mathbf{a}) + \mathcal{F}(\mathbf{b}) \\ \mathcal{F}(c\mathbf{a}) &= c\mathcal{F}(\mathbf{a}) \end{aligned}$$

for all vectors **a** and **b** in the domain and all real numbers c.

Thus, if **a** and **b** are two vectors related by a linear vector function $\mathcal{F}$

$$\mathbf{a} = \mathcal{F}(\mathbf{b}) \tag{3.1}$$

we can express this relationship by the tensorial operation

$$\mathbf{a} = \mathbf{C} \cdot \mathbf{b} \tag{3.2}$$

where **C** is a 2-nd order tensor. In a CaORS, the above product may be indicated as

$$\mathbf{a} = \mathbf{C}\mathbf{b}$$

and algebraically it implies the classical matrix vector product

$$a_i = C_{ik} b_k$$

obeying the rule

$$\mathbf{a}^T = (\mathbf{C}\mathbf{b})^T = \mathbf{b}^T \mathbf{C}^T$$

where the superscript T denotes the transpose of the tensor, that is

$$\begin{aligned} \mathbf{a}^T &= \{a_1, a_2, a_3\} \\ \mathbf{C}^T &= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \end{aligned}$$

3.2.1 Addition of tensors

By definition, the *sum* of two tensors **A** and **B** of the same kind, is the single tensor

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

which operates on any vector **v** to yield the sum of the two vectors obtained by operating with **A** and **B** separately. In a CaORS, the sum is algebraically performed as

$$C_{ij} = A_{ij} + B_{ij}$$

Tensor addition obeys the *commutative and associative laws*, so that

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= \mathbf{B} + \mathbf{A} \\ (\mathbf{A} + \mathbf{B}) + \mathbf{C} &= \mathbf{A} + (\mathbf{B} + \mathbf{C}) \end{aligned}$$

By definition, for each tensor **A**, there exists another $-\mathbf{A}$ such that

$$\mathbf{A} + (-\mathbf{A}) = \mathbf{O}$$

A tensor **A** may be *multiplied by a scalar* m to yield a new tensor $m\mathbf{A}$. Multiplication of a tensor by a scalar obeys the *associative and distributive laws*, so that

$$\begin{aligned} m(n\mathbf{A}) &= (mn)\mathbf{A} = n(m\mathbf{A}) \\ (m+n)\mathbf{A} &= (n+m)\mathbf{A} = n\mathbf{A} + m\mathbf{A} \\ m(\mathbf{A}+\mathbf{B}) &= m(\mathbf{B}+\mathbf{A}) = m\mathbf{A} + n\mathbf{B} \end{aligned}$$

3.2.2 Tensor products

By definition, the *product* of two tensors **A** and **B**, usually denoted as $\mathbf{A}\cdot\mathbf{B}$, is the composition of two operations **A** and **B**, with **B** performed first, defined by the requirement that

$$(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{v} = \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{v})$$

for all vectors **v**. In a CaORS, the product of two tensors may be indicated as

$$\mathbf{C} = \mathbf{AB}$$

and algebraically it implies the classical row by column matrix product

$$C_{ij} = A_{ik}B_{kj}$$

The product obeys the *associative* and *distributive laws*, so that

$$\begin{aligned} (\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} &= \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C}) \\ \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) &= \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \\ (\mathbf{A}+\mathbf{B})\cdot\mathbf{C} &= \mathbf{A}\cdot\mathbf{C} + \mathbf{B}\cdot\mathbf{C} \\ m(\mathbf{A}\cdot\mathbf{B}) &= (m\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot(m\mathbf{B}) \end{aligned}$$

The tensor product is *not commutative*, so that

$$\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A}$$

In a CaORS, however,

$$\begin{aligned} \mathbf{AB} &= (\mathbf{B}^T\mathbf{A}^T)^T \\ (\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T \end{aligned}$$

where the superscript T denotes the transpose of the tensor.

The *scalar product* of two tensors **A** and **B** is the scalar

$$c = \mathbf{A} : \mathbf{B} = \mathbf{B} : \mathbf{A}$$

which, in a CaORS, is algebraically expressed as

$$c = A_{ij} B_{ij}$$

The scalar product obeys the *associative* and *distributive laws*, so that

$$\begin{aligned} (\mathbf{A} : \mathbf{B}) : \mathbf{C} &= \mathbf{A} : (\mathbf{B} : \mathbf{C}) \\ \mathbf{A} : (\mathbf{B} + \mathbf{C}) &= \mathbf{A} : \mathbf{B} + \mathbf{A} : \mathbf{C} \\ (\mathbf{A} + \mathbf{B}) : \mathbf{C} &= \mathbf{A} : \mathbf{C} + \mathbf{B} : \mathbf{C} \\ m(\mathbf{A} : \mathbf{B}) &= (m\mathbf{A}) : \mathbf{B} = \mathbf{A} : (m\mathbf{B}) \end{aligned}$$

Note that,

$$\mathbf{A} : \mathbf{A} > 0$$

unless for $\mathbf{A} \equiv \mathbf{O}$. An alternative *scalar product* of two tensors **A** and **B** is defined as

$$c = \mathbf{A} \cdot\cdot \mathbf{B} = \mathbf{B} \cdot\cdot \mathbf{A}$$

which, in a CaORS, is algebraically expressed as

$$c = A_{ij} B_{ji} = B_{ij} A_{ji}$$

3.3 Symmetric and Skew-symmetric Tensors

A tensor whose Cartesian representation is of the type

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

is called *symmetric tensor*. Note that

$$\mathbf{A} = \mathbf{A}^T$$

On the other hand, a tensor whose Cartesian representation is of the type

$$\mathbf{B} = \begin{bmatrix} 0 & B_{12} & B_{13} \\ -B_{12} & 0 & B_{23} \\ -B_{13} & -B_{23} & 0 \end{bmatrix}$$

is called *skew-symmetric tensor*. Note that

$$\mathbf{B} = -\mathbf{B}^T$$

It is easy to verify that:

- Any tensor $\mathbf{C}$ can be uniquely decomposed into the sum of a symmetric tensor $\mathbf{A}$ and of a skew-symmetric tensor $\mathbf{B}$, namely

$$\mathbf{C} = \mathbf{A} + \mathbf{B} \tag{3.3}$$

where

$$\begin{aligned} \mathbf{A} &= \frac{1}{2}[\mathbf{C} + \mathbf{C}^T] \\ \mathbf{B} &= \frac{1}{2}[\mathbf{C} - \mathbf{C}^T] \end{aligned}$$

- The scalar product between a symmetric tensor $\mathbf{A}$ and a skew-symmetric tensor $\mathbf{B}$ is equal to zero, namely

$$\mathbf{A} : \mathbf{B} = \mathbf{A} \cdot \cdot \mathbf{B} = 0 \tag{3.4}$$

- The product

$$\mathbf{a} = \mathbf{B} \cdot \mathbf{b} \tag{3.5}$$

where $\mathbf{B}$ is a skew-symmetric matrix, is equivalent to the vector product

$$\mathbf{a} = \boldsymbol{\omega} \times \mathbf{b} \tag{3.6}$$

where

$$\boldsymbol{\omega} = \left\{ \begin{array}{c} \omega_1 \\ \omega_2 \\ \omega_2 \end{array} \right\} = -\frac{1}{2} \left\{ \begin{array}{c} (B_{23} - B_{32}) \\ (B_{31} - B_{13}) \\ (B_{12} - B_{21}) \end{array} \right\}$$

that is

$$\omega_i = -\frac{1}{2} \epsilon_{ijk} B_{jk}$$

3.4 Tensor Transformation Rules

Let $F = Ox_1x_2x_3$ be a CaORS and $F' = O\xi_1\xi_2\xi_3$ be a CuRS, both fixed in space and of unit base vectors $(\hat{\mathbf{e}}^{(1)}, \hat{\mathbf{e}}^{(2)}, \hat{\mathbf{e}}^{(3)})$ and $(\hat{\mathbf{e}}'^{(1)}, \hat{\mathbf{e}}'^{(2)}, \hat{\mathbf{e}}'^{(3)})$, respectively. Let $\mathbf{A}$ be the transformation matrix $\mathbf{x} \to \boldsymbol{\xi}$, Section 2.7, relating the unit base vectors as

$$\hat{\mathbf{e}}'^{(i)} = A_{ji}\hat{\mathbf{e}}^{(j)} \tag{3.7}$$

Then:

- The rule of changing the vector components under a change of reference system is given by

  $$\mathbf{u} = \mathbf{A}\mathbf{u}' \tag{3.8}$$

  that is

  $$u_i = A_{ij}u'_j$$

  where u_i and u'_j represent the values of the components that the same vector takes in F and F', respectively.

- The rule of changing the tensor components under a change of reference system is given by

  $$\mathbf{T}' = \mathbf{A}^{-1}\mathbf{T}\mathbf{A} \tag{3.9}$$

  that is

  $$T'_{ij} = A^{-1}_{ik}T_{kl}A_{lj}$$

  where T_{ij} and T'_{kl} represent the values of the components that the same tensor takes in F and F', respectively. We recall that if $\hat{\mathbf{e}}'^{(i)}$ form an orthogonal triad, as in the case of a CaORS, then $\mathbf{A}^{-1} = \mathbf{A}^T$ and, consequently,

  $$\mathbf{T}' = \mathbf{A}^T\mathbf{T}\mathbf{A}$$

In fact, let

$$\begin{aligned} \mathbf{u} &= u_j\hat{\mathbf{e}}^{(j)} \\ \mathbf{u}' &= u'_i\hat{\mathbf{e}}'^{(i)} \end{aligned}$$

be the tensorial representation of the same vector in F and F', respectively. According to Eq. 3.7,

$$\mathbf{u}' = u'_i\hat{\mathbf{e}}'^{(i)} = u'_i\left(A_{ji}\hat{\mathbf{e}}^{(j)}\right) = \left(u'_iA_{ji}\right)\hat{\mathbf{e}}^{(j)}$$

from which we identify the relationship in Eq. 3.8.

Let

$$\mathbf{u} = \mathbf{T}\mathbf{v}$$

be a linear vector operation performed with respect to F and

$$\mathbf{u}' = \mathbf{T}'\mathbf{v}'$$

be the same linear vector operation but performed with respect to F'. Then, according to Eq. 3.8,

$$\begin{aligned} \mathbf{v} &= \mathbf{A}\mathbf{v}' \\ \mathbf{u}' &= \mathbf{A}^{-1}\mathbf{u} = \mathbf{A}^{-1}\mathbf{T}\mathbf{v} = \mathbf{A}^{-1}\mathbf{T}\mathbf{A}\mathbf{v}' \end{aligned}$$

and, equalizing the two expressions for $\mathbf{u}'$, we obtain

$$\left(\mathbf{T}' - \mathbf{A}^{-1}\mathbf{T}\mathbf{A}\right)\mathbf{v}' = \mathbf{o}$$

for all $\mathbf{v}'$. This implies that

$$\mathbf{T}' - \mathbf{A}^{-1}\mathbf{T}\mathbf{A} = \mathbf{O}$$

from which we can establish the relationship in Eq. 3.9.

It is important to underline that the above transformation rules can be applied only between two reference systems fixed in space. The extension to the case of CaORS in relative motion one respect to the other is presented in Section 4.4.2.

3.5 Principal Values

In Continuum Mechanics, the three eigenvalues t_i associated to a symmetric Cartesian tensor

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \tag{3.10}$$

are called the *principal values* of $\mathbf{T}$. It is known from Linear Algebra, Chapter 5 in [38], that:

- The three eigenvalues t_i are the solutions of the standard eigenproblem

$$(\mathbf{T} - t\mathbf{I})\mathbf{a} = \mathbf{o} \tag{3.11}$$

- The eigenvalues can be calculated as the roots of the *characteristic equation* defined as

$$\det(\mathbf{T} - t\mathbf{I}) = 0 \tag{3.12}$$

- If $\mathbf{T}$ is a symmetric matrix, then the three roots are all real.
- If $\mathbf{T}$ is a symmetric and positive definite matrix, then the three roots are all real and positive.

The three principal values t_i are usually ordered as

$$t_1 \geq t_2 \geq t_3 \tag{3.13}$$

and the corresponding normalized eigenvectors

$$\hat{\mathbf{a}}^{(i)} = \frac{\mathbf{a}^{(i)}}{|\mathbf{a}^{(i)}|} \tag{3.14}$$

for, $i = 1, 2, 3$. are called *principal directions* of $\mathbf{T}$.

It is possible to prove that, Theorem 5.2.2 in [38], the three principal directions form a right-handed triad and they satisfy the following orthogonal properties,

$$\hat{\mathbf{a}}^{(i)T}\hat{\mathbf{a}}^{(j)} = \delta_{ij} = \begin{cases} 1; & \text{for } i = j. \\ 0; & \text{for } i \neq j. \end{cases} \tag{3.15}$$

$$\hat{\mathbf{a}}^{(i)T}\mathbf{T}\hat{\mathbf{a}}^{(j)} = \delta_{ij}t_i = \begin{cases} t_i; & \text{for } i = j. \\ 0; & \text{for } i \neq j. \end{cases} \tag{3.16}$$

Consequently, the matrix defined as

$$\hat{\mathbf{A}} = \left[\hat{\mathbf{a}}^{(1)}, \hat{\mathbf{a}}^{(2)}, \hat{\mathbf{a}}^{(3)}\right] \tag{3.17}$$

satisfies the following orthogonal properties,

$$\begin{aligned} \hat{\mathbf{A}}^T\hat{\mathbf{A}} &= \mathbf{I} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \\ \hat{\mathbf{A}}^T\mathbf{T}\hat{\mathbf{A}} &= \hat{\mathbf{T}} = \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix} \end{aligned}$$

The CaORS of unit base vector $\hat{\mathbf{a}}^{(i)}$ is called *principal reference system*.

It is possible to verify that the characteristic equation associated to the symmetric tensor $\mathbf{T}$ is given by

$$t^3 - I_1 t^2 + I_2 t - I_3 = 0 \tag{3.18}$$

where

$$\begin{aligned}
I_1 &= \operatorname{tr}\mathbf{T} = T_{ii} = T_{11} + T_{22} + T_{33} \\
I_2 &= \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ij}\right) = \\
&= \begin{vmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{23} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{13} & T_{33} \end{vmatrix} = \\
&= \left(T_{11}T_{22} + T_{11}T_{33} + T_{22}T_{33}\right) - \left(T_{12}^2 + T_{13}^2 + T_{23}^2\right) \\
I_3 &= \det\mathbf{T} = \epsilon_{ijk}T_{1i}T_{2j}T_{3k} = \\
&= T_{11}T_{22}T_{33} + 2T_{12}T_{13}T_{23} - \left(T_{11}T_{23}^2 + T_{22}T_{13}^2 + T_{33}T_{12}^2\right)
\end{aligned}$$

The principal values t_i and the values of the coefficients I_i, also known as *tensor invariants*, are invariant under any change of CaORS. In fact, according to Eq. 3.9, the components of a tensor under a change of CaORS can be calculated as

$$\mathbf{T}' = \mathbf{A}^{-1}\mathbf{T}\mathbf{A}$$

Algebraically, this represents a *similarity transformation* for which it is possible to prove that, Theorem 5.2.3 in [38], the characteristic equations relative to the standard eigenproblems associated to $\mathbf{T}'$ and $\mathbf{T}$, and thus the eigenvalues of $\mathbf{T}'$ and $\mathbf{T}$, coincide.

Finally, let $\mathbf{T}$ be a tensor defined in a CaORS of base vector $\vec{\mathbf{e}}^{(i)}$ and

$$\mathbf{T}' = \mathbf{A}^T\mathbf{T}\mathbf{A}$$

the transformed tensor in another CaORS of base vectors $\vec{\mathbf{e}}'^{(i)}$. According to Eq. 2.66 the relative transformation matrix is defined as

$$\mathbf{A} = \left[\vec{\mathbf{e}}'^{(1)}, \vec{\mathbf{e}}'^{(2)}, \vec{\mathbf{e}}'^{(3)}\right] \tag{3.19}$$

Then, from the orthogonal properties in Eq. 3.16, it is possible to prove that:

- If one of the $\vec{\mathbf{e}}'^{(i)}$ is equal to the principal direction $\hat{\mathbf{a}}^{(i)}$, then the nondiagonal terms in the i-th row and column of $\mathbf{T}'$ are equal to zero and $T'_{ii} = t_i$. For example, for $\hat{\mathbf{e}}'^{(1)} \equiv \hat{\mathbf{a}}^{(1)}$,

$$\mathbf{T}' = \begin{bmatrix} T_{11} & & \\ & T_{22} & T_{23} \\ & T_{23} & T_{33} \end{bmatrix} \tag{3.20}$$

where $T_{11} = t_1$. Tensors of this form are called *plane tensors*.

- If $\hat{\mathbf{e}}'^{(i)} \equiv \hat{\mathbf{a}}^{(i)}$ for all $i = 1, 2, 3$, then $\mathbf{T}'$ becames a diagonal matrix of diagonal values equal to the principal value of $\mathbf{T}$,

$$\mathbf{T}' = \tilde{\mathbf{T}} = \begin{bmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{bmatrix} \tag{3.21}$$

Incidentally, this diagonal representation of $\mathbf{T}$ allows to identify that, in terms of principal values, the tensor invariant expressions reduce to

$$\begin{aligned} I_1 &= t_1 + t_2 + t_3 \\ I_2 &= t_1 t_2 + t_1 t_3 + t_2 t_3 \\ I_3 &= t_1 t_2 t_3 \end{aligned}$$

3.6 Deviatoric tensor

Associated to a symmetric tensor $\mathbf{T}$, Eq. 3.10, we can define a *deviatoric tensor*

$$\mathbf{D} = \mathbf{T} - \mathbf{I}\frac{I_1}{3} \tag{3.22}$$

where I_1 is the first tensor invariant of $\mathbf{T}$. Similarly to the case of $\mathbf{T}$, the eigenvalues d_i solutions of the standard eigenproblem

$$(\mathbf{D} - d\mathbf{I})\mathbf{b} = \mathbf{o} \tag{3.23}$$

are called *principal deviatoric tensor*. We recognize that, algebraically, this eigenproblem in Eq. 3.23 represents a *shifted eigenproblem* with respect to that in Eq. 3.11 and the shift value is equal to $I_1/3$. Consequently, we can establish that, Theorem 5.6.4 in [38], the principal values d_i and directions $\hat{\mathbf{b}}^{(i)}$ of $\mathbf{D}$ are related to that of $\mathbf{T}$ as

$$d_i = t_i - \frac{I_1}{3} \tag{3.24}$$

$$\hat{\mathbf{b}}^{(i)} = \hat{\mathbf{a}}^{(i)} \tag{3.25}$$

for all $i = 1,2,3$. However, the principal values d_i can be directly calculated as the roots of the characteristic equations

$$\det(\mathbf{D} - d\mathbf{I}) = 0 \tag{3.26}$$

It is possible to verify that this characteristic equation has the following explicit expression

$$d^3 - J_2 d - J_3 = 0 \tag{3.27}$$

where the *deviatoric tensor invariants* J_i result to be equal to

$$\begin{aligned}
J_1 &= \operatorname{tr}\mathbf{D} = D_{ii} = D_{11} + D_{22} + D_{33} = 0 \\
J_2 &= -\frac{1}{2}\left(D_{ii}D_{jj} - D_{ij}D_{ij}\right) = \frac{1}{2}D_{ij}D_{ij} = \\
&= \frac{1}{2}\left(D_{11}^2 + D_{22}^2 + D_{33}^2\right) + D_{12}^2 + D_{13}^2 + D_{23}^2 = \\
&= -\left(D_{11}D_{22} + D_{11}D_{33} + D_{22}D_{33}\right) + D_{12}^2 + D_{13}^2 + D_{23}^2 \\
&= \frac{1}{6}\left\{\left[(T_{11} - T_{22})^2 + (T_{11} - T_{33})^2 + (T_{22} - T_{33})^2\right] + \right. \\
&\quad \left. +6\left(T_{12}^2 + T_{13}^2 + T_{23}^2\right)\right\} \\
J_3 &= \det\mathbf{D} = \epsilon_{ijk}D_{1i}D_{2j}D_{3k} = \\
&= D_{11}D_{22}D_{33} + 2D_{12}D_{13}D_{23} - \left(D_{11}D_{23}^2 + D_{22}D_{13}^2 + D_{33}D_{12}^2\right)
\end{aligned}$$

In the principal directions of $\mathbf{D}$, that is of $\mathbf{T}$, these tensor invariant expressions reduce to

$$\begin{aligned}
J_1 &= 0 \\
J_2 &= \frac{1}{2}\left(d_1^2 + d_2^2 + d_3^2\right) = \\
&= -\left(t_1t_2 + t_1t_3 + t_2t_3\right) = \\
&= \frac{1}{6}\left\{(t_1 - t_2)^2 + (t_1 - t_3)^2 + (t_2 - t_3)^2\right\} \\
J_3 &= d_1d_2d_3
\end{aligned}$$

Finally, the tensor invariants of $\mathbf{D}$ result to be related to that of $\mathbf{T}$ as

$$\begin{aligned}
J_2 &= \frac{1}{3}\left(I_1^2 - 3I_2\right) \\
J_3 &= \frac{1}{27}\left(2I_1^3 - 9I_1I_2 + 27I_3\right)
\end{aligned}$$

In fact, in analogy with the case of $\mathbf{T}$, the characteristic equation associated to the deviatoric tensor $\mathbf{D}$ results to be of the form

$$\det(\mathbf{D} - d\mathbf{I}) = d^3 - J_1 d^2 - J_2 d - J_3$$

where

$$\begin{aligned} J_1 &= D_{ii} \\ J_2 &= -\frac{1}{2}(D_{ii}D_{jj} - D_{ij}D_{ij}) \\ J_3 &= \det \mathbf{D} \end{aligned}$$

However,

$$D_{ii} = D_{11} + D_{22} + D_{33} = \left(T_{11} - \frac{I_1}{3}\right) + \left(T_{22} - \frac{I_1}{3}\right) + \left(T_{33} - \frac{I_1}{3}\right) = 0$$

and, consequently,

$$\begin{aligned} J_1 &= 0 \\ J_2 &= \frac{1}{2}D_{ij}D_{ij} = \frac{1}{2}\left(D_{11}^2 + D_{22}^2 + D_{33}^2\right) + D_{12}^2 + D_{13}^2 + D_{23}^2 = \\ &= \frac{1}{2}\left\{\left[\left(D_{11}^2 + D_{22}^2 + D_{33}^2\right) + 2(D_{11}D_{22} + D_{11}D_{33} + D_{22}D_{33})\right] + \right. \\ &\left. -2\,(D_{11}D_{22} + D_{11}D_{33} + D_{22}D_{33})]\right\} + D_{12}^2 + D_{13}^2 + D_{23}^2 = \\ &= \frac{1}{2}(D_{11} + D_{22} + D_{33})^2 - (D_{11}D_{22} + D_{11}D_{33} + D_{22}D_{33}) + \\ &+ D_{12}^2 + D_{13}^2 + D_{23}^2 \\ &= -(D_{11}D_{22} + D_{11}D_{33} + D_{22}D_{33}) + D_{12}^2 + D_{13}^2 + D_{23}^2 \end{aligned}$$

It is interesting to note that in a plane tensor, with

$$D_{13} = D_{23} = 0$$

the tensor invariant expressions reduce to

$$\begin{aligned} J_2 &= \frac{1}{2}\left(D_{11}^2 + D_{22}^2 + D_{33}^2\right) + D_{12}^2 = \\ &= -(D_{11}D_{22} + D_{11}D_{33} + D_{22}D_{33}) + D_{12}^2 \\ J_3 &= D_{11}D_{22}D_{33} - D_{33}D_{12}^2 = D_{33}\left(D_{11}D_{22} - D_{12}^2\right) \end{aligned}$$

Moreover, J_3 can be also expressed as

$$J_3 = D_{33}\left(D_{33}^2 - J_2\right)$$

In fact, solving the J_2 expression for $D_{11}D_{22}$, we find

$$\begin{aligned} D_{11}D_{22} &= -J_2 - (D_{11}D_{33} + D_{22}D_{33}) + D_{12}^2 = \\ &= -J_2 - D_{33}(D_{11} + D_{22}) + D_{12}^2 \end{aligned}$$

Since $J_1 = 0$, we can set

$$D_{11} + D_{22} = -D_{33}$$

and obtain

$$D_{11}D_{22} = -J_2 + D_{33}^2 + D_{12}^2$$

Consequently,

$$\begin{aligned} J_3 &= D_{33}\left(D_{11}D_{22} - D_{12}^2\right) = \\ &= D_{33}\left(-J_2 + D_{33}^2 + D_{12}^2 - D_{12}^2\right) = \\ &= D_{33}\left(D_{33}^2 - J_2\right) \end{aligned}$$

3.7 Analytical Solution for Principal Values

The principal values t_i of the symmetric Cartesian tensor $\mathbf{T}$ in Eq. 3.10 can be calculated as

$$\begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix} = \frac{I_1}{3}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} + 2\left(\frac{J_2}{3}\right)^{\frac{1}{2}}\begin{Bmatrix} \sin\left(\theta + \frac{2}{3}\pi\right) \\ \sin\theta \\ \sin\left(\theta - \frac{2}{3}\pi\right) \end{Bmatrix} \qquad (3.28)$$

where

$$\theta = \frac{1}{3}\arcsin\left(-\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right)$$

with

$$-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$$

In fact, through simple trigonometry, it is possible to verify the following identity

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta \tag{3.29}$$

Identical expression can be obtained by setting

$$\begin{aligned} d_i &= r\sin\theta_i \\ \frac{J_2}{r^2} &= \frac{3}{4} \\ -\frac{J_3}{r^3} &= \frac{\sin 3\theta_i}{4} \\ \theta_i &= \theta + \frac{2}{3}k_i\pi \end{aligned}$$

in the characteristic equation in Eq. 3.27, associated to the deviatoric tensor **D** in Eq. 3.22, that is

$$d^3 = J_2 d + J_3 \tag{3.30}$$

Hence, the principal values d_i of **D** can be calculated as

$$d_i = r\sin\left(\theta + \frac{2}{3}k_i\pi\right) \tag{3.31}$$

where

$$\begin{aligned} r &= 2\left(\frac{J_2}{3}\right)^{\frac{1}{2}} \\ k_i &= -1, 0, 1 \end{aligned}$$

Then, according to Eq. 3.24, the principal values t_i of **T** in Eq. 3.10 can be calculated as

$$t_i = \frac{I_1}{3} + d_i$$

from which it follows the analytical solution in Eq. 3.28

We notice that the angle θ is function of two tensor invariants so that it can be consider a tensor invariant too. We will see in next Section 3.8 that θ has a precise geometrical meaning in the principal CaORS of **T**.

3.8 Geometrical Meaning of the Tensor Components

In general, each column of a symmetric Cartesian tensor

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \tag{3.32}$$

may be seen as the component of the vector

$$\mathbf{u} = \{T_{1j}, T_{2,j}, T_{3j}\}$$

resultant from the linear tensor operation

$$\mathbf{u} = \mathbf{T}\hat{\mathbf{e}}^{(j)} \tag{3.33}$$

where $\hat{\mathbf{e}}^{(j)}$ is a base vector of the CaORS, see Fig. 3.1.

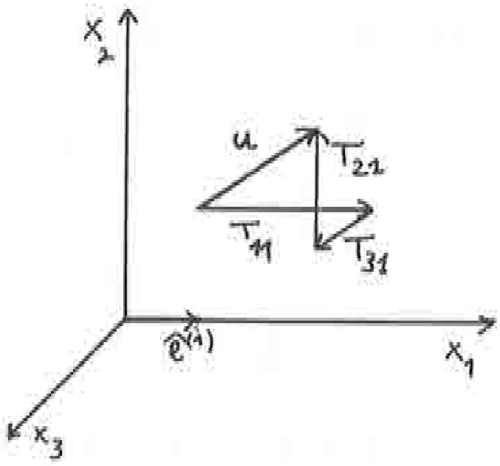


Figure 3.1: The vector components of $\mathbf{u} = \mathbf{T}\hat{\mathbf{e}}^{(1)}$

Let $F = Ox_1x_2x_3$ be the principal CaORS of $\mathbf{T}$ of unit base vectors $\hat{\mathbf{e}}^{(i)}$ and

$$\mathbf{T} = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \tag{3.34}$$

be the relative Cartesian representation of $\mathbf{T}$ in F where

$$t_1 \geq t_2 \geq t_3$$

are the principal values of $\mathbf{T}$. In this space F, Fig. 3.2, let define:

- the *spatial diagonal* as the unit vector

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}\{1, 1, 1\}^T \tag{3.35}$$

  equally inclined with respect to the axes of the reference system.

- the *octahedral plane* as the plane $\overline{\pi}$ orthogonal to $\hat{\mathbf{n}}$ and passing through the origin O of F, namely

$$\pi: \; x_1 + x_2 + x_3 = 0 \tag{3.36}$$

- the *octahedral axes* $\boldsymbol{\xi}^{(i)}$, for $i = 1, 2, 3$, as the axes resulting from the intersection of the planes $(\hat{\mathbf{n}}, \hat{\mathbf{e}}^{(i)})$ with the octahedral plane $\overline{\pi}$.

- the *octahedral reference system* as the refence system $\overline{F} = O\xi_1\xi_2\xi_3$ on the octahedral plane made by the three octahedral axes, Fig. 3.2b.

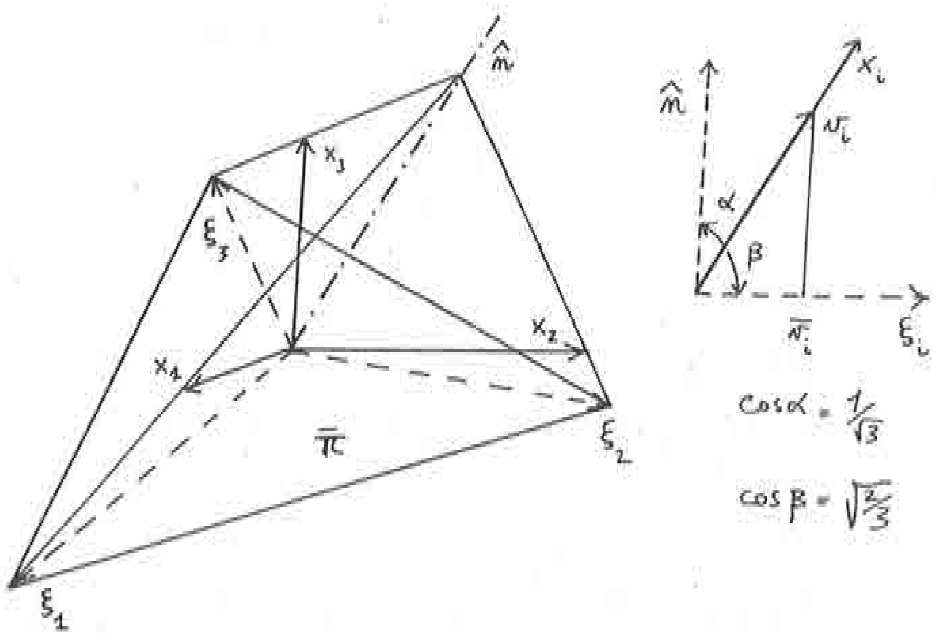
We notice that the $\boldsymbol{\xi}^{(i)}$-axes define on the $\overline{\pi}$-plane six sectors of 60° each; we will refer to the bisects of these sectors as the θ-axis. Let $\hat{\boldsymbol{\xi}}^{(i)}$ be the unit vectors in the direction of the $\boldsymbol{\xi}^{(i)}$ axes. It is immediate to verify that with respect to F

$$\begin{aligned}
\hat{\boldsymbol{\xi}}^{(1)} &= \frac{1}{\sqrt{6}}\{2, -1, -1\}^T \\
\hat{\boldsymbol{\xi}}^{(2)} &= \frac{1}{\sqrt{6}}\{-1, 2, -1\}^T \\
\hat{\boldsymbol{\xi}}^{(3)} &= \frac{1}{\sqrt{6}}\{-1, -1, 2\}^T
\end{aligned}$$

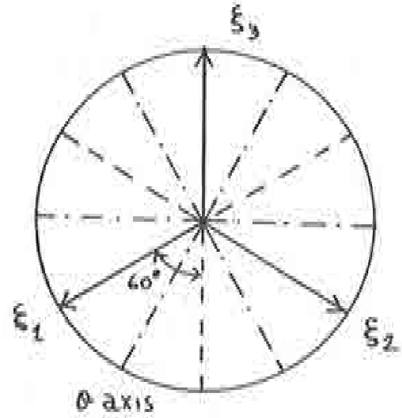
Figure 3.3 shows the graphical procedure for projecting a vector $\mathbf{v}$ from the F space into the $\overline{\pi}$-plane. Incidentally, the value of the components v_i of $\mathbf{v}$ in the F reference system are related to those in the $\overline{F}$ coordinate reference system as, Fig. 3.2,

$$\overline{v}_i = \sqrt{\frac{2}{3}}v_i \tag{3.37}$$

Then, with respect to the spatial diagonal $\hat{\mathbf{n}}$ we define:

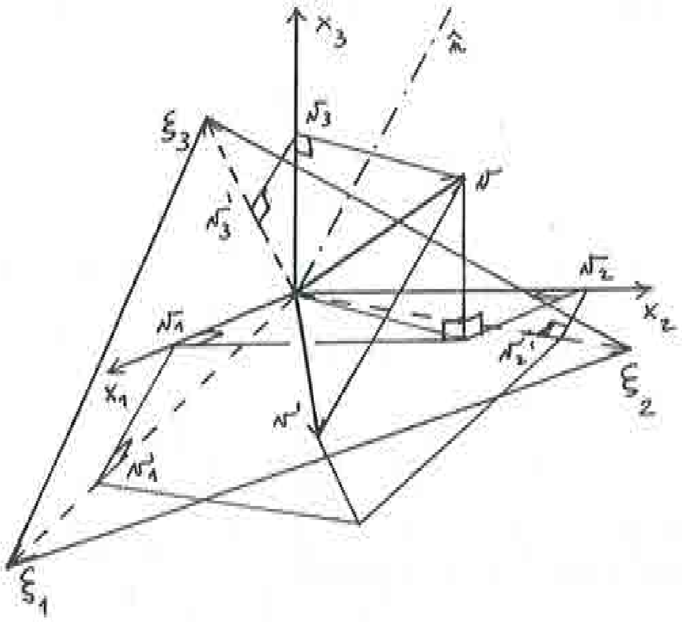


(a) Principal reference system.

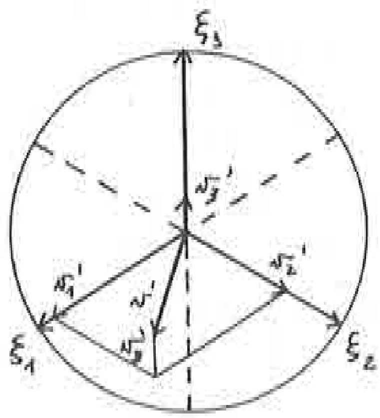


(b) Octahedral plane.

Figure 3.2: Octahedral plane in the principal CaORS



(a) Principal reference system.



(b) Octahedral plane.

Figure 3.3: Projection of a vector into the octahedral plane

- the *octahedral vector* representation of $\mathbf{T}$ as, Fig. 3.4,

$$\overline{\chi} = \mathbf{T}\hat{\mathbf{n}} = \{\overline{\chi}_1, \overline{\chi}_2, \overline{\chi}_3\}^T \tag{3.38}$$

where

$$\overline{\chi}_i = \frac{t_i}{\sqrt{3}}$$

- the *normal octahedral vector* $\overline{\boldsymbol{\nu}}$ as the projection of $\overline{\chi}$ on $\hat{\mathbf{n}}$, namely

$$\overline{\boldsymbol{\nu}} = \overline{\nu}\hat{\mathbf{n}} = \frac{\overline{\nu}}{\sqrt{3}}\{1,1,1\}^T \tag{3.39}$$

where

$$\overline{\nu} = |\overline{\chi}^T\hat{\mathbf{n}}| = \frac{t_1 + t_2 + t_3}{3} = \frac{I_1}{3}$$

- the *deviatoric octahedral vector* $\overline{\boldsymbol{\delta}}$ as the vector lying on $\overline{\pi}$ resultant of the difference between $\overline{\chi}$ and $\overline{\boldsymbol{\nu}}$, namely

$$\overline{\boldsymbol{\delta}} = \overline{\chi} - \overline{\boldsymbol{\nu}} = \left\{\overline{\delta}_1, \overline{\delta}_2, \overline{\delta}_3\right\}^T \tag{3.40}$$

where

$$\begin{aligned} \overline{\delta}_i &= \frac{1}{\sqrt{3}}\left(t_i - \frac{I_1}{3}\right) = \frac{d_i}{\sqrt{3}} \\ \overline{\delta} &= |\overline{\boldsymbol{\delta}}| = \left(\frac{2}{3}J_2\right)^{\frac{1}{2}} \end{aligned}$$

It is possible to prove that the value of the angle between $\overline{\boldsymbol{\delta}}$ and the θ-axis, Fig. 3.4, corresponds to the θ value defined in Eq. 3.28, that is

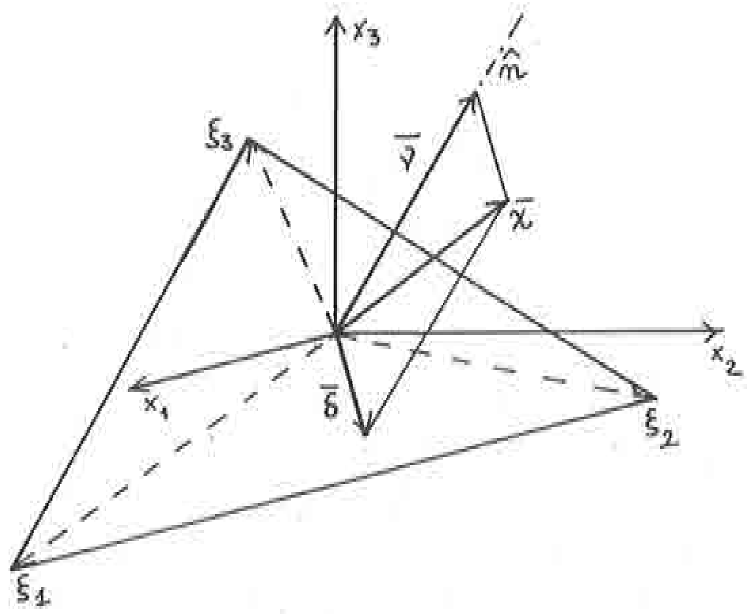
$$\theta = \frac{1}{3}\arcsin\left(-\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right) \tag{3.41}$$

In fact, the cosine of the angle α shown in Fig. 3.4 can be calculated as

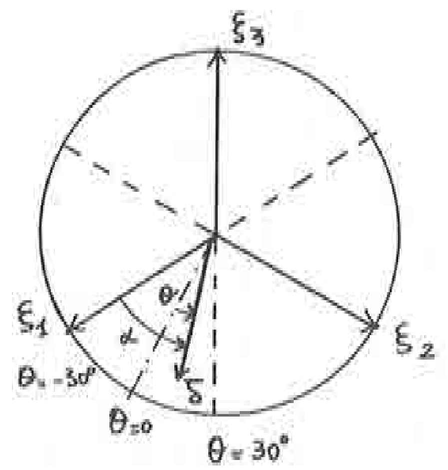
$$\cos\alpha = \frac{\overline{\xi}^{(1)T}\overline{\boldsymbol{\delta}}}{|\overline{\boldsymbol{\delta}}|} = \frac{1}{2\sqrt{3J_2}}(2d_1 - d_2 - d_3)$$

Since $J_1 = 0$, we can set

$$d_2 + d_3 = -d_1$$



(a) Principal reference system.



(b) Octahedral plane.

Figure 3.4: Octahedral vector representation of a tensors

and obtain

$$\cos\alpha = \frac{\sqrt{3}}{2}\frac{d_1}{J_2^{1/2}}$$

This expression substituted in the identity

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

yields to

$$\begin{aligned}\cos 3\alpha &= 4\left(\frac{\sqrt{3}}{2}\frac{d_1}{J_2^{1/2}}\right)^3 - 3\left(\frac{\sqrt{3}}{2}\frac{d_1}{J_2^{1/2}}\right) = \\ &= \frac{3\sqrt{3}}{2J_2^{3/2}}(d_1^3 - d_1 J_2)\end{aligned}$$

and, being

$$J_2 = -(d_1 d_2 + d_2 d_3 + d_3 d_1)$$

we obtain

$$\cos 3\alpha = \frac{3\sqrt{3}}{2J_2^{3/2}}\left[d_1^3 + d_1^2(d_2 + d_3) + d_1 d_2 d_3\right]$$

Then, being

$$\begin{aligned}d_2 + d_3 &= -d_1 \\ J_3 &= d_1 d_2 d_3\end{aligned}$$

we get

$$\cos 3\alpha = \frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}$$

Finally, since

$$\begin{aligned}\sin 3\theta &= \sin 3(\alpha - 30^o) = \\ &= \sin 3\alpha\cos 90^o - \cos 3\alpha\sin 90^o = \\ &= -\cos 3\alpha\end{aligned}$$

we can prove the statement in Eq. 3.41

We notice that the modulus of the normal octahedral vector $\overline{\nu}$ is proportional to the first tensor invariant I_1, the modulus of the deviatoric octahedral vector $\overline{\delta}$ is function of the 2-nd deviatoric tensor invariant J_2 while

Table 3.1: 2-nd order tensor invariants

Type of tensor invariants	Definition
Principal values of **T**	t_1, t_2, t_3
Coefficients of the $p(t)$ associated to **T**	I_1, I_2, I_3
Coefficients of the $p(d)$ associated to **D**	I_1, J_2, J_3
Octahedral components	$\overline{\nu}, \overline{\delta}, \theta$

the value of θ is function of the 3-rd deviatoric tensor invariant J_3. Thus, $(\overline{\nu}, \overline{\delta}, \theta)$ are themselves three independent tensor invariants of **T**.

At this regard it is worthwhile to remark that a 2-nd order Cartesian tensor has only three (independent) tensor invariants I_1, I_2, I_3. However, any three independent combinations of them can be elected as the invariants of a tensor. The most common alternative definitions of tensor invariants are reported in Table 3.1.

3.9 Mohr Circles

The Mohr circles is a very popular method for representing graphically a 2-nd order tensorial relationship. Let be:

- **T** a Cartesian symmetric tensor of principal values

$$t_1 \geq t_2 \geq t_3 \tag{3.42}$$

- $F = Ox_1x_2x_3$ the principal CaORS associated to **T**. We recall that with respect to F

$$\mathbf{T} = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix} \tag{3.43}$$

- $\hat{\mathbf{n}}$ a unit vector defined in F of components

$$\begin{aligned} \hat{\mathbf{n}} &= \{n_1, n_2, n_3\}^T &(3.44)\\ |\hat{\mathbf{n}}| &= \left(n_1^2 + n_2^2 + n_3^2\right)^{\frac{1}{2}} = 1 &(3.45)\end{aligned}$$

- π a plane orthogonal to $\hat{\mathbf{n}}$ and passing through the origin O of F.

Then, the tensorial relationship

$$\chi = \mathbf{T}\hat{\mathbf{n}} = \{t_1 n_1, t_2 n_2, t_3 n_3\}^T \quad (3.46)$$

associates to $\hat{\mathbf{n}}$ a vector χ whose projections on $\hat{\mathbf{n}}$ and π identify the following two orthogonal vectors, respectively,

$$\begin{aligned} \boldsymbol{\nu} &= \nu\hat{\mathbf{n}} = \nu\{n_1, n_2, n_3\}^T &(3.47)\\ \boldsymbol{\delta} &= \chi - \boldsymbol{\nu} = \{(t_1 - \nu)n_1, (t_2 - \nu)n_2, (t_3 - \nu)n_3\}^T &(3.48)\end{aligned}$$

It is easy to verify that the moduli of these three vectors are respectively equal to

$$\begin{aligned} \chi &= |\chi| = (t_1^2 n_1^2 + t_2^2 n_2^2 + t_3^2 n_3^2)^{\frac{1}{2}} &(3.49)\\ \nu &= |\boldsymbol{\nu}| = |\chi^T \hat{\mathbf{n}}| = |t_1 n_1^2 + t_2 n_2^2 + t_3 n_3^2| &(3.50)\\ \delta &= |\boldsymbol{\delta}| = (\chi^2 - \nu^2)^{\frac{1}{2}} &(3.51)\end{aligned}$$

According to the *Mohr Circles Method*, the value of the moduli ν, δ and χ can be graphically determined in a diagram δ vs. ν as the intersection of the following three circles

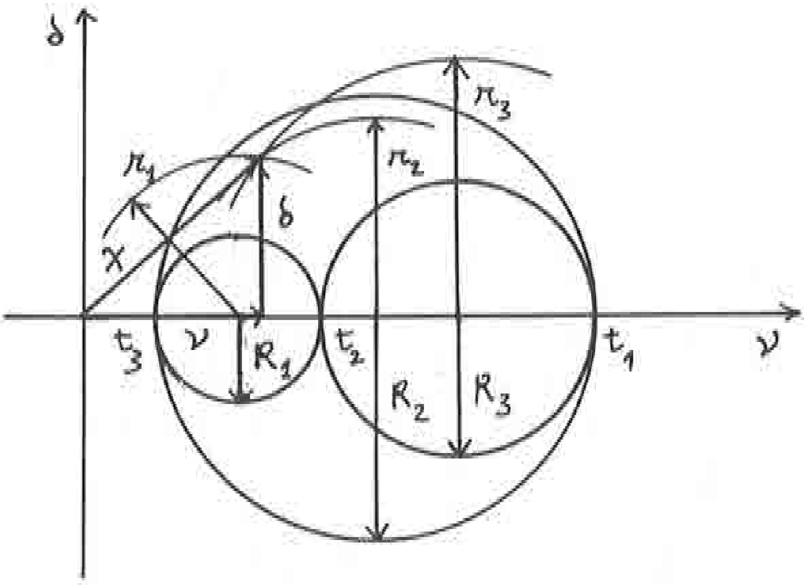
$$\begin{aligned} (\nu - C_1)^2 + \delta^2 &= r_1^2\\ (\nu - C_2)^2 + \delta^2 &= r_2^2 \qquad (3.52)\\ (\nu - C_3)^2 + \delta^2 &= r_3^2 \end{aligned}$$

where the center C_i and the radius r_i values are reported in Fig. 3.5.

3.9.1 Derivation

For a given set of principal values (t_1, t_2, t_3), Eqs. 3.45, 3.49, 3.50 and 3.51 can be combined to form the system of three equations

$$\begin{aligned} 1 &= n_1^2 + n_2^2 + n_3^2\\ \nu &= t_1 n_1^2 + t_2 n_2^2 + t_3 n_3^2\\ \nu^2 + \delta^2 &= t_1^2 n_1^2 + t_2^2 n_2^2 + t_3^2 n_3^2 \end{aligned}$$



Nº	C_i	r_i^2	R_i^2
1	$\frac{t_2+t_3}{2}$	$\left(\frac{t_2-t_3}{2}\right)^2 + n_1^2(t_1-t_2)(t_1-t_3)$	$\left(\frac{t_2-t_3}{2}\right)^2$
2	$\frac{t_1+t_3}{2}$	$\left(\frac{t_1-t_3}{2}\right)^2 + n_2^2(t_2-t_1)(t_2-t_3)$	$\left(\frac{t_1-t_3}{2}\right)^2$
3	$\frac{t_1+t_2}{2}$	$\left(\frac{t_1-t_2}{2}\right)^2 + n_3^2(t_3-t_1)(t_3-t_2)$	$\left(\frac{t_1-t_2}{2}\right)^2$

Figure 3.5: The Mohr circles

in the three unknowns (n_1, n_2, n_3). Solving this system for n_i^2 we obtain

$$\begin{aligned} n_1^2 &= \frac{\nu^2 + \delta^2 - \nu(t_2 + t_3) + t_2 t_3}{(t_2 - t_1)(t_3 - t_1)} \\ n_2^2 &= \frac{\nu^2 + \delta^2 - \nu(t_3 + t_1) + t_3 t_1}{(t_3 - t_2)(t_1 - t_2)} \\ n_3^2 &= \frac{\nu^2 + \delta^2 - \nu(t_1 + t_2) + t_1 t_2}{(t_1 - t_3)(t_2 - t_3)} \end{aligned}$$

Clearing these equations of their denominators we obtain

$$\begin{aligned} \nu^2 + \delta^2 - \nu(t_2 + t_3) &= n_1^2(t_2 - t_1)(t_3 - t_1) - t_2 t_3 \\ \nu^2 + \delta^2 - \nu(t_3 + t_1) &= n_2^2(t_3 - t_2)(t_1 - t_2) - t_3 t_1 \\ \nu^2 + \delta^2 - \nu(t_1 + t_2) &= n_3^2(t_1 - t_3)(t_2 - t_3) - t_1 t_2 \end{aligned}$$

from which, adding on both sides the terms

$$\frac{1}{4}(t_2 + t_3)^2, \ \frac{1}{4}(t_3 + t_1)^2, \text{ and } \frac{1}{4}(t_1 + t_2)^2$$

on each equation, respectively, we finally obtain the Mohr's circle equations in Eq. 3.52.

3.9.2 Remarks

It is easy to verify that since

$$\begin{aligned} &n_1^2, n_2^2, n_3^2 \geq 0 \\ &t_1 \geq t_2 \geq t_3 \end{aligned}$$

the following inequalities hold contemporaneously

$$\begin{aligned} (\nu - C_1)^2 + \delta^2 &\geq R_1 \\ (\nu - C_2)^2 + \delta^2 &\leq R_2 \\ (\nu - C_3)^2 + \delta^2 &\geq R_3 \end{aligned}$$

where the radius R_i values are reported in Fig. 3.5.

Moreover, for any unit vector $\hat{\mathbf{n}}$ lying on the plane orthogonal to a principal direction, the relative ν and δ values belonging to one of the boundary circles, that is:

- the χ values relative to all the unit vectors

$$\hat{\mathbf{n}} = \{0, n_2, n_3\}$$

orthogonal to the x_1 principal direction, lie on the Mohr circle

$$(\nu - C_1)^2 + \delta^2 = R_1$$

- the χ values relative to all the unit vectors

$$\hat{\mathbf{n}} = \{n_1, 0, n_3\}$$

orthogonal to the x_2 principal direction, lie on the Mohr circle

$$(\nu - C_2)^2 + \delta^2 = R_2$$

- the χ values relative to all the unit vectors

$$\hat{\mathbf{n}} = \{n_1, n_2, 0\}$$

orthogonal to the x_3 principal direction, lie on the Mohr circle

$$(\nu - C_3)^2 + \delta^2 = R_3$$

This implies that, for any given direction, the values of χ, and thus of ν and δ, are bounded by the Mohr circles. The region defined by the Mohr circles for all possible value of χ is also known as the *Mohr's Arbello.*

3.10 Applications

Let $F = Ox_1x_2x_3$ be a CaORS where the direction of the x_3-axis coincides with the 3-rd principal direction ξ_3 of $\mathbf{T}$. In this space F, the Cartesian symmetric tensor $\mathbf{T}$ is represented by a matrix of the type

$$\mathbf{T} = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{21} & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \tag{3.53}$$

where

$$\begin{aligned} T_{33} &\equiv t_3 \\ T_{21} &\equiv T_{12} \end{aligned}$$

and t_3 is the principal value associated to the $x_3 \equiv \xi_3$ direction. The Mohr circles relative to this *plane* tensor can be immediately constructed according to the graphical procedure shown in Fig. 3.6. This because:

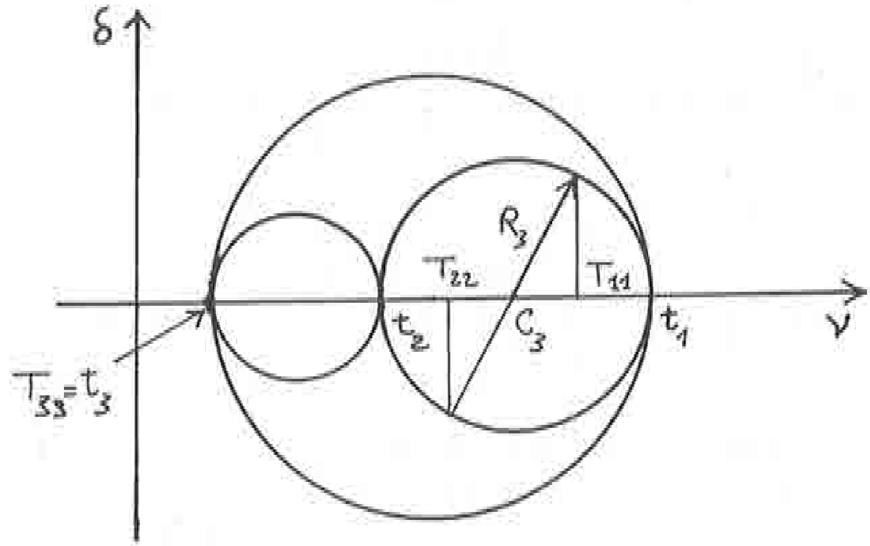


Figure 3.6: Construction of the Mohr circle relative to a plane tensor

- The radius and the center of the Mohr circle

$$(\nu - C_3)^2 + \delta^2 = R_3 \qquad (3.54)$$

describing the χ values relative to all the unit vectors

$$\hat{\mathbf{n}} = \{n_1, n_2, 0\}$$

orthogonal to the x_3 principal direction, can be calculated as

$$C_3 = \frac{t_1 + t_2}{2} = \frac{T_{11} + T_{22}}{2} \qquad (3.55)$$

$$R_3^2 = \frac{t_1 - t_2}{2} = \left(\frac{T_{11} - T_{22}}{2}\right)^2 + T_{12}^2 \qquad (3.56)$$

- The two principal values t_1 and t_2 of $\mathbf{T}$ can then be determined as the intersection of this Mohr circle with the ν axis; we recall that $T_{33} = t_3$.

- Knowing all the principal values t_1, t_2 and t_3, the other two circles can be easily constructed.

An analogous procedure can be followed for any *plane* tensor.

In fact, according to the relationships reported in Fig. 3.5, the Mohr circle associated with the directions

$$\hat{\mathbf{n}} = \{n_1, n_2, 0\}^T \tag{3.57}$$

orthogonal to the x_3 principal direction has equation

$$(\nu - C_3)^2 + \delta^2 = R_3^2 \tag{3.58}$$

where

$$\begin{aligned} C_3 &= \frac{t_1 + t_2}{2} \\ R_3 &= \frac{t_1 - t_2}{2} \end{aligned}$$

Since in this case

$$\begin{aligned} I_1 &= t_1 + t_2 + t_3 = T_{11} + T_{22} + T_{33} \\ t_3 &= T_{33} \end{aligned}$$

the center C_3 may be alternatively calculated as reported in Eq. 3.55. On the other hand, the octahedral vector associated with the $\hat{\mathbf{n}}$ direction is given by

$$\chi = \mathbf{T}\hat{\mathbf{n}} = \{(T_{11}n_1 + T_{12}n_2), (T_{21}n_1 + T_{22}n_2), 0\} \tag{3.59}$$

and the moduli of its octahedral vector components result to be equal to

$$\begin{aligned} \nu &= |\chi^T\hat{\mathbf{n}}| = |(T_{11}n_1 + T_{12}n_2)n_1 + (T_{21}n_1 + T_{22}n_2)n_2| \\ \delta &= \left(\chi^2 - \nu^2\right)^{1/2} \end{aligned}$$

In case $\hat{\mathbf{n}}$ has the direction of the $\mathbf{x}^{(1)}$ reference axis then

$$\begin{aligned} \nu &= |T_{11}| \\ \delta &= |T_{12}| = |T_{21}| \end{aligned}$$

Substituting these values and the C_3 expression in the Mohr circle equation in Eq. 3.58 we obtain the alternative expression for the radius reported in Eq. 3.56.

The Mohr-Circles method becomes extremely useful for studying the linear tensor function

$$\mathbf{b} = \mathbf{T}\mathbf{a} = a\mathbf{T}\hat{\mathbf{n}} \tag{3.60}$$

where

$$\begin{aligned} a &= |\mathbf{a}| \\ \hat{\mathbf{n}} &= \frac{\mathbf{a}}{a} \end{aligned}$$

and

- $\mathbf{T}$ is a plane tensor, i.e. its Cartesian representation is defined in a CaORS with one Cartesian axis coincident with one of the principal directions.
- $\hat{\mathbf{n}}$ varies in the plane orthogonal to that principal direction.

In fact, for example, let $\mathbf{T}$ be the plane tensor reported in Eq. 3.53 defined in a CaORS with the x_3-axis oriented as the 3-rd principal direction of $\mathbf{T}$. Following the procedure above reported, we can easily construct the Mohr circle describing the χ values relative to all the unit vectors

$$\hat{\mathbf{n}} = \{n_1, n_2, 0\} \quad .$$

orthogonal to the x_3 principal direction, Fig. 3.7a. Then, let us proceed as follows:

- Orient the ν and δ axes as the x_1 and x_2 reference axes, respectively.
- Assume the sign convention shown in Fig. 3.7a by which vectors oriented in counter-clock-wise versus have their modulus reported in the positive direction of δ-axis.
- Identify on the Mohr circle the so-called *Pole* as the intersection of the trace of the coordinate plane drawn from the (ν, δ) point associated to these planes.

With these positions, we can graphically determine the χ, and thus the ν and δ, values associated to any direction orthogonal to the x_3 direction as shown in Fig. 3.7b. Moreover, we can easily determine the principal values t_1 and t_2 as well as their principal directions.

In practice, the value of χ is found by drawing from the pole a line parallel to trace of the plane orthogonal to the given $\hat{\mathbf{n}}$. The intersection of this line with the Mohr circle identify the sought for χ, and thus the ν and δ, values.

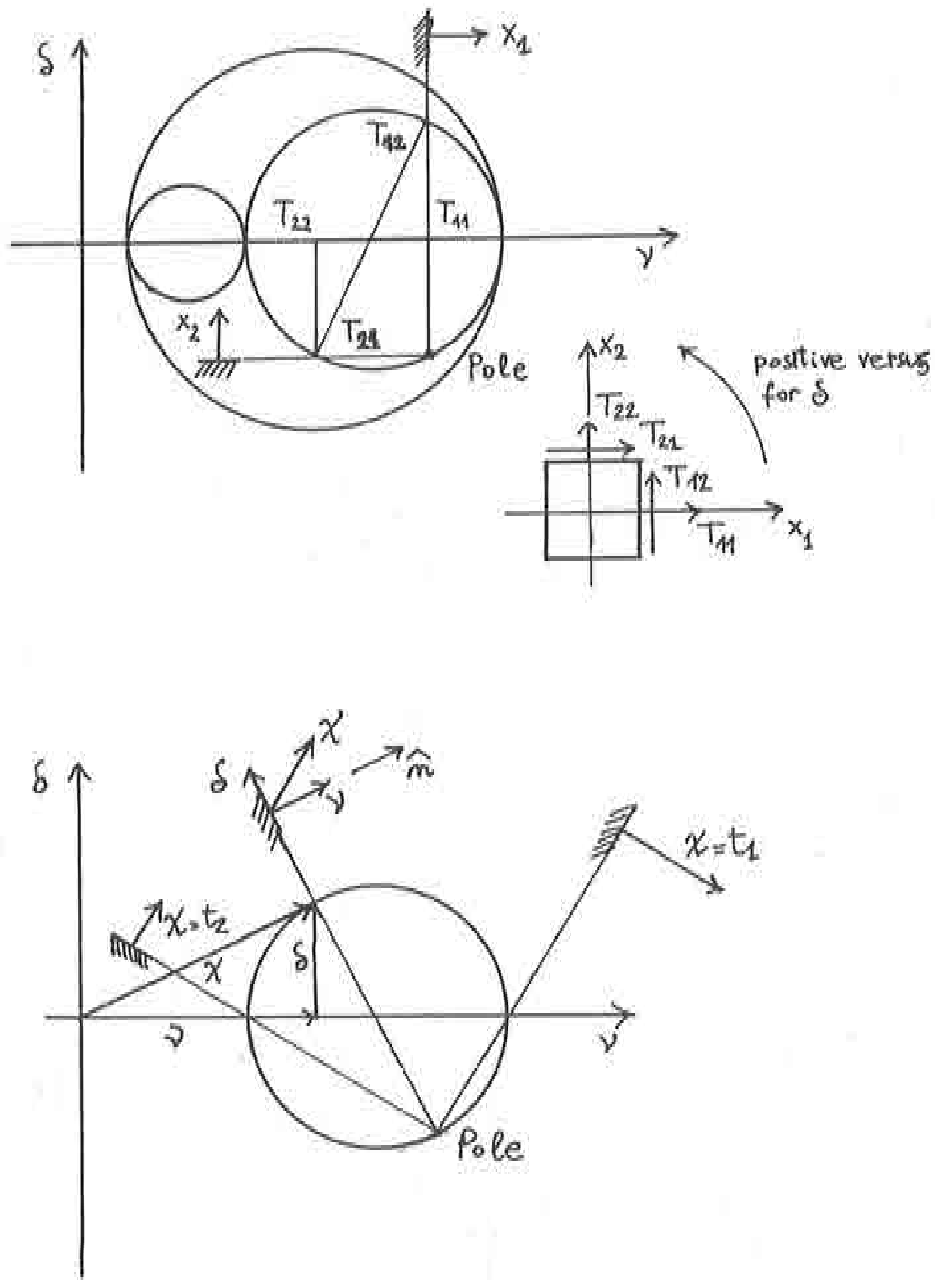


Figure 3.7: Graphical procedure on the Mohr circle for determining $\chi = \mathbf{Tn}$ when $\mathbf{T}$ is a plane tensor

Chapter 4

Mechanics of a Material Particle

4.1 Introduction

The Continuum Mechanics Theory deals with a medium made by a material that is assumed to be continuum. However, at the microscope level, any real material is made by a finite, although extremely large, number of discrete interacting material particles. Then, what the Continuum Mechanics Theory tries to describe mathematically is the final result of all interacting mechanisms of these material particles. The branch of physic that concerns the motion and change in position of interacting but distinct material particles is known as the Particles Mechanics Theory.

In this Chapter we review the fundamental kinematic and mechanical laws governing a material particle. In the next Chapters these laws will be then extended to a system of discrete interacting particle and, finally, to a continuum medium.

The classical Theoretical Mechanics is found on the mathematical abstraction of the existence of an elementary material object, called *particle*, which can be considered as occupying a point in space and perhaps moving as time goes by. This particle is matter that has a unique, positive measure known as the *mass* of the particle. This mass is a material property constant in space and in time.

With respect to a given frame of reference we can easily describe the motion of a material object, Section 4.2. However, we should be warned that the reference frame from which we make our observation may itself travel in space. In fact, if we observe the same material point from another frame in relative motion with respect to the first one, its motion may appear quite different, Section 4.3.

Among the different choices of frame of reference, two are of fundamental importance in the Particles Mechanics Theory: the *absolute* and the *inertial frame of reference*, Section 4.4. The absolute frame of reference, F^*, is assumed to be fixed in space, i.e. absolutely at rest. With respect to this frame the observed space or motion is absolute. Any other frame of reference which is moving at constant velocity (and no rotation) with respect to F^* is known as an inertial frame of reference.

With respect to F^* the motion of a material point is governed by the three Newton's laws of motion, Section 4.5. These laws are valid with respect to any other inertial frame of reference as well. Newton's laws of motion introduce the other fundamental mathematical concept of the Mechanics Theory: the *force*. Forces are eventually the cause of the change of state of rest or of uniform motion of a material point.

4.2 Motion

Let $F = Ox_1x_2x_3$ be a CaORS of unit base vectors $\widehat{\mathbf{e}}^{(i)}$, for $i = 1, 2, 3$, and P be a material point that from an initial position P_o of rest starts moving continuously with time along a path or curve C, Fig. 4.1. With respect to F, the initial position P_o at time $t = t_o$ and the current one P at time t can be identified by the position vectors

$$\mathbf{X} = X_i\widehat{\mathbf{e}}^{(i)} \tag{4.1}$$
$$\mathbf{x}(t) = x_i\widehat{\mathbf{e}}^{(i)} \tag{4.2}$$

Then, we define:

- The *displacement* of the particle as

$$\mathbf{u}(t) = \mathbf{x}(t) - \mathbf{X} \tag{4.3}$$

- The *velocity*, or *instantaneous velocity*, of the particle at P as

$$\mathbf{v} = \frac{d\mathbf{x}}{dt} = \frac{dx_i}{dt}\widehat{\mathbf{e}}^{(i)} \tag{4.4}$$

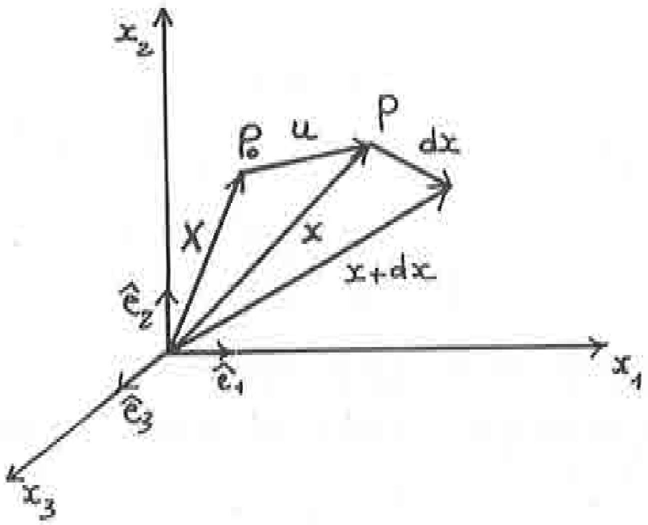


Figure 4.1: Motion in space of a material point P

- The *acceleration*, or *instantaneous acceleration*, of the particle at P as

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2 x_i}{dt^2}\hat{\mathbf{e}}^{(i)} \tag{4.5}$$

In the following we will represent the operation of total derivation with respect of time superpositioning a *dot* on the time variable, so that

$$\mathbf{v} = \dot{\mathbf{x}} = \dot{x}_i\hat{\mathbf{e}}^{(i)} \tag{4.6}$$
$$\mathbf{a} = \ddot{\mathbf{x}} = \ddot{x}_i\hat{\mathbf{e}}^{(i)} \tag{4.7}$$

and their relative moduli can be indicated as

$$v = |\mathbf{v}| = \left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right)^{\frac{1}{2}} \tag{4.8}$$
$$a = |\mathbf{a}| = \left(\ddot{x}_1^2 + \ddot{x}_2^2 + \ddot{x}_3^2\right)^{\frac{1}{2}} \tag{4.9}$$

The magnitude of the velocity is called *speed* and it can be also expressed as

$$v = |\mathbf{v}| = \sqrt{\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2} = \frac{ds}{dt} \tag{4.10}$$

where ds is the arc length, measured along the path C, covered by the particle in dt.

According to Eq. 4.3, the current position vector $\mathbf{x}(t)$ can equivalently be expressed as

$$\mathbf{x}(t) = \mathbf{X} + \mathbf{u}(t) \tag{4.11}$$

Since the initial position $\mathbf{X}$ is independent on time, it follows that the velocity and the acceleration can equivalently be expressed as

$$\mathbf{v} = \dot{\mathbf{u}} = \dot{u}_i \hat{\mathbf{e}}^{(i)} \tag{4.12}$$

$$\mathbf{a} = \ddot{\mathbf{u}} = \ddot{u}_i \hat{\mathbf{e}}^{(i)} \tag{4.13}$$

4.2.1 Tangential and normal acceleration

At the current position P of the particle, we can define a CaORS of unit base vectors, Fig. 4.2,

$$\hat{\mathbf{t}} = \frac{d\mathbf{x}}{ds} \tag{4.14}$$

$$\hat{\mathbf{n}} = R\frac{d\hat{\mathbf{t}}}{ds} \tag{4.15}$$

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} \tag{4.16}$$

where

- $\hat{\mathbf{t}}$ is the tangent to the curve C at P;
- s is the arc length from the initial point P_o to P.
- and
$$R = \left|\frac{d\hat{\mathbf{t}}}{ds}\right|$$
is the *radius of curvature* of C at P.

In terms of this unit base vectors, the velocity and the acceleration vectors are given by

$$\mathbf{v} = v\hat{\mathbf{t}} \tag{4.17}$$

$$\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{t}} + \frac{v^2}{R}\hat{\mathbf{n}} \tag{4.18}$$

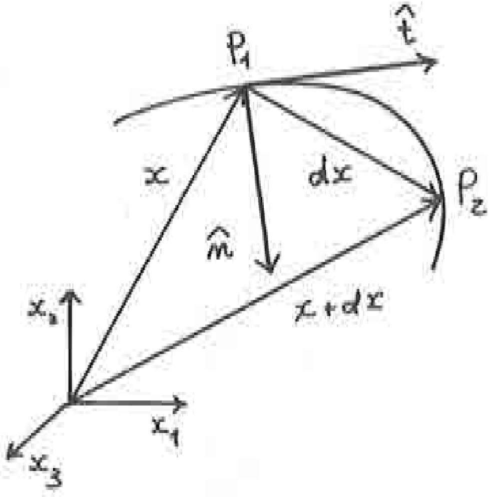


Figure 4.2: Tangential and normal acceleration

where the acceleration terms are commonly known as the *tangential acceleration* and the *normal or centripetal acceleration*, respectively.

The mathematical definition of the unit base vectors of the moving CaORS may be proved as follow. It is known from elementary geometry that the unit vector tangent to a curve C of equation

$$\mathbf{x} = \mathbf{x}(t)$$

is given by

$$\hat{\mathbf{t}} = \frac{\mathbf{t}}{|\mathbf{t}|}$$

where, according to Eqs. 4.4 and Eq. 4.10,

$$\mathbf{t} = \frac{d\mathbf{x}}{dt} = \mathbf{v} \tag{4.19}$$

$$|\mathbf{t}| = |\mathbf{v}| = v = \frac{ds}{dt} \tag{4.20}$$

Hence,

$$\hat{\mathbf{t}} = \frac{d\mathbf{x}}{dt}\frac{dt}{ds} = \frac{d\mathbf{x}}{ds}$$

Since $\hat{\mathbf{t}}$ is a unit vector, that is

$$|\hat{\mathbf{t}}| = \hat{\mathbf{t}} \cdot \hat{\mathbf{t}} = 1$$

then

$$\begin{aligned} 0 &= \frac{d|\hat{\mathbf{t}}|}{ds} = \hat{\mathbf{t}} \cdot \frac{d\hat{\mathbf{t}}}{ds} + \frac{d\hat{\mathbf{t}}}{ds} \cdot \hat{\mathbf{t}} = \\ &= 2\hat{\mathbf{t}} \cdot \frac{d\hat{\mathbf{t}}}{ds} \end{aligned}$$

Consequently,

$$\hat{\mathbf{t}} \cdot \hat{\mathbf{n}} = 0$$

where

$$\hat{\mathbf{n}} = R\frac{d\hat{\mathbf{t}}}{ds}$$

The velocity expression in Eq. 4.17 follows immediately from Eqs. 4.19 and 4.20. Consequently,

$$\begin{aligned} \mathbf{a} &= \frac{d}{dt}(v\hat{\mathbf{t}}) = \frac{dv}{dt}\hat{\mathbf{t}} + v\frac{d\mathbf{t}}{dt} = \\ &= \frac{dv}{dt}\hat{\mathbf{t}} + v\frac{d\hat{\mathbf{t}}}{ds}\frac{ds}{dt} = \frac{dv}{dt}\hat{\mathbf{t}} + v^2\frac{d\hat{\mathbf{t}}}{ds} = \\ &= \frac{dv}{dt}\hat{\mathbf{t}} + \frac{v^2}{R}\hat{\mathbf{n}} \end{aligned}$$

4.2.2 Circular motion

If the path C of the particle is a circle of radius R, then the arc length measured along C between two points P and Q is given by

$$s = R\theta \tag{4.21}$$

where θ is the corresponding angle subtended at the center of the circle, Fig. 4.3. In this case, the velocity and the acceleration vectors in Eqs. 4.17 and 4.18 reduce to

$$\mathbf{v} = R\omega\hat{\mathbf{t}} = \mathbf{r} \times \omega \tag{4.22}$$

$$\mathbf{a} = R\left(\frac{d\omega}{dt}\hat{\mathbf{t}} + \omega^2\hat{\mathbf{n}}\right) \tag{4.23}$$

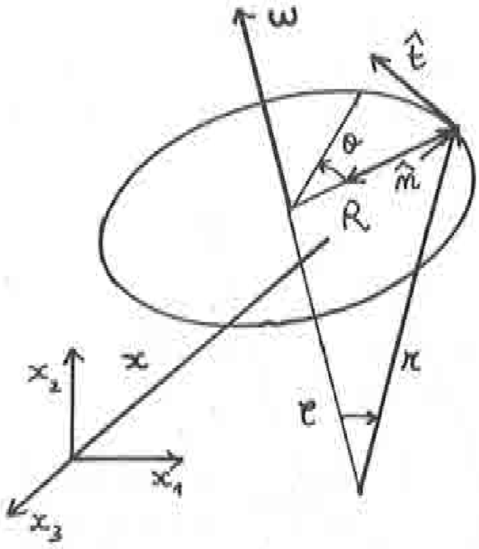


Figure 4.3: Circular motion

where

$$\omega = |\boldsymbol{\omega}| = \frac{d\theta}{dt}$$

and the vectors $\boldsymbol{\omega}$ and $\mathbf{r}$ are shown in Fig. 4.3. In fact, being

$$\begin{aligned} s &= R\theta \\ R &= |\mathbf{r}|\sin\varphi = r\sin\varphi \end{aligned}$$

it follows that

$$\begin{aligned} \mathbf{v} &= v\hat{\mathbf{t}} = \frac{ds}{dt}\hat{\mathbf{t}} = \frac{d}{dt}(R\theta)\hat{\mathbf{t}} = R\frac{d\theta}{dt}\hat{\mathbf{t}} \\ &= R\omega\hat{\mathbf{t}} \\ v &= |\mathbf{v}| = R\omega \end{aligned}$$

and

$$\mathbf{v} = R\omega\hat{\mathbf{t}} = r(\sin\varphi)\omega\hat{\mathbf{t}} = \mathbf{r}\times\boldsymbol{\omega}$$

4.3 Equivalent Motion

The motion of the same material point P, as seen from two reference systems, is called *equivalent motion*. Then, let $F = Ox_1x_2x_3$ and $F' = O'\xi_1\xi_2\xi_3$ be two CaORS in relative motion. According to Eq. 2.64, we can establish

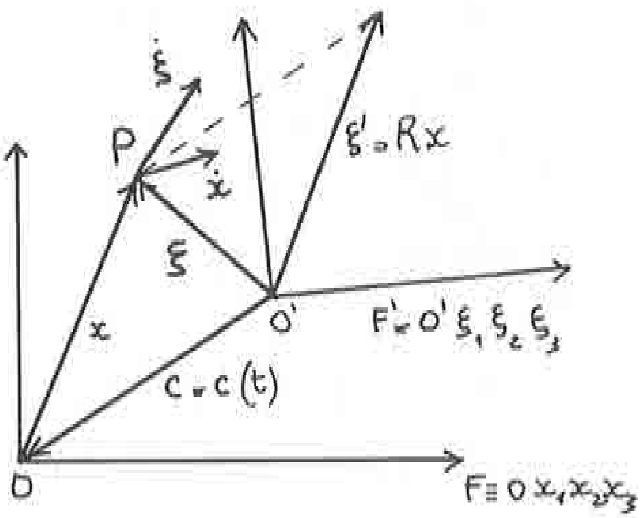


Figure 4.4: Motion of a material particle as seen by two reference systems traveling and rotating one with respect to the other.

that at a fixed time t

$$\xi = \mathbf{c} + \mathbf{R}\mathbf{x} \tag{4.24}$$

where

- $\mathbf{x} = \mathbf{x}(t)$ is the location of P with respect to F.
- $\xi = \xi(t)$ is the location of P with respect to F'.
- $\mathbf{c} = \mathbf{c}(t)$ is the distance between the origins O' and O.
- $\mathbf{R} = \mathbf{R}(t)$ is the orthogonal tensor which rotates F' to the orientation of F.

Consequently, the velocity and the acceleration of P with respect to F' may be calculated as

$$\begin{aligned}
\dot{\boldsymbol{\xi}} &= \dot{\mathbf{c}} + \mathbf{W}(\boldsymbol{\xi} - \mathbf{c}) + \mathbf{R}\dot{\mathbf{x}} = && (4.25)\\
&= \dot{\mathbf{c}} + \omega \times (\boldsymbol{\xi} - \mathbf{c}) + \mathbf{R}\dot{\mathbf{x}} && (4.26)\\
\ddot{\boldsymbol{\xi}} &= \ddot{\mathbf{c}} + (\dot{\mathbf{W}} - \mathbf{W}^2)(\boldsymbol{\xi} - \mathbf{c}) + 2\mathbf{W}(\dot{\boldsymbol{\xi}} - \dot{\mathbf{c}}) + \mathbf{R}\ddot{\mathbf{x}} = && (4.27)\\
&= \ddot{\mathbf{c}} + \dot{\omega} \times (\boldsymbol{\xi} - \mathbf{c}) - \omega \times [\omega \times (\boldsymbol{\xi} - \mathbf{c})] + 2\omega \times (\dot{\boldsymbol{\xi}} - \dot{\mathbf{c}}) + \mathbf{R}\ddot{\mathbf{x}} && (4.28)
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{W} &= \dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T\\
\dot{\mathbf{W}} &= \ddot{\mathbf{R}}\mathbf{R}^T + \dot{\mathbf{R}}\dot{\mathbf{R}}^T
\end{aligned}$$

represent the angular velocity and acceleration of F relative to F', respectively, and

$$\begin{aligned}
\omega &= -\left\{\frac{1}{2}\epsilon_{ijk}W_{jk}\right\}^T\\
\dot{\omega} &= \frac{d}{dt}\omega
\end{aligned}$$

represent the angular velocity and acceleration of F relative to F', respectively. Finally, the acceleration terms

$$\begin{aligned}
&\dot{\omega} \times (\boldsymbol{\xi} - \mathbf{c})\\
&\omega \times [\omega \times (\boldsymbol{\xi} - \mathbf{c})]\\
&2\omega \times (\dot{\boldsymbol{\xi}} - \dot{\mathbf{c}})
\end{aligned}$$

in the $\ddot{\boldsymbol{\xi}}$ expression are known as the *linear*, *centripetal* and *Coriolis* accelerations, respectively.

In fact,

$$\begin{aligned}
\dot{\boldsymbol{\xi}} &= \frac{d\boldsymbol{\xi}}{dt} = \frac{d}{dt}(\mathbf{c} + \mathbf{R}\mathbf{x}) =\\
&= \dot{\mathbf{c}} + \dot{\mathbf{R}}\mathbf{x} + \mathbf{R}\dot{\mathbf{x}} =\\
&= \dot{\mathbf{c}} + \dot{\mathbf{R}}\mathbf{R}^T(\boldsymbol{\xi} - \mathbf{c}) + \mathbf{R}\dot{\mathbf{x}} =\\
&= \dot{\mathbf{c}} + \mathbf{W}(\boldsymbol{\xi} - \mathbf{c}) + \mathbf{R}\dot{\mathbf{x}}
\end{aligned}$$

since, according to Eq. 4.24,

$$\mathbf{x} = \mathbf{R}^T(\boldsymbol{\xi} - \mathbf{c})$$

Then,

$$
\begin{aligned}
\ddot{\xi} &= \frac{d^2\xi}{dt^2} = \frac{d}{dt}(\dot{\mathbf{c}} + \mathbf{W}(\xi - \mathbf{c}) + \mathbf{R}\dot{\mathbf{x}}) = \\
&= \ddot{\mathbf{c}} + \dot{\mathbf{W}}(\xi - \mathbf{c}) + \mathbf{W}(\dot{\xi} - \dot{\mathbf{c}}) + \dot{\mathbf{R}}\dot{\mathbf{x}} + \mathbf{R}\ddot{\mathbf{x}} \\
&= \ddot{\mathbf{c}} + \dot{\mathbf{W}}(\xi - \mathbf{c}) + \mathbf{W}(\dot{\xi} - \dot{\mathbf{c}}) + \mathbf{W}(\dot{\xi} - \dot{\mathbf{c}}) - \mathbf{W}^2(\xi - \mathbf{c}) + \mathbf{R}\ddot{\mathbf{x}} = \\
&= \ddot{\mathbf{c}} + (\dot{\mathbf{W}} - \mathbf{W}^2)(\xi - \mathbf{c}) + 2\mathbf{W}(\dot{\xi} - \dot{\mathbf{c}}) + \mathbf{R}\ddot{\mathbf{x}}
\end{aligned}
$$

since, according to Eq. 4.25,

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{R}^T \left[(\dot{\xi} - \dot{\mathbf{c}}) - \mathbf{W}(\xi - \mathbf{c})\right] \\
\dot{\mathbf{R}}\dot{\mathbf{x}} &= \dot{\mathbf{R}}\mathbf{R}^T \left[(\dot{\xi} - \dot{\mathbf{c}}) - \mathbf{W}(\xi - \mathbf{c})\right] = \\
&= \mathbf{W}(\dot{\xi} - \dot{\mathbf{c}}) - \mathbf{W}^2(\xi - \mathbf{c})
\end{aligned}
$$

We note that $\mathbf{W}$, and thus $\dot{\mathbf{W}}$, are skewsymmetric matrices. In fact, since $\mathbf{R}$ is an orthogonal matrix, then

$$
\begin{aligned}
\mathbf{R}\mathbf{R}^T &= \mathbf{I} \\
\frac{d}{dt}(\mathbf{R}\mathbf{R}^T) &= \frac{d}{dt}\mathbf{I} = \mathbf{O} \\
\frac{d}{dt}(\mathbf{R}\mathbf{R}^T) &= \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \mathbf{W} + \mathbf{W}^T
\end{aligned}
$$

from which we can establish that

$$\mathbf{W} = -\mathbf{W}^T$$

Consequently, according to Eq. 3.5, we can set

$$
\begin{aligned}
\mathbf{W}(\xi - \mathbf{c}) &= \omega \times (\xi - \mathbf{c}) \\
\dot{\mathbf{W}}(\xi - \mathbf{c}) &= \dot{\omega} \times (\xi - \mathbf{c}) \\
\mathbf{W}^2(\xi - \mathbf{c}) &= \mathbf{W} \cdot [\mathbf{W}(\xi - \mathbf{c})] = \omega \times [\omega \times (\xi - \mathbf{c})] \\
\mathbf{W}(\dot{\xi} - \dot{\mathbf{c}}) &= \omega \times (\dot{\xi} - \dot{\mathbf{c}})
\end{aligned}
$$

4.4 Inertial Frame of References

The classical Theoretical Mechanics assumes the existence of a frame of reference which is fixed in space, i.e. absolutely at rest. This reference system is called ***absolute reference system*** and all the measurements or observations made in this reference system are said to be *absolute*.

Let $F^* = Ox_1x_2x_3$ be the absolute CaORS and $F' = O'\xi_1\xi_2\xi_3$ be another CaORS in relative motion with respect to F^*. If F' is moving with a constant velocity with respect to F^* in the sense that, Section 4.3,

$$\begin{aligned} \dot{\mathbf{c}} &= \dot{\mathbf{c}}(t) = \text{constant} \\ \mathbf{R} &= \mathbf{R}(t) = \text{constant} \end{aligned}$$

then F' is said to be an *inertial frame of reference*. For inertial reference systems, the equivalent motion equations reported in Section 4.3 simplify as

$$\xi = \mathbf{c} + \mathbf{R}(t)\mathbf{x} \tag{4.29}$$

$$\dot{\xi} = \dot{\mathbf{c}} + \mathbf{R}(t)\dot{\mathbf{x}} \tag{4.30}$$

$$\ddot{\xi} = \mathbf{R}\ddot{\mathbf{x}} \tag{4.31}$$

In fact, as consequence of the assumptions in Eqs. 4.29 and 4.29, we have that

$$\begin{aligned} \ddot{\mathbf{c}} &= \mathbf{o} \\ \dot{\mathbf{R}} &= \mathbf{W} = \mathbf{O} \\ \omega &= \dot{\omega} = \mathbf{o} \end{aligned}$$

4.4.1 Relative motion between inertial frame of references

Let $F^* = Ox_1x_2x_3$ be the absolute CaORS and $F' = Ox'_1x'_2x'_3$ and $\tilde{F} = \tilde{O}\tilde{x}_1\tilde{x}_2\tilde{x}_3$ be two inertial CaORS. Indicate the laws of motion of F' with respect to F^* as, Section 4.3,

$$\mathbf{x}' = \mathbf{c}' + \mathbf{R}'\mathbf{x} \tag{4.32}$$

$$\dot{\mathbf{x}}' = \dot{\mathbf{c}}' + \mathbf{R}'\dot{\mathbf{x}} \tag{4.33}$$

$$\ddot{\mathbf{x}}' = \mathbf{R}'\ddot{\mathbf{x}} \tag{4.34}$$

while that of $\tilde{F}$ with respect to F^* as

$$\tilde{\mathbf{x}} = \tilde{\mathbf{c}} + \tilde{\mathbf{R}}\mathbf{x} \tag{4.35}$$

$$\dot{\tilde{\mathbf{x}}} = \dot{\tilde{\mathbf{c}}} + \tilde{\mathbf{R}}\dot{\mathbf{x}} \tag{4.36}$$

$$\ddot{\tilde{\mathbf{x}}} = \tilde{\mathbf{R}}\ddot{\mathbf{x}} \tag{4.37}$$

Then, the law of motion of F' with respect to $\overline{F}$ are given by

$$\begin{aligned} \mathbf{x}' &= \mathbf{c}" + \mathbf{R}"\bar{\mathbf{x}} && (4.38)\\ \dot{\mathbf{x}}' &= \dot{\mathbf{c}}" + \mathbf{R}"\dot{\bar{\mathbf{x}}} && (4.39)\\ \ddot{\mathbf{x}}' &= \mathbf{R}"\ddot{\bar{\mathbf{x}}} && (4.40) \end{aligned}$$

where

$$\begin{aligned} \mathbf{c}" &= \mathbf{c}' - \mathbf{R}"\bar{\mathbf{c}}\\ \dot{\mathbf{c}}" &= \dot{\mathbf{c}}' - \mathbf{R}"\dot{\bar{\mathbf{c}}}\\ \mathbf{R}" &= \mathbf{R}'\bar{\mathbf{R}}^T \end{aligned}$$

and $\mathbf{R}"$ is an orthogonal tensor which rotates $\bar{F}$ to the orientation of F'. In fact, according to Eqs. 4.32 and 4.35,

$$\begin{aligned} \mathbf{x} &= \mathbf{R}'^T(\mathbf{x}' - \mathbf{c}')\\ \mathbf{x} &= \bar{\mathbf{R}}^T(\bar{\mathbf{x}} - \bar{\mathbf{c}}) \end{aligned}$$

and, equalizing these two alternative expressions of $\mathbf{x}$, we obtain

$$\begin{aligned} \mathbf{x}' &= \mathbf{c}' + \mathbf{R}'\bar{\mathbf{R}}^T(\bar{\mathbf{x}} - \bar{\mathbf{c}}) =\\ &= (\mathbf{c}' - \mathbf{R}'\bar{\mathbf{R}}^T\bar{\mathbf{c}}) + \mathbf{R}'\bar{\mathbf{R}}^T\bar{\mathbf{x}} =\\ &= \mathbf{c}'' + \mathbf{R}''\bar{\mathbf{x}} \end{aligned}$$

4.4.2 Transformation laws between inertial frame of references

Let $F = Ox_1x_2x_3$ and $F' = O'\xi_1\xi_2\xi_3$ be two inertial CaORS and let their relative motion be described by

$$\begin{aligned} \boldsymbol{\xi} &= \mathbf{c} + \mathbf{R}\mathbf{x} && (4.41)\\ \dot{\boldsymbol{\xi}} &= \dot{\mathbf{c}} + \mathbf{R}\dot{\mathbf{x}} && (4.42)\\ \ddot{\boldsymbol{\xi}} &= \mathbf{R}\ddot{\mathbf{x}} && (4.43) \end{aligned}$$

where $\mathbf{R}$ is the orthogonal matrix which rotate F' to the orientation of F.

We know from Section 2.7 that the Jacobian matrix $\mathbf{J}$ and the transformation matrix $\mathbf{A}$ between CaORS are related to the rotation matrix $\mathbf{R}$ as

$$\mathbf{J} = \mathbf{A} = \mathbf{R}^T \qquad (4.44)$$

Similarly to the case of fixed CaORS, Section 3.4, the tensor transformation obeys the following rules:

- The rule of changing the vector components under a change of inertial CaORS is given by
$$\mathbf{u}' = \mathbf{R}\mathbf{u} \tag{4.45}$$
that is
$$u'_i = R_{ij}u_j$$
where u_i and u'_j represent the values of the components that the same vector takes in F and F', respectively.

- The rule of changing the tensor components under a change of CaORS is given by
$$\mathbf{T}' = \mathbf{R}\mathbf{T}\mathbf{R}^T \tag{4.46}$$
that is
$$T'_{ij} = R_{ik}T_{kl}R_{jl}$$
where T_{ij} and T'_{kl} represent the values of the components that the same tensor takes in F and F', respectively.

In fact, with reference to F, let identify a vector $\mathbf{u}$ as the difference of the position vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, namely
$$\mathbf{u} = \mathbf{x}^{(2)} - \mathbf{x}^{(1)}$$
According to Eq. 4.41,
$$\mathbf{x}^{(i)} = \mathbf{R}^T(\boldsymbol{\xi}^{(i)} - \mathbf{c})$$
where $\boldsymbol{\xi}^{(i)}$ locate from F' the same points located by $\mathbf{x}$ from F. Then,
$$\begin{aligned}\mathbf{u} &= \mathbf{x}^{(2)} - \mathbf{x}^{(1)} = \\ &= \mathbf{R}^T(\boldsymbol{\xi}^{(2)} - \mathbf{c}) - \mathbf{R}^T(\boldsymbol{\xi}^{(1)} - \mathbf{c}) = \\ &= \mathbf{R}^T(\boldsymbol{\xi}^{(2)} - \boldsymbol{\xi}^{(1)}) = \\ &= \mathbf{R}^T\mathbf{u}'\end{aligned}$$
The transformation law for 2nd-order tensors follows from that of vectors in the same way as reported in Section 3.4.

4.5 Newton's Laws of Motion

With respect to the absolute reference system F^*, Sir Isaac Newton formulated the following three laws which are considered the axioms of mechanics:

1. Every particle persists in a state of rest or of uniform motion in a straight line (i.e. with constant velocity) unless acted upon by a force.

2. Let be $\mathbf{f}$ the (external) force acting on a particle of mass m which, as a consequence, is moving with velocity $\mathbf{v}$, Fig. 4.5a. Then,

$$\mathbf{f} = \frac{d}{dt}\mathbf{p} \tag{4.47}$$

   where

$$\mathbf{p} = m\mathbf{v}$$

   is called *momentum*. If m is independent of time t, this becomes

$$\mathbf{f} = m\frac{d}{dt}\mathbf{v} = m\mathbf{a} \tag{4.48}$$

   where $\mathbf{a}$ is the acceleration of the particle.

3. In a system of particles that interact with each other let be, Fig. 4.5b:

   - $\mathbf{f}^{(ij)}$ and $\mathbf{f}^{(ji)}$ the forces of interaction exerted by a particle P_i on a particle P_j and viceversa, respectively.
   - $\mathbf{d}^{(i)}$ and $\mathbf{d}^{(j)}$ the relative distances of P_i and P_j with respect to any arbitrary point D in the space.

   Then,

$$\begin{aligned} \mathbf{f}^{(ij)} &= -\mathbf{f}^{(ji)} \\ \mathbf{d}^{(i)} \times \mathbf{f}^{(ij)} &= -\mathbf{d}^{(j)} \times \mathbf{f}^{(ji)} \end{aligned}$$

   that is

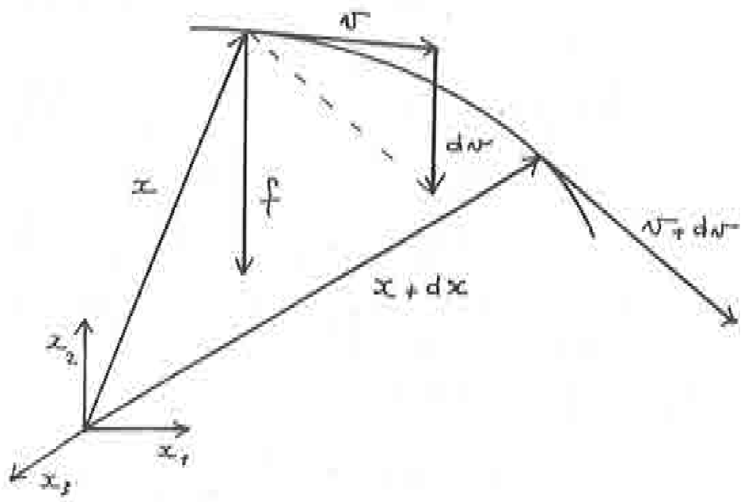
$$\mathbf{f}^{(ij)} + \mathbf{f}^{(ji)} = \mathbf{o} \tag{4.49}$$

$$(\mathbf{d}^{(i)} - \mathbf{d}^{(j)}) \times \mathbf{f}^{(ij)} = \mathbf{o} \tag{4.50}$$

   In words, to every action there is an equal and opposite reaction.

The 3rd Newton's law, also known as the *principle of action and reaction*, has been herein reported under the most common assumption that the forces of interaction are always directed toward the line joining the particles, i.e. they are assumed to be *central forces*.

The concepts of *force* and *mass* used in the above axioms are not yet defined. Intuitively we have some ideas of mass as a *measure of the quantity*



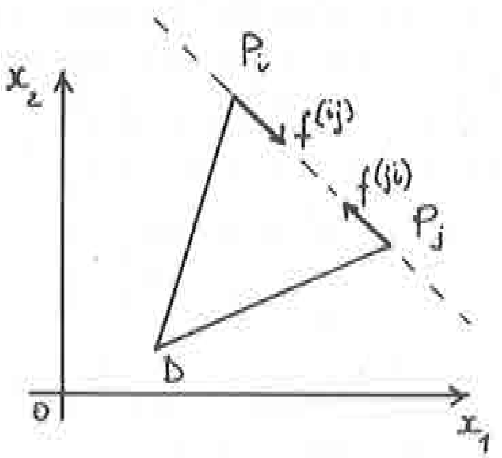


Figure 4.5: Force acting on a particle

of matter in a particle and force as a *a measure of the push or pull on an particle.*

According to the classical theoretical mechanics, we will always assume that the *mass of a particle is constant in time.* Forces acting on a particle may be classified into two kinds:

- *Contact forces* are those to which most properly include the intuitive concept of push or pull. In a system of particles, an example of contact forces are those developed by the collision of two particles.

- *Action-at-a-distance forces* are those to which a particle is subjected without any contact with the external agent. An example of such a type of forces acting at distance have been postulated by Newton in the law of gravitation, Section 4.7.

According to the 2nd Newton's law

$$\mathbf{f} = m\mathbf{a} = \left[\text{Mass}\ \frac{\text{Length}}{(\text{Time})^2}\right] \tag{4.51}$$

Standard unit of mass is the *grams* (gm) while standard unit of force is the *dyne*: by definition, a dyne is that force which will give a 1 gm mass an acceleration of 1 cm/sec^2. A *Newton* (Nw) is that force which will give a 1 kgm mass an acceleration of 1 m/sec^2.

We recall that Newton's laws of motion have been postulated under the assumption that all measurements are taken with respect to the absolute reference system. However, it can be shown that Newton's laws hold in any inertial reference system. For example, let $F^* = Ox_1x_2x_3$ and $F' = O'x'_1x'_2x'_3$ be the absolute and an inertial CaORS, respectively, and let the relative motion of F' with respect to F^* be described by, Section 4.4.2,

$$\begin{aligned} \mathbf{x}' &= \mathbf{c} + \mathbf{R}\mathbf{x} \\ \dot{\mathbf{x}}' &= \dot{\mathbf{c}} + \mathbf{R}\dot{\mathbf{x}} \\ \ddot{\mathbf{x}}' &= \mathbf{R}\ddot{\mathbf{x}} \end{aligned}$$

where $\mathbf{c}$ is the distance of O' from O and $\mathbf{R}$ is the orthogonal matrix which rotates F' to the orientation of F. If an observer in F^* measures on particle P of mass m a force and a torque equal to

$$\begin{aligned} \mathbf{f} &= m\mathbf{a} \\ \boldsymbol{\lambda} &= \mathbf{d} \times \mathbf{f} \end{aligned}$$

where $\mathbf{d}$ is the distance of P from any point D in space, then an observer in F' measures on P a force and a torque equal to

$$\begin{aligned} \mathbf{f}' &= m\mathbf{a}' \\ \boldsymbol{\lambda}' &= \mathbf{d}' \times \mathbf{f}' \end{aligned}$$

where $\mathbf{d}'$ is the distance of P from the same point D, which are related to the measurements in F^* as

$$\mathbf{f}' = \mathbf{R}\mathbf{f} \tag{4.52}$$

$$\boldsymbol{\lambda}' = \mathbf{R}\boldsymbol{\lambda} \tag{4.53}$$

Moreover, scalar values resulting from a scalar product of force times distance remain unchanged, that is

$$\begin{aligned} w &= \mathbf{d}^T\mathbf{f} \\ w' &= \mathbf{d}'^T\mathbf{f}' \end{aligned}$$

result to be equal

$$w' = w \tag{4.54}$$

In fact, according to the laws of relative motion,

$$\begin{aligned} \mathbf{f} &= m\mathbf{a} = m\mathbf{R}^T\mathbf{a}' = \mathbf{R}^T\mathbf{f}' \\ \boldsymbol{\lambda} &= \mathbf{d} \times \mathbf{f} = \mathbf{R}^T(\mathbf{d}' \times \mathbf{f}') = \mathbf{R}^T\boldsymbol{\lambda}' \\ w &= \mathbf{d}^T\mathbf{f} = \mathbf{d}'^T\mathbf{R}^T\mathbf{R}\mathbf{f}' = \mathbf{d}'^T\mathbf{f}' \end{aligned}$$

4.6 Kinetic and Mechanical Quantities

Newton's laws of motion establish that the mechanical sources (force) cause the change of the kinetic (momentum) of a material point. In general, the kinetic quantities which can be associated to a moving particle P of mass m are, Fig. 4.6:

- The *momentum* defined as product between the mass and the velocity $\dot{\mathbf{x}}$ of P, namely

$$\mathbf{p} = m\dot{\mathbf{x}}$$

- The *angular momentum* about any given point D defined as the moment of momentum of P,

$$\boldsymbol{\gamma} = \mathbf{d} \times \mathbf{p} = m(\mathbf{d} \times \dot{\mathbf{x}})$$

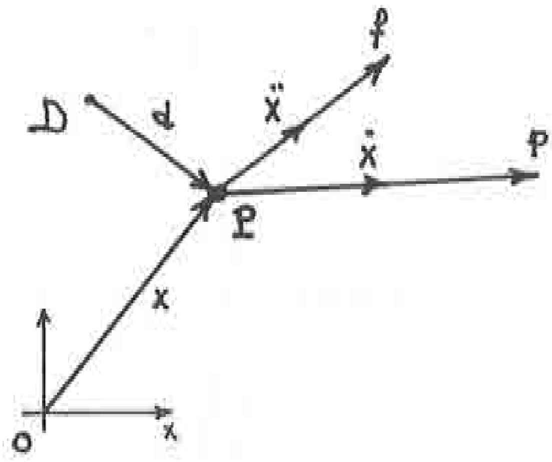


Figure 4.6: Mechanical quantities acting on a particle

- The *time rate of momentum* defined as the time derivative of the momentum, namely

$$\dot{\mathbf{p}} = \frac{d}{dt}\mathbf{p} = m\ddot{\mathbf{x}}$$

- The *time rate of angular momentum* about a given point D defined as the time derivative of the angular momentum, namely

$$\dot{\gamma} = \frac{d}{dt}\gamma = m(\dot{\mathbf{d}} \times \dot{\mathbf{x}} + \mathbf{d} \times \ddot{\mathbf{x}})$$

- The *kinetic energy* defined as the scalar component of $\mathbf{p}$ in the direction of the motion $\dot{\mathbf{x}}$, namely

$$e^{(k)} = \frac{1}{2}\mathbf{p}^T\dot{\mathbf{x}} = \frac{1}{2}m\dot{x}^2$$

The mechanical quantities which can be associated to a force vector $\mathbf{f}$ acting on a moving particle P of mass m are, Fig. 4.6:

- The *torque* about any given point D defined as the moment of the force $\mathbf{f}$, namely

$$\lambda = \mathbf{d} \times \mathbf{f}$$

- The *work* defined as the scalar component of $\mathbf{f}$ in the direction of the motion $d\mathbf{x}$ of P, namely

$$dw = \mathbf{f}^T d\mathbf{x}$$

- The *total work* defined as the work done by the force $\mathbf{f}$ on the particle P between two points along a curve C, namely

$$w = \int_C \mathbf{f}^T d\mathbf{x}$$

- The *(instantaneous) power* defined as the time rate of doing work, namely,

$$\dot{w} = \frac{dw}{dt} = \mathbf{f}^T \dot{\mathbf{x}}$$

According to the 2nd Newton's law

$$\mathbf{f} = \dot{\mathbf{p}} = m\ddot{\mathbf{x}} \tag{4.55}$$

from which we can establish the following relationships between the above defined mechanical (cause) and kinetical (effect) quantities

$$\begin{aligned}
\boldsymbol{\lambda} &= \mathbf{d} \times \mathbf{f} = \dot{\boldsymbol{\gamma}} - m(\dot{\mathbf{d}} \times \mathbf{x}) && (4.56)\\
dw &= \mathbf{f}^T d\mathbf{x} = de^{(k)} && (4.57)\\
w &= \int_C \mathbf{f}^T d\mathbf{x} = \frac{1}{2} m\dot{x}^2 \Big|_{t_1}^{t_2} = e^{(k)} \Big|_{t_1}^{t_2} && (4.58)\\
\dot{w} &= \mathbf{f}^T \dot{\mathbf{x}} = \dot{e}^{(k)} && (4.59)
\end{aligned}$$

In fact,

$$\begin{aligned}
\boldsymbol{\lambda} &= \mathbf{d} \times \mathbf{f} = m(\mathbf{d} \times \ddot{\mathbf{x}}) = m\left[\dot{\mathbf{d}} \times \dot{\mathbf{x}} + \mathbf{d} \times \ddot{\mathbf{x}} - \dot{\mathbf{d}} \times \dot{\mathbf{x}}\right] = \\
&= \dot{\boldsymbol{\gamma}} - m\dot{\mathbf{d}} \times \dot{\mathbf{x}} \\
dw &= \mathbf{f}^T d\mathbf{x} = m\ddot{\mathbf{x}}^T d\mathbf{x} = m\ddot{\mathbf{x}}^T \frac{d\mathbf{x}}{dt} dt = m\ddot{\mathbf{x}}^T \dot{\mathbf{x}} dt = \\
&= m\frac{d}{dt}\left(\frac{1}{2}\dot{\mathbf{x}}^T\dot{\mathbf{x}}\right) dt = d\left(\frac{1}{2}m\dot{x}^2\right) = de^{(k)} \\
\dot{w} &= \frac{dw}{dt} = \frac{d}{dt}(\mathbf{f}^T\mathbf{x}) = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2\right) = \dot{e}^{(k)}
\end{aligned}$$

4.7 Conservative Force Fields

A force is said to be *conservative* if there exists a continuously differential scalar function

$$e^{(p)} = e^{(p)}(\mathbf{x}) \tag{4.60}$$

known as the *potential energy*, such that

$$\mathbf{f} = -\operatorname{grad} e^{(p)} \tag{4.61}$$

According to the properties of the gradient operator reported in Section 2.5.1, we have that

$$\begin{aligned} \operatorname{grad} e^{(p)} &= \frac{\partial e^{(p)}}{\partial \mathbf{x}} = \nabla e^{(p)} \\ de^{(p)} &= \left(\frac{\partial e^{(p)}}{\partial \mathbf{x}}\right)^T d\mathbf{x} = \nabla e^{(p)T} d\mathbf{x} \\ \nabla \times \nabla e^{(p)} &= \mathbf{o} \end{aligned}$$

Using these properties we can easly prove that a force field $\mathbf{f}$ is conservative if and only if:

- the work done for moving a particle is equal to the variation of a potential energy, that is

$$dw = \mathbf{f}^T d\mathbf{x} = -de^{(p)} \tag{4.62}$$

- or alternatively, the total work done for moving a particle from a position at P_1 to P_2 is independent of the path C:

$$w = \int_C \mathbf{f}^T d\mathbf{x} = -e^{(p)}\Big|_{P_1}^{P_2} \tag{4.63}$$

- or alternatively, the total work done in moving a particle around any closed path on a non intersecting curve C (i.e. simple closed curve) is zero:

$$w = \oint \mathbf{f}^T d\mathbf{x} = 0 \tag{4.64}$$

- or alternatively,

$$\operatorname{curl} \mathbf{f} = \nabla \times \mathbf{f} = \mathbf{o} \tag{4.65}$$

Examples of conservative forces are some kinds of action-at-distance forces by which two particles P_1 and P_2 attract or repel each other with intensity function of their relative distance. In mathematical terms they can be generally expressed by

$$\mathbf{f} = \Psi(r)\mathbf{r} \tag{4.66}$$

where, Fig. 4.7,

$$\begin{aligned} \mathbf{r} &= \mathbf{x}^{(2)} - \mathbf{x}^{(1)} \\ r &= |\mathbf{r}| \end{aligned}$$

Identifying $d\mathbf{x} \equiv d\mathbf{r}$, it follows that

$$\begin{aligned} de^{(p)} &= \mathbf{f}^T d\mathbf{x} = \Psi(r)\mathbf{r}^T d\mathbf{r} = \frac{1}{2}\Psi(r)d(r^2) = \\ &= \frac{1}{2}\Psi(r)rdr \qquad (4.67) \\ e^{(p)} &= \frac{1}{2}\int_{r_o}^{r} \Psi(r)rdr \qquad (4.68) \end{aligned}$$

where r_o is any value of r in which the integral is convergent.

The existence of one kind of such conservative forces has been postulated by Newton in the *law of gravitation* which states that any particle of mass m_1 exerts a gravity force on a particle of mass m_2 equal to

$$\mathbf{f} = -\mathcal{G}\frac{m_1 m_2}{r^3}\mathbf{r} \tag{4.69}$$

where r is the modulus of $\mathbf{r}$, Fig. 4.7, and $\mathcal{G}$ is the *universal gravitational constant*. Experimental measures indicate an approximate value of

$$\mathcal{G} = 6.673 \times 10^{-11} \qquad [Nm^2/Kg^2] \tag{4.70}$$

From this law one can prove that any material object of mass m on the earth is subjected to a *body force*

$$\mathbf{f} = m\mathbf{b} \tag{4.71}$$

where $\mathbf{b}$ is the *gravity acceleration vector* defined as

$$\mathbf{b} = \frac{\mathcal{G}M}{R^2}\hat{\mathbf{b}} = g\hat{\mathbf{b}} \tag{4.72}$$

in which M and R represent the mass and the radius of the earth, respectively, while $\hat{\mathbf{b}}$ a unit vector pointing toward the center of the earth. An average value of the gravity acceleration g on the earth is

$$g = 9.81 \qquad [m/sec^2] \tag{4.73}$$

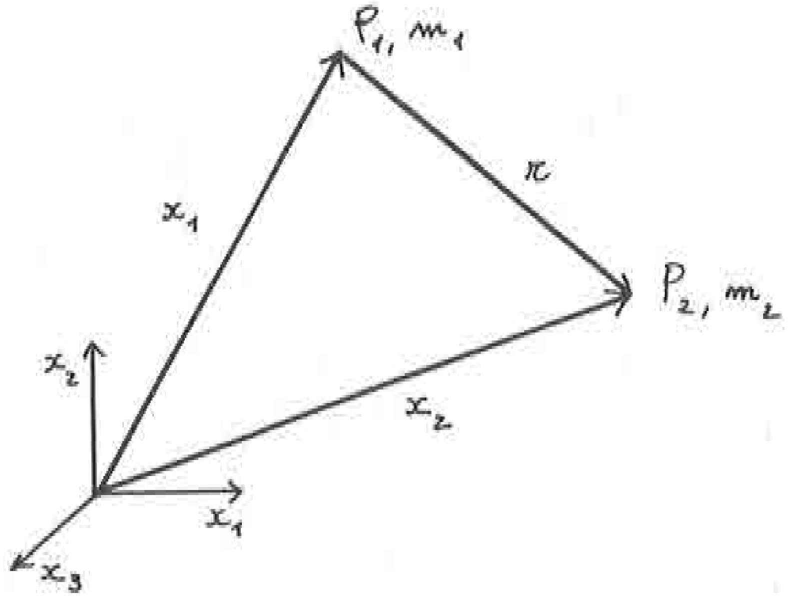


Figure 4.7: Action-at-a-distance forces between two material particles

Common engineering problems concern material objects of mass $m << M$ and which are located at distance from the center of the earth $|\mathbf{r}| \approx R$. Then, it is reasonable to assume that:

- gravity forces exerted by a material particle on any other material particles negligible compared to the body forces;
- constant gravity acceleration vector $\mathbf{b}$. This implies that

$$\mathbf{f} = -\frac{\partial e^{(p)}}{\partial \mathbf{x}} = m\mathbf{b} \tag{4.74}$$

where

$$\begin{aligned} e^{(p)} &= -m\mathbf{b}^T\mathbf{x} \\ de^{(p)} &= \left(\frac{\partial e^{(p)}}{\partial \mathbf{x}}\right)^T d\mathbf{x} = -m\mathbf{b}^T d\mathbf{x} = -dw \\ d^2 e^{(p)} &= dx_i \frac{\partial^2 e^{(p)}}{\partial x_i \partial x_j} dx_j = -m dx_i \frac{\partial b_i}{\partial x_j} dx_j = 0 \end{aligned}$$

4.8 Mechanical Principles

From the Newton's laws of motion and the relative relationships reported in Section 4.6, we can easly verify the following mechanical principles.

Theorem 4.8.1 (Linear Momentum Principle). *The net external force acting on a particle equals the time rate of change of the momentum of the particle, that is*

$$\mathbf{f} = \frac{d}{dt}(m\dot{\mathbf{x}})$$

Theorem 4.8.2 (Principle of Conservation of Momentum). *If the net external force acting on a particle is zero, the momentum remains constant, that is*

$$m\dot{\mathbf{x}} = \ const$$

In such a case the particle preserves its state of rest or motion with constant velocity.

Theorem 4.8.3 (Angular Momentum Principle). *The net external torque acting on a particle about a fixed point in space, or about any point traveling with the same velocity of the particle, equals the time rate of change of the angular moment, that is*

$$\mathbf{x} \times \mathbf{f} = \frac{d}{dt} m(\mathbf{x} \times \dot{\mathbf{x}})$$

Theorem 4.8.4 (Principle of Conservation of Angular Momentum). *If the net external torque acting on a particle about a fixed point in space, or about any point traveling with the same velocity of the particle, is equal to zero, then the angular momentum remains constant, that is*

$$m(\mathbf{x} \times \dot{\mathbf{x}}) = const$$

Theorem 4.8.5 (Kinetic Energy vs. Total Work). *The total work done by a force in moving the particle from one state where the kinetic energy is equal to $e_1^{(k)}$ to another where the kinetic energy is equal to $e_2^{(k)}$ is given by*

$$w_{12} = e_2^{(k)} - e_1^{(k)}$$

Theorem 4.8.6 (Principle of Conservation of Energy.) *If the external force is conservative, Section 4.7, i.e. there exists a potential energy* $e^{(p)}$ *such that*

$$\mathbf{f} = -\operatorname{grad} e^{(p)}$$

then the total energy

$$e = e^{(k)} + e^{(p)} = \frac{1}{2} m \dot{x}^2 + e^{(p)}$$

is constant.

Proof: In fact, according to Eqs. 4.58 and 4.63,

$$w = e_2^{(k)} - e_1^{(k)} = e_1^{(p)} - e_2^{(p)}$$

which may be rearranged to define a total energy as

$$e = e_1^{(k)} + e_1^{(p)} = e_2^{(k)} + e_2^{(p)}$$

Chapter 5

System of Interacting Particles

5.1 Introduction

A collection of material particles forms a system of particles which is discreet or continuous depending on the possibility to consider the particles as separated from each other or not. In this Chapter we are primarily concerned about a discrete system of particles; however, all the presented results will be in the next Chapters straightforward extended to the case of a continuum system.

Applied to each single particle of the system, a set of forces may act and, according to Newton's laws, it may induce a change of momentum of the particle. The issue is to establish the relationship between the force field and the rate of change of momentum of the whole system of particles. This can be accomplished on the basis of the resultant sum of each force and momentum contributions, Section 5.2.

Associated to a system of particles, we can define a so-called *center of mass* of the system, Section 5.3. With respect to this center of mass, all the kinetic quantities associated to a system of particles may be decomposed into two distinct terms:

- one term concerns the motion of the center of mass in which all the masses of the system are supposed to be concentrated;
- the second term concerns the relative motion of each individual particle with respect to the center of mass.

Each particle of a system may interact with the other particles, Section 5.5. Once the contribution to motion due to these interacting forces is identified, we can finally establish the mechanical principles governing the motion of a system of particles under a set of external forces, Section 5.6.

5.2 Kinetic and Mechanical Quantities

In general, associated to a system of n particles P_i of masses $m_1, m_2, \ldots, m_n$, we can define the *total mass* as

$$m = \sum_{i=1}^{n} m_i \tag{5.1}$$

and the following total kinetic quantities, Fig. 5.1:

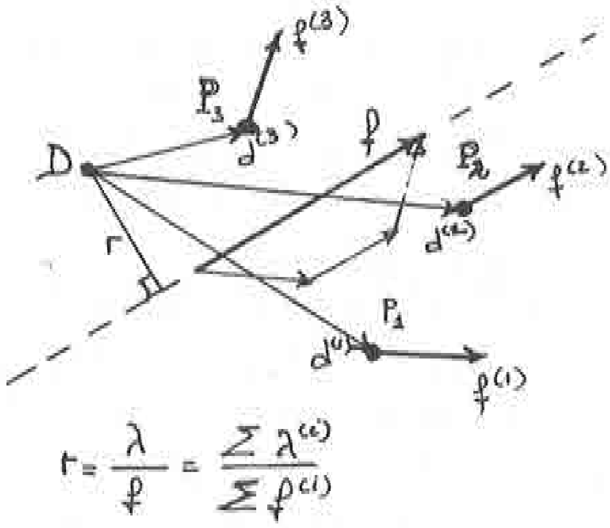


Figure 5.1: System of particles

- The *total momentum* defined as

$$\mathbf{p} = \sum_{i=1}^{n} \mathbf{p}^{(i)} = \sum_{i=1}^{n} m_i \dot{\mathbf{x}}^{(i)}$$

- The *total angular momentum* about a given point D defined as

$$\gamma = \sum_{i=1}^{n} \gamma^{(i)} = \sum_{i=1}^{n} m_i(\mathbf{d}^{(i)} \times \dot{\mathbf{x}}^{(i)})$$

- The *total time rate of momentum* defined as

$$\dot{\mathbf{p}} = \sum_{i=1}^{n} \dot{\mathbf{p}}^{(i)} = \sum_{i=1}^{n} m_i \ddot{\mathbf{x}}^{(i)}$$

- The *total time rate of angular momentum* about a given point D defined as

$$\dot{\gamma} = \sum_{i=1}^{n} \dot{\gamma}^{(i)} = \sum_{i=1}^{n} m_i(\dot{\mathbf{d}}^{(i)} \times \dot{\mathbf{x}}^{(i)} + \mathbf{d}^{(i)} \times \ddot{\mathbf{x}}^{(i)})$$

- The *total kinetic energy* defined as

$$e^{(k)} = \sum_{i=1}^{n} e_i^{(k)} = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{x}_i^2$$

Associated to a system of n forces $\mathbf{f}^{(i)}$ acting on n particles P_i of masses $m_1, m_2, \ldots, m_n$, we can define the following total mechanical quantities:

- The *total force*, or the *resultant* of $\mathbf{f}^{(i)}$, defined as

$$\mathbf{f} = \sum_{i=1}^{n} \mathbf{f}^{(i)}$$

- The *total torque* about a given point D defined as

$$\lambda = \sum_{i=1}^{n} \lambda^{(i)} = \sum_{i=1}^{n} (\mathbf{d}^{(i)} \times \mathbf{f}^{(i)})$$

- The *work* of the system of forces $\mathbf{f}^{(i)}$ defined as

$$dw = \sum_{i=1}^{n} dw_i = \sum_{i=1}^{n} (\mathbf{f}^{(i)T} d\mathbf{x}^{(i)})$$

- The *total work* of the system of forces $\mathbf{f}^{(i)}$ defined as

$$w = \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} \left(\int_C \mathbf{f}^{(i)T} d\mathbf{x}^{(i)} \right)$$

- The *(instantaneous) power* of the system of forces $\mathbf{f}^{(i)}$ defined as

$$\dot{w} = \frac{dw}{dt} = \sum_{i=1}^{n} \frac{dw_i}{dt} = \sum_{i=1}^{n} (\mathbf{f}^{(i)T} d\dot{\mathbf{x}}^{(i)})$$

We immediately note that

$$\mathbf{f} = \dot{\mathbf{p}} = \sum_{i=1}^{n} m_i \ddot{\mathbf{x}}^{(i)} \tag{5.2}$$

from which we can establish the following relationships between the above defined total mechanical (cause) and total kinetic (effect) quantities

$$\lambda = \dot{\gamma} - \sum_{i=1}^{n} m_i (\dot{\mathbf{d}}^{(i)} \times \dot{\mathbf{x}}^{(i)}) \tag{5.3}$$

$$dw = \sum_{i=1}^{n} (\mathbf{f}^{(i)T} d\mathbf{x}^{(i)}) = de^{(k)} \tag{5.4}$$

$$w = \sum_{i=1}^{n} \left(\int_C \mathbf{f}^{(i)T} d\mathbf{x}^{(i)} \right) = \frac{1}{2} \sum_{i=1}^{n} m_i \dot{x}_i^2 \,|_{t_1}^{t_2} = e^{(k)} \Big|_{t_1}^{t_2} \tag{5.5}$$

$$\dot{w} = \sum_{i=1}^{n} (\mathbf{f}^{(i)T} d\dot{\mathbf{x}}^{(i)}) = \dot{e}^{(k)} \tag{5.6}$$

The procedure for proving these relationships is analogous to that presented in Section 4.6 for the case of a single particle.

5.3 Center of Mass

Let $\mathbf{x}^{(i)}$, for $i = 1, 2, \ldots, n$, be the position vectors of a system of n particles of masses m_i. The *center of mass* or *centroid* of the system of particle is defined as that point C having position vector, Fig. 5.2

$$\mathbf{c} = \frac{\sum_{i=1}^{n} m_i \mathbf{x}^{(i)}}{\sum_{i=1}^{n} m_i} = \frac{1}{m} \sum_{i=1}^{n} m_i \mathbf{x}^{(i)} \tag{5.7}$$

It is easy to prove that if $\xi^{(i)}$ are the relative position of each particle P_i

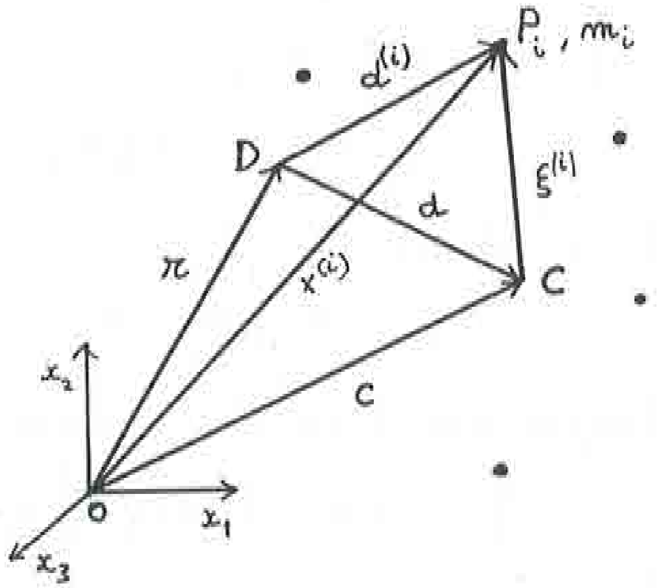


Figure 5.2: Center of mass in a system of particles

from the center of mass C, so that

$$\mathbf{x}^{(i)} = \mathbf{c} + \boldsymbol{\xi}^{(i)}$$

the following quantities are null

$$\sum_{i=1}^{n} m_i \boldsymbol{\xi}^{(i)} = \sum_{i=1}^{n} m_i \dot{\boldsymbol{\xi}}^{(i)} = \sum_{i=1}^{n} m_i \ddot{\boldsymbol{\xi}}^{(i)} = \mathbf{o}$$

In fact,

$$\begin{aligned} m\mathbf{c} &= \sum_{i=1}^{n} m_i \mathbf{x}^{(i)} = \sum_{i=1}^{n} m_i(\mathbf{c} + \boldsymbol{\xi}^{(i)}) = \mathbf{c}\sum_{i=1}^{n} m_i + \sum_{i=1}^{n} m_i \boldsymbol{\xi}^{(i)} = \\ &= m\mathbf{c} + \sum_{i=1}^{n} m_i \boldsymbol{\xi}^{(i)} \end{aligned}$$

Consequently, all the total kinetic quantities associated to a system of masses defined in the previous Section 5.2 can be expressed as follows, Fig. 5.2:

- Total momentum
$$\mathbf{p} = \sum_{i=1}^{n} \mathbf{p}^{(i)} = m\dot{\mathbf{c}}$$
- Total angular momentum about a given point D
$$\boldsymbol{\gamma} = \sum_{i=1}^{n} \boldsymbol{\gamma}^{(i)} = m(\mathbf{d} \times \dot{\mathbf{c}}) + \sum_{i=1}^{n} m_i(\boldsymbol{\xi}^{(i)} \times \dot{\boldsymbol{\xi}})$$
- Total time rate of momentum
$$\dot{\mathbf{p}} = \sum_{i=1}^{n} \dot{\mathbf{p}}^{(i)} = m\ddot{\mathbf{c}}$$
- Total time rate of angular momentum about a given point D defined as
$$\dot{\boldsymbol{\gamma}} = \sum_{i=1}^{n} \dot{\boldsymbol{\gamma}}^{(i)} = m(\mathbf{d} \times \ddot{\mathbf{c}}) + m(\dot{\mathbf{d}} \times \dot{\mathbf{c}}) + \sum_{i=1}^{n} m_i(\boldsymbol{\xi}^{(i)} \times \ddot{\boldsymbol{\xi}}^{(i)})$$
- Total kinetic energy as
$$e^{(k)} = \sum_{i=1}^{n} e_i^k = \frac{1}{2}m\dot{c}^2 + \frac{1}{2}\sum_{i=1}^{n} m_i\dot{\xi}_i^2$$

We note that all the above formulas contain two distinct terms:

- the first one concerns the motion of the center of mass in which the total mass m of the system is supposed to be concentrated;
- the second one concerns the relative motion of each individual particle with respect to the center of mass.

Then, the relationships from mechanical and kinetic quantities in Eqs. 5.2-5.6 can be alternatively expressed as

$$\mathbf{f} = \sum_{i=1}^{n} \mathbf{f}^{(i)} = m\ddot{\mathbf{c}} \tag{5.8}$$
$$\boldsymbol{\lambda} = \sum_{i=1}^{n} \boldsymbol{\lambda}^{(i)} = \sum_{i=1}^{n}(\mathbf{d}^{(i)} \times \mathbf{f}^{(i)}) \tag{5.9}$$
$$= \dot{\boldsymbol{\gamma}} - m(\dot{\mathbf{d}} \times \dot{\mathbf{c}}) \tag{5.10}$$

$$
\begin{aligned}
dw &= \sum_{i=1}^{n}(\mathbf{f}_i^T d\mathbf{x}^{(i)}) = de^{(k)} && (5.11)\\
w &= \sum_{i=1}^{n}\left(\int_C \mathbf{f}^{(i)T} d\mathbf{x}^{(i)}\right) = \frac{1}{2}m\dot{c}^2 + \sum_{i=1}^{n}\frac{1}{2}m_i\dot{\xi}^{(i)2} = e^{(k)}\,|_{t_1}^{t_2} && (5.12)\\
\dot{w} &= \sum_{i=1}^{n}(\mathbf{f}^{(i)T} d\dot{\mathbf{x}}^{(i)}) = \dot{e}^{(k)} && (5.13)
\end{aligned}
$$

In fact,

$$
\begin{aligned}
\mathbf{p} &= \sum_{i=1}^{n} m_i\dot{\mathbf{x}}^{(i)} = \sum_{i=1}^{n} m_i\left(\dot{\mathbf{c}} + \dot{\boldsymbol{\xi}}^{(i)}\right) = m\dot{\mathbf{c}} + \sum_{i=1}^{n} m_i\dot{\boldsymbol{\xi}}^{(i)} = m\dot{\mathbf{c}}\\
\boldsymbol{\gamma} &= \sum_{i=1}^{n} m_i\left(\mathbf{d}^{(i)}\times\dot{\mathbf{x}}^{(i)}\right) = \sum_{i=1}^{n} m_i\left(\mathbf{d}+\boldsymbol{\xi}^{(i)}\right)\times\left(\dot{\mathbf{c}}+\dot{\boldsymbol{\xi}}^{(i)}\right) =\\
&= \sum_{i=1}^{n} m_i\left[(\mathbf{d}\times\dot{\mathbf{c}}) + \left(\mathbf{d}\times\dot{\boldsymbol{\xi}}^{(i)}\right) + \left(\boldsymbol{\xi}^{(i)}\times\dot{\mathbf{c}}\right) + \left(\boldsymbol{\xi}^{(i)}\times\dot{\boldsymbol{\xi}}^{(i)}\right)\right] =\\
&= m(\mathbf{d}\times\dot{\mathbf{c}}) + \sum_{i=1}^{n} m_i\left(\boldsymbol{\xi}^{(i)}\times\dot{\boldsymbol{\xi}}^{(i)}\right)\\
\dot{\boldsymbol{\gamma}} &= \frac{d}{dt}\left[m(\mathbf{d}\times\dot{\mathbf{c}}) + \sum_{i=1}^{n} m_i\left(\boldsymbol{\xi}^{(i)}\times\dot{\boldsymbol{\xi}}^{(i)}\right)\right] =\\
&= m(\dot{\mathbf{d}}\times\dot{\mathbf{c}}) + m(\mathbf{d}\times\ddot{\mathbf{c}}) + \sum_{i=1}^{n} m_i\left(\boldsymbol{\xi}^{(i)}\times\ddot{\boldsymbol{\xi}}^{(i)}\right)\\
e^{(k)} &= \sum_{i=1}^{n}\frac{1}{2}m_i\dot{x}^{(i)2} = \sum_{i=1}^{n}\frac{1}{2}m_i\left(\dot{\mathbf{c}}+\dot{\boldsymbol{\xi}}^{(i)}\right)^T\left(\dot{\mathbf{c}}+\dot{\boldsymbol{\xi}}^{(i)}\right) =\\
&= \sum_{i=1}^{n}\frac{1}{2}m_i\left[\dot{\mathbf{c}}^T\dot{\mathbf{c}} + 2\dot{\mathbf{c}}^T\dot{\boldsymbol{\xi}}^{(i)} + \dot{\boldsymbol{\xi}}^{(i)T}\dot{\boldsymbol{\xi}}^{(i)}\right] =\\
&= \frac{1}{2}m\dot{c}^2 + \sum_{i=1}^{n}\frac{1}{2}m_i\dot{\xi}^{(i)2}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{f} &= \sum_{i=1}^{n} m_i\ddot{\mathbf{x}}^{(i)} = m\frac{\sum_{i=1}^{n} m_i\ddot{\mathbf{x}}^{(i)}}{m} = m\ddot{\mathbf{c}}\\
\boldsymbol{\lambda} &= \sum_{i=1}^{n}\boldsymbol{\lambda}^{(i)} = \sum_{i=1}^{n}(\mathbf{d}^{(i)}\times\mathbf{f}^{(i)}) =
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{i=1}^{n}[(\mathbf{d}+\boldsymbol{\xi}^{(i)})\times\mathbf{f}^{(i)}] = \mathbf{d}\times\sum_{i=1}^{n}\mathbf{f}^{(i)} + \sum_{i=1}^{n}[\boldsymbol{\xi}^{(i)}\times m_i\ddot{\mathbf{x}}^{(i)}] = \\
&= m(\mathbf{d}\times\ddot{\mathbf{c}}) + \sum_{i=1}^{n} m_i[\boldsymbol{\xi}^{(i)}\times(\ddot{\mathbf{c}}+\ddot{\boldsymbol{\xi}}^{(i)})] = m(\mathbf{d}\times\ddot{\mathbf{c}}) + \sum_{i=1}^{n} m_i(\boldsymbol{\xi}^{(i)}\times\ddot{\boldsymbol{\xi}}^{(i)}) \\
&= \dot{\gamma} - m(\dot{\mathbf{d}}\times\dot{\mathbf{c}})
\end{aligned}$$

5.4 Equivalence of Force Fields

Let $\mathbf{f}^{(i)}$, for $i=1,2,\ldots,n$, be a force field and let indicate the relative force resultant and the torque resultant about a point D as

$$\mathbf{f} = \sum_{i=1}^{n}\mathbf{f}^{(i)} \tag{5.14}$$

$$\boldsymbol{\lambda} = \sum_{i=1}^{n}\boldsymbol{\lambda}^{(i)} \tag{5.15}$$

where

$$\boldsymbol{\lambda}^{(i)} = \mathbf{d}^{(i)}\times\mathbf{f}^{(i)}$$

where $\mathbf{d}^{(i)}$ is the distance of the point of application of $\mathbf{f}^{(i)}$ from D.

In general, $\mathbf{f}$ may be defined in magnitude and in direction but cannot be located in space. Only in the special case that all the direction $\mathbf{f}^{(i)}$ cross a unique point, say A, we can locate $\mathbf{f}$ at A. Consider a point Q at a distance $\mathbf{q}^{(i)}$ from $\mathbf{f}^{(i)}$, for $i=1,2,\ldots,n$, Fig. 5.3. The relative torque resultant about Q is given by

$$\gamma = \sum_{i=1}^{n}\gamma^{(i)} \tag{5.16}$$

where

$$\gamma^{(i)} = \mathbf{q}^{(i)}\times\mathbf{f}^{(i)}$$

It is easy to verify that if $\mathbf{s}$ is the distance of Q from D then

$$\gamma = \boldsymbol{\lambda} + \mathbf{s}\times\mathbf{f} \tag{5.17}$$

In fact, since the distance $\mathbf{q}^{(i)}$ can be expressed as, Fig. 5.3,

$$\mathbf{q}^{(i)} = \mathbf{s} + \mathbf{d}^{(i)}$$

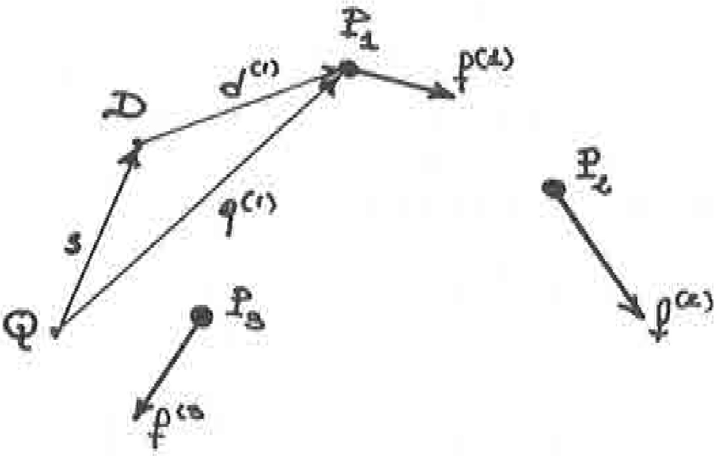


Figure 5.3: Force field

it follows that

$$\begin{aligned}
\gamma &= \sum_{i=1}^{n}\gamma^{(i)} = \sum_{i=1}^{n}\left(\mathbf{q}^{(i)} \times \mathbf{f}^{(i)}\right) = \\
&= \sum_{i=1}^{n}\left((\mathbf{s}+\mathbf{d}^{(i)}) \times \mathbf{f}^{(i)}\right) = \mathbf{s} \times \sum_{i=1}^{n}\mathbf{f}^{(i)} + \sum_{i=1}^{n}(\mathbf{d}^{(i)} \times \mathbf{f}^{(i)}) = \\
&= \mathbf{s} \times \mathbf{f} + \lambda
\end{aligned}$$

The scalar product between the forces resultant and the torque resultant is invariant with respect to the point about which the total torque is calculated. In fact,

$$\begin{aligned}
\gamma^T\mathbf{f} &= (\mathbf{s} \times \mathbf{f} + \lambda)^T\mathbf{f} = \mathbf{s}^T(\mathbf{f} \times \mathbf{f}) + \lambda^T\mathbf{f} = \\
&= \lambda^T\mathbf{f}
\end{aligned}$$

From a physical point of view, $\mathbf{f}$ may represent the total effort applied in pushing or pulling a material object. We know from experience that the effect produced on a rigid object does not only depend on the total effort applied. Different force fields with the same resultant $\mathbf{f}$ may produce different effects. On the other hand, it is possible to prove that different force fields but with the same $\mathbf{f}$ and $\boldsymbol{\lambda}$ about an arbitrary point, produce the same effect on a rigid material object. Accordingly, we have the following definition of equivalence on force fields.

Definition 5.4.1 *Let $\mathbf{f}^{(i)}$, for $i = 1, 2, \ldots, n$, and $\overline{\mathbf{f}}^{(j)}$, for $j = 1, 2, \ldots, p$, be two force fields and let indicate the relative force resultants and the torque resultants about a same point D as*

$$\mathbf{f} = \sum_{i=1}^{n} \mathbf{f}^{(i)}; \qquad \boldsymbol{\lambda} = \sum_{i=1}^{n} \boldsymbol{\lambda}^{(i)}$$

and

$$\overline{\mathbf{f}} = \sum_{j=1}^{p} \overline{\mathbf{f}}^{(j)}; \qquad \overline{\boldsymbol{\lambda}} = \sum_{i=1}^{n} \overline{\boldsymbol{\lambda}}^{(j)}$$

respectively. Then, these two vector fields are said to be equivalent if

$$\mathbf{f} = \overline{\mathbf{f}} \text{ and } \boldsymbol{\lambda} = \overline{\boldsymbol{\lambda}}$$

for any point D.

According to this definition, it follows that any force field is equivalent to its force resultant applied to an arbitrary point and to the torque resultant calculated about that point. As special cases we have:

- If all vectors concur to a same point, then the vector field may be represented only by its vector resultant applied at that point;
- If the vector resultant is zero, then the vector field may be represented only by its (in this case) constant torque resultant.

5.5 System of Interacting Particles

Let $\mathbf{f}^{(i)}$, for $i = 1, 2, \ldots, n$, be the vectors acting on a system of n particle P_i. In general, each particle may exert a force on the other particles. Consequently, we may express each $\mathbf{f}^{(i)}$ as the resultant of two distinct contributions, namely

$$\mathbf{f}^{(i)} = \overline{\mathbf{f}}^{(i)} + \tilde{\mathbf{f}}^{(i)} \tag{5.18}$$

where

$$\begin{aligned}
\overline{\mathbf{f}}^{(i)} &= \sum_{j=1}^{n} \overline{\mathbf{f}}^{(ij)} \\
\tilde{\mathbf{f}}^{(i)} &= \sum_{s=1}^{n} \tilde{\mathbf{f}}^{(is)}
\end{aligned}$$

and

- $\overline{\mathbf{f}}^{(ij)}$ is the *interacting* force that the particle j exerts on the particle i;
- $\tilde{\mathbf{f}}^{(is)}$ is the *non interacting* or *external* force that acts on the particle i.

Accordingly, the torque of $\mathbf{f}^{(i)}$ about any point D at distance $\mathbf{d}^{(i)}$ apart from P_i can be expressed as

$$\boldsymbol{\lambda}^{(i)} = \overline{\boldsymbol{\lambda}}^{(i)} + \tilde{\boldsymbol{\lambda}}^{(i)} \tag{5.19}$$

where

$$\begin{aligned}
\boldsymbol{\lambda}^{(i)} &= \mathbf{d}^{(i)} \times \mathbf{f}^{(i)} \\
\overline{\boldsymbol{\lambda}}^{(i)} &= \mathbf{d}^{(i)} \times \overline{\mathbf{f}}^{(i)} \\
\tilde{\boldsymbol{\lambda}}^{(i)} &= \mathbf{d}^{(i)} \times \tilde{\mathbf{f}}^{(i)}
\end{aligned}$$

Consequently, we can express the force and the torque resultants of the force field $\mathbf{f}^{(i)}$ as

$$\begin{aligned}
\mathbf{f} &= \overline{\mathbf{f}} + \tilde{\mathbf{f}} && (5.20) \\
\boldsymbol{\lambda} &= \overline{\boldsymbol{\lambda}} + \tilde{\boldsymbol{\lambda}} && (5.21)
\end{aligned}$$

where

$$\begin{aligned}
\overline{\mathbf{f}} &= \sum_{i=1}^{n} \overline{\mathbf{f}}^{(i)} \\
\overline{\boldsymbol{\lambda}} &= \sum_{i=1}^{n} \overline{\boldsymbol{\lambda}}^{(i)}
\end{aligned}$$

are the resultants of the interacting force field and

$$\begin{aligned}
\tilde{\mathbf{f}} &= \sum_{i=1}^{n} \tilde{\mathbf{f}}^{(i)} \\
\tilde{\boldsymbol{\lambda}} &= \sum_{i=1}^{n} \tilde{\boldsymbol{\lambda}}^{(i)}
\end{aligned}$$

are the resultants of the external force field. It can be proved that the field of interacting forces is equivalent to a null force field, so that

$$\mathbf{f} = \tilde{\mathbf{f}} \tag{5.22}$$

$$\boldsymbol{\lambda} = \tilde{\lambda} \tag{5.23}$$

However, the field of interacting forces still contributes to the total work of the system particles unless the relative distance between particles is constant in time, i.e. the system is rigid. In fact, with regard to the force resultant of the interacting forces, we have

$$\begin{aligned}\overline{\mathbf{f}} &= \sum_{i=1}^{n} \overline{\mathbf{f}}^{(i)} = \sum_{j=1}^{n} \mathbf{f}^{(ij)} \\ &= (\overline{\mathbf{f}}^{(11)} + \cdots + \overline{\mathbf{f}}^{(1i)} + \cdots + \overline{\mathbf{f}}^{(1n)}) + \cdots + \\ &\quad +(\overline{\mathbf{f}}^{(i1)} + \cdots + \overline{\mathbf{f}}^{(ii)} + \cdots + \overline{\mathbf{f}}^{(in)}) + \cdots + \\ &\quad +(\overline{\mathbf{f}}^{(n1)} + \cdots + \overline{\mathbf{f}}^{(ni)} + \cdots + \overline{\mathbf{f}}^{(nn)}) = \\ &= \mathbf{o}\end{aligned}$$

since, whenever $\overline{\mathbf{f}}^{(ij)}$ appears, $\overline{\mathbf{f}}^{(ji)}$ appears also, they add up to zero according to the 3rd Newton's law. Incidentally, particles do not exert any force on themselves, so that we have set in the above derivation

$$\overline{\mathbf{f}}^{(11)} = \ldots = \overline{\mathbf{f}}^{(ii)} = \ldots = \overline{\mathbf{f}}^{(nn)} = \mathbf{o}$$

With regard to the total torque of the interacting forces, we have

$$\begin{aligned}\overline{\boldsymbol{\lambda}} &= \sum_{i=1}^{n} \mathbf{d}^{(i)} \times \overline{\mathbf{f}}^{(i)} = \sum_{i=1}^{n} \mathbf{d}^{(i)} \times \left(\sum_{j=1}^{n} \overline{\mathbf{f}}^{(ij)}\right) \\ &= \mathbf{d}^{(1)} \times (\overline{\mathbf{f}}^{(11)} + \cdots + \overline{\mathbf{f}}^{(1i)} + \cdots + \overline{\mathbf{f}}^{(1n)}) + \cdots + \\ &\quad +\mathbf{d}^{(i)} \times (\overline{\mathbf{f}}^{(i1)} + \cdots + \overline{\mathbf{f}}^{(ii)} + \cdots + \overline{\mathbf{f}}^{(in)}) + \cdots + \\ &\quad +\mathbf{d}^{(n)} \times (\overline{\mathbf{f}}^{(n1)} + \cdots + \overline{\mathbf{f}}^{(ni)} + \cdots + \overline{\mathbf{f}}^{(nn)}) \\ &= \mathbf{o}\end{aligned}$$

since, whenever $\mathbf{d}^{(i)} \times \overline{\mathbf{f}}^{(ij)}$ appears, $\mathbf{d}^{(j)} \times \overline{\mathbf{f}}^{(ji)}$ appears also, they add up to zero according to the 3rd Newton's law.

Finally, according to the decomposition in Eq. 5.18 the work done by $\mathbf{f}^{(i)}$ on P_i can be expressed as

$$dw_i = d\overline{w}_i + d\tilde{w}_i \tag{5.24}$$

where

$$\begin{aligned} dw_i &= \mathbf{f}^{(i)T} d\mathbf{x}^{(i)} \\ d\overline{w}_i &= \overline{\mathbf{f}}^{(i)T} d\mathbf{x}^{(i)} \\ d\tilde{w}_i &= \tilde{\mathbf{f}}^{(i)T} d\mathbf{x}^{(i)} \end{aligned}$$

Consequently, the total work done by the force field on the system of particles can be expressed as

$$dw = d\overline{w} + d\tilde{w} \tag{5.25}$$

where

$$\begin{aligned} dw &= \sum_{i=1}^{n} \mathbf{f}^{(i)T} d\mathbf{x}^{(i)} \\ d\overline{w} &= \sum_{i=1}^{n} \overline{\mathbf{f}}^{(i)T} d\mathbf{x}^{(i)} \\ d\tilde{w} &= \sum_{i=1}^{n} \tilde{\mathbf{f}}^{(i)T} d\mathbf{x}^{(i)} \end{aligned}$$

In particular,

$$\begin{aligned} d\overline{w} &= \sum_{i=1}^{n} \overline{\mathbf{f}}^{(i)T} d\mathbf{x}^{(i)} = \sum_{i=1}^{n} d\mathbf{x}^{(i)T} \left(\sum_{j=1}^{n} \overline{\mathbf{f}}^{(ij)} \right) = \\ &= d\mathbf{x}^{(1)T}(\overline{\mathbf{f}}^{(11)} + \cdots + \overline{\mathbf{f}}^{(1i)} + \cdots + \overline{\mathbf{f}}^{(1n)}) + \cdots + \\ &\quad + d\mathbf{x}^{(i)T}(\overline{\mathbf{f}}^{(i1)} + \cdots + \overline{\mathbf{f}}^{(ii)} + \cdots + \overline{\mathbf{f}}^{(in)}) + \cdots + \\ &\quad + d\mathbf{x}^{(n)T}(\overline{\mathbf{f}}^{(n1)} + \cdots + \overline{\mathbf{f}}^{(ni)} + \cdots + \overline{\mathbf{f}}^{(nn)}) = \\ &= \overline{\mathbf{f}}^{(11)T} d\mathbf{x}^{(1)} + \cdots + \overline{\mathbf{f}}^{(1i)T}(d\mathbf{x}^{(1)} - d\mathbf{x}^{(i)}) + \cdots + \\ &\quad + \overline{\mathbf{f}}^{(1n)T}(d\mathbf{x}^{(1)} - d\mathbf{x}^{(n)}) + \cdots + + \overline{\mathbf{f}}^{(ii)T} d\mathbf{x}^{(i)} + \cdots + \\ &\quad + \overline{\mathbf{f}}^{(in)T}(d\mathbf{x}^{(i)} - d\mathbf{x}^{(n)}) + \cdots + \overline{\mathbf{f}}^{(nn)T} d\mathbf{x}^{(n)} \end{aligned}$$

is not null in general. This work can be equal to zero when the system of particle is *rigid*, i.e. when the distance between any two particles does not change so that

$$\begin{aligned} \mathbf{q}^{(ij)} &= \mathbf{x}^{(i)} - \mathbf{x}^{(j)} = \text{const} \\ d\mathbf{q}^{(ij)} &= \mathbf{o} \end{aligned}$$

for all $i, j = 1, 2, \ldots, n$.

5.6 Mechanical Principles

According to the findings reported in the previous Section, we can establish the following mechanical principles governing the motion of a system of interacting particles.

Theorem 5.6.1 (Linear Momentum Principle). *The resultant external force acting on a given set of n interacting particles equals the time rate of change of the total momentum of the particle of this set, that is*

$$\sum_{i=1}^{n} \mathbf{f}^{(i)} = \frac{d}{dt} \sum_{i=1}^{n} (m_i \dot{\mathbf{x}}^{(i)})$$

or, equivalently,

$$\sum_{i=1}^{n} \mathbf{f}^{(i)} = \frac{d}{dt} (m\dot{\mathbf{c}})$$

where $\mathbf{c}$ *represents the location of the center of mass in which we may suppose to concentrate the total mass*

$$m = \sum_{i=1}^{n} m_i$$

Theorem 5.6.2 (Principle of Conservation of Momentum). *If the resultant external force acting on a system of n particles is zero, then the total momentum remains constant, that is*

$$\sum_{i=1}^{n} (m_i \dot{\mathbf{x}}^{(i)}) = \ const$$

or, equivalently,

$$m\dot{\mathbf{c}} = \ const$$

where $\mathbf{c}$ *represent the location of the center of mass in which we may suppose to concentrate the total mass*

$$m = \sum_{i=1}^{n} m_i$$

In such case, therefore, the center of mass of the system preserves its state of rest or motion with constant velocity.

Theorem 5.6.3 (Angular Momentum Principle). *The total external torque acting on a given set of n interacting particles calculated about a fixed point in space, or about the center of mass, equals the time rate of change of the angular moment, that is*

$$\sum_{i=1}^{n}(\mathbf{x}^{(i)} \times \mathbf{f}^{(i)}) = \frac{d}{dt}\sum_{i=1}^{n} m_i(\mathbf{x}^{(i)} \times \dot{\mathbf{x}}^{(i)})$$

Theorem 5.6.4 (Principle of Conservation of Angular Momentum). *If the resultant external torque acting on a given set of n interacting particles calculated about a fixed point in space, or about the center of mass, is equal to zero, then the angular momentum remains constant, that is*

$$\sum_{i=1}^{n} m_i(\mathbf{x}^{(i)} \times \dot{\mathbf{x}}^{(i)}) = \ const$$

Theorem 5.6.5 (Kinetic Energy vs. Total Work). *The total work due by all forces (external and interacting) in moving a system of n interacting particles from one state where the total kinetic energy is $e_1^{(k)}$ to another where the total kinetic energy is $e_2^{(k)}$, is equal to*

$$w_{12} = e_2^{(k)} - e_1^{(k)}$$

and the total kinetic energy may be expressed as

$$e^{(k)} = \frac{1}{2}m\dot{c}^2 + \frac{1}{2}\sum_{i=1}^{n} m_i \dot{\xi}^{(i)\,2}$$

where **c** *represents the location of the center of mass in which we may suppose to concentrate the total mass*

$$m = \sum_{i=1}^{n} m_i$$

and $\dot{\xi}^{(i)}$ is the relative velocity of the i-th particle of mass m_i with respect to the center of mass.

Theorem 5.6.6 (Principle of Conservation of Energy). *If all forces (external and internal) acting on a system of particles are conservative, i.e. there exists a potential function (energy) $e^{(p_i)}$ such that:*

$$\mathbf{f}^{(i)} = -\operatorname{grad} e_i^p$$

then the total energy

$$e = e^{(k)} + e^{(p)}$$

is constant.

Chapter 6

Motion and Deformation in a Continuum

6.1 Introduction

A continuous system of material particles occupying a region of space is a *continuous medium* or a *continuum*. In a continuum the material particle, which have been postulated to occupy a point in space, are *tight one to the other without gaps nor overlapping*.

Accordingly, all the physical quantities defined in Chapter 4 as a consequence of the three Newton laws must be understood as physical quantities which may vary continuously in the medium. The extension of the concepts of force and mass to a continuum is postpone to the next Chapter, since it is first necessary to present the mathematical tools available for describing the motion and the deformation of a continuum.

The Continuum Mechanics Theory has two alternative formulations for describing the motion and the deformation of a continuum: the *spatial* and the *material* descriptions. The spatial description, Section 6.3, is particulary useful to formulate the equilibrium conditions in a solid medium, whereas the material description, Section 6.4, is serviceable to formulate the equilibrium conditions in a fluid medium. However, both formulations are the two sides of the same problem, Section 6.5. A particular case of these two formulations is the so-called *small deformation theory*, Section 6.6.

6.2 Generality

Let $F = Ox_1x_2x_3$ be a CaORS of unit base vectors $\hat{\mathbf{e}}^{(i)}$, for $i = 1, 2, 3$, and $\mathcal{M}$ be a continuum that from an initial configuration $\mathcal{M}_o$ of rest at time $t = 0$ starts moving and deforming continuously with time, Fig. 6.1. With

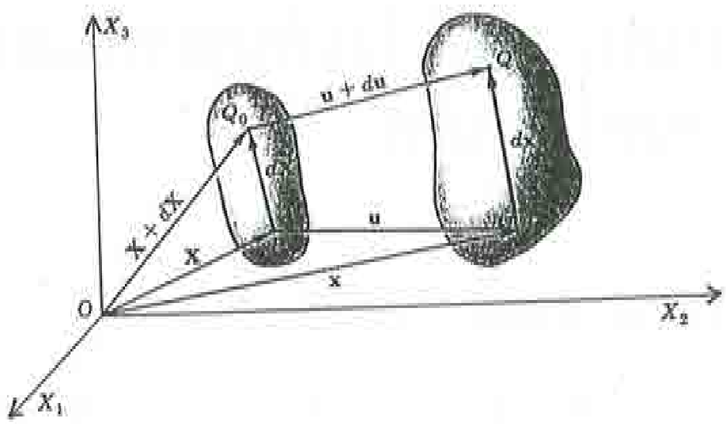


Figure 6.1: Motion in space of a continuous medium

respect to F, the initial position P_o at time $t = t_o$ and the current one P at time t of any material point of $\mathcal{M}$ can be identified by the position vectors

$$\mathbf{X} = X_i\hat{\mathbf{e}}^{(i)} \quad (6.1)$$

$$\mathbf{x} = x_i\hat{\mathbf{e}}_i \quad (6.2)$$

Then, for each material point P in $\mathcal{M}$, we define:

- The *displacement* of the particle as

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \quad (6.3)$$

- The *velocity*, or *instantaneous velocity*, of the particle at P as

$$\mathbf{v} = \dot{\mathbf{x}} = \dot{\mathbf{u}} \quad (6.4)$$

- The *acceleration*, or the *instantaneous acceleration*, of the particle at P as
$$\mathbf{a} = \ddot{\mathbf{x}} = \ddot{\mathbf{u}} \tag{6.5}$$
- The *relative displacement vector* of the particle P with respect to a neighboring point
$$d\mathbf{u} = d\mathbf{x} - d\mathbf{X} \tag{6.6}$$

where the *dot* indicates the operation of total differentiation with respect of time.

In continuum mechanics, there are problems in which it is interesting to establish the evolution in time of the whole medium. For example, in solid mechanics we are interested in describing the deformation process of the medium and this can be accomplished by tracing the displacement of each point of the medium. Thus, we require displacement functions of the type
$$\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$$
For such a type of problems, it is convenient to adopt the *spatial description*, also known as the *Lagrangian formulation*, according to which the motion of any particle P in $\mathcal{M}$ is expressed as
$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{6.7}$$
where the independent variables are the initial position $\mathbf{X}$ of the particle and the current time t.

Other problems concern the monitoring of physical quantities in a certain part of the continuum. For example, in fluid mechanics we are interested in monitoring the fluid velocity at given fixed points. Thus, we require velocity functions of the type
$$\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$$
For such a type of problems, it is convenient to adopt the *spatial description* also known as the *Eulerian formulation* according to which the motion of any particle P in $\mathcal{M}$ is expressed as
$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{6.8}$$
where the independent variables are the current position $\mathbf{x}$ of the particle and the current time t.

The Lagrangian formulation in Eq. 6.7 may be interpreted as a mapping of the initial configuration in the current one. On the contrary, the Eulerian formulation in Eq. 6.8 may be interpreted as one which provides a tracing to its original position of the particle that now occupies the location $\mathbf{x}$.

Under the assumption that $\mathcal{M}$ remains a continuum, i.e. without gaps and overlapping of matter, the correspondence between $\mathbf{X}$ and $\mathbf{x}$ has to be one-to-one and continuous over the spatial domain of $\mathcal{M}$. Consequently, the continuum mechanics theory assumes that the functions in Eqs. 6.7 and 6.8 are continuous with continuous partial derivatives to whatever order is required. Moreover, it assumes that the Jacobians

$$\tilde{J} = \det\left[\frac{\partial x_i}{\partial X_j}\right]; \qquad \hat{J} = \det\left[\frac{\partial X_i}{\partial x_j}\right] \tag{6.9}$$

do not vanish, which is the necessary and sufficient condition for the two mapping in Eqs. 6.7 and 6.8 to be the unique inverse of one another, Section 6.5.

6.3 Material Description: Lagrangian Formulation

The spatial description concentrates on a given body of matter of the continuum $\mathcal{M}$ and the motion of any particle P is expressed by equations of the form

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{6.10}$$

where $\mathbf{X}$ is the location of P at the initial time $t = 0$ while $\mathbf{x}$ is the location of P at the current time t.

This description, where the independent variables are the initial location $\mathbf{X}$ and the current time t, may be viewed as a mapping of P from its original position into the current configuration. It is assumed that such a tracing is one-to-one and continuous, with continuous partial derivatives to whatever order is required. According to Eq. 6.10, we have that

$$d\mathbf{x} = \widetilde{\mathcal{F}} d\mathbf{X} + \frac{\partial \mathbf{x}}{\partial t} dt \tag{6.11}$$

where

$$\widetilde{\mathcal{F}} = \left[\frac{\partial x_i}{\partial X_j}\right]$$

is known as the *Lagrangian Deformation Gradient*. The first term on the right, namely

$$\delta\mathbf{x} = \widetilde{\mathcal{F}} d\mathbf{X} \tag{6.12}$$

is the relative distance at the current (and fixed) time of two neighboring points originally located at a relative distance $d\mathbf{X}$. The condition of one-to-one mapping requires the invertibility of $\tilde{\mathcal{F}}$ so that the correspondence of any $\delta\mathbf{X}$ into the current configuration is unique, namely:

$$d\mathbf{X} = \tilde{\mathcal{F}}^{-1}\delta\mathbf{x} \tag{6.13}$$

A necessary and sufficient condition for the inverse of $\tilde{\mathcal{F}}$ to exist is that the determinant

$$\tilde{J} = \det\tilde{\mathcal{F}} \tag{6.14}$$

should not vanish for any material point in the current configuration. Then, for each material point in $\mathcal{M}$, we have that, Section 6.2:

- The displacement of the particle is given by

$$\mathbf{u}(\mathbf{X},t) = \mathbf{x}(\mathbf{X},t) - \mathbf{X} \tag{6.15}$$

- The relative displacement vector of the particle P with respect to a neighboring point is given by

$$d\mathbf{u}(\mathbf{X},t) = \tilde{\mathcal{U}}d\mathbf{X} + \frac{\partial\mathbf{u}(\mathbf{X},t)}{\partial t}dt \tag{6.16}$$

  where

$$\tilde{\mathcal{U}} = \left[\frac{\partial u_i}{\partial X_j}\right]$$

  is known as the *Lagrange Displacement Gradient* tensor.

- The velocity of the particle at P is given by

$$\mathbf{v}(\mathbf{X},t) = \dot{\mathbf{u}}(\mathbf{X},t) = \frac{\partial\mathbf{u}(\mathbf{X},t)}{\partial t} \tag{6.17}$$

- The acceleration of the particle at P is given by

$$\mathbf{a}(\mathbf{X},t) = \ddot{\mathbf{u}}(\mathbf{X},t) = \frac{\partial\mathbf{v}(\mathbf{X},t)}{\partial t} \tag{6.18}$$

6.3.1 Lagrangian tensors

The material formulation leads to define the following tensors:

- The *Lagrange Displacement Gradient* tensor defined as

$$\tilde{\mathcal{U}} = [u_{i,j}] = \left[\frac{\partial u_i}{\partial X_j}\right]$$

- The *Lagrange Deformation Gradient* tensor defined as

$$\tilde{\mathcal{F}} = [x_{i,j}] = \left[\frac{\partial x_i}{\partial X_j}\right]$$

  which can be alternatively expressed as

$$\tilde{\mathcal{F}} = \mathbf{I} + \tilde{\mathcal{U}} = [\delta_{ij} + u_{i,j}]$$

- The *Green Strain* tensor defined as

$$\tilde{\mathcal{G}} = \tilde{\mathcal{F}}^T \tilde{\mathcal{F}} = [x_{k,i} x_{k,j}]$$

  which can be alternatively expressed as

$$\tilde{\mathcal{G}} = \mathbf{I} + \tilde{\mathcal{U}} + \tilde{\mathcal{U}}^T + \tilde{\mathcal{U}}^T \tilde{\mathcal{U}} = [\delta_{ij} + u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}]$$

- The *Finite Lagrange Strain* tensor defined as

$$\tilde{\mathcal{L}} = \frac{1}{2}\left(\tilde{\mathcal{G}} - \mathbf{I}\right) = \frac{1}{2}[x_{k,i} x_{k,j} - \delta_{ij}]$$

  which can be alternatively expressed as

$$\tilde{\mathcal{L}} = \frac{1}{2}\left(\tilde{\mathcal{U}} + \tilde{\mathcal{U}}^T + \tilde{\mathcal{U}}^T \tilde{\mathcal{U}}\right) = \frac{1}{2}[u_{i,j} + u_{j,i} + u_{k,i} u_{k,j}]$$

- The *Lagrange Velocity Gradient* tensor defined as

$$\tilde{\mathcal{V}} = [v_{i,i}] = \left[\frac{\partial v_i}{\partial X_j}\right]$$

All the above reported alternative expressions are obtained taking into account that according to Eq. 6.15

$$x_{i,j} = [X_i + u_i]_{,j} = \delta_{ij} + u_{i,j}$$

It is immediate to verify that the time rate of the above tensors can be expressed as

$$\begin{aligned} \dot{\tilde{\mathcal{U}}} &= \tilde{\mathcal{V}} \\ \dot{\overline{\mathcal{F}}} &= \dot{\tilde{\mathcal{U}}} = \tilde{\mathcal{V}} \\ \dot{\tilde{\mathcal{G}}} &= \dot{\overline{\mathcal{F}}}^T \overline{\mathcal{F}} + \overline{\mathcal{F}}^T \dot{\overline{\mathcal{F}}} = \tilde{\mathcal{V}}^T \overline{\mathcal{F}} + \overline{\mathcal{F}}^T \tilde{\mathcal{V}} \\ \dot{\tilde{\mathcal{L}}} &= \frac{\dot{\tilde{\mathcal{G}}}}{2} \end{aligned}$$

6.3.2 Measure of Deformation

Let $\mathcal{M}$ be a continuum and P, Q and R be three neighboring material points in $\mathcal{M}$, Fig. 6.2. Let us indicate the distance between these points as:

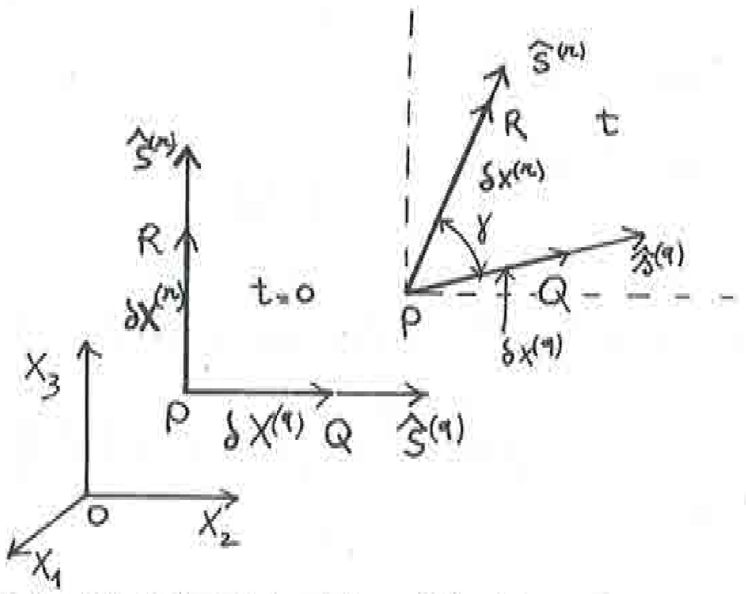


Figure 6.2: Material description of the deformation process

- $d\mathrm{X} \equiv d\mathrm{X}^{(q)}$ and $\delta\mathrm{x} \equiv \delta\mathrm{x}^{(q)}$ the relative distance between P and Q at time $t = 0$ and t, respectively.

- $d\mathrm{X}^{(r)}$ and $\delta\mathrm{x}^{(r)}$ the relative distance between P and R at time $t = 0$ and t, respectively.

- and, for $i = q, r$,

$$\hat{\mathrm{S}}^{(i)} = \frac{d\mathrm{X}^{(i)}}{|d\mathrm{X}^{(i)}|}; \qquad \hat{\mathrm{s}}^{(i)} = \frac{\delta\mathrm{x}^{(i)}}{|\delta\mathrm{x}^{(i)}|}$$

  the unit vectors in the direction of $d\mathrm{X}^{(i)}$ and $\delta\mathrm{x}^{(i)}$, respectively.

Then, according to the Lagrangian formulation, we can measure their deformation process as follows:

- The *Undeformed Length* defined as

$$dS^2 = d\mathrm{X}^T d\mathrm{X}$$

- The *Deformed Length* defined as

$$\delta s^2 = \delta\mathrm{x}^T \delta\mathrm{x}$$

  can be calculated as

$$\delta s^2 = d\mathrm{X}^T \tilde{\mathcal{G}} d\mathrm{X}$$

  and its time rate as

$$\frac{d}{dt}(\delta s^2) = d\mathrm{X}^T \dot{\tilde{\mathcal{G}}} d\mathrm{X}$$

  In fact,

$$\delta s^2 = \delta\mathrm{x}^T \delta\mathrm{x} = \left(d\mathrm{X}^T \overline{\mathcal{F}}^T\right)\left(\overline{\mathcal{F}} d\mathrm{X}\right) = d\mathrm{X}^T \left(\overline{\mathcal{F}}^T \overline{\mathcal{F}}\right) d\mathrm{X} = d\mathrm{X}^T \tilde{\mathcal{G}} d\mathrm{X}$$

  and

$$\frac{d}{dt}\left(\delta s^2\right) = \frac{d}{dt}\left(d\mathrm{X}^T \tilde{\mathcal{G}} d\mathrm{X}\right) = d\mathrm{X}^T \dot{\tilde{\mathcal{G}}} d\mathrm{X} = 2 d\mathrm{X}^T \dot{\tilde{\mathcal{C}}} d\mathrm{X}$$

- The *Stretch* defined as

$$\lambda^2 = \frac{\delta s^2}{dS^2}$$

  can be calculated as

$$\lambda^2 = \hat{\mathrm{S}}^T \tilde{\mathcal{G}} \hat{\mathrm{S}}$$

and its time rate as

$$\frac{d}{dt}(\lambda^2) = \hat{\mathrm{S}}^T \dot{\tilde{\mathcal{G}}} \hat{\mathrm{S}}$$

In fact,

$$\lambda^2 = \frac{\delta s^2}{dS^2} = \frac{\delta \mathrm{x}^T \delta \mathrm{x}}{d\mathrm{X}^T d\mathrm{X}} = \frac{d\mathrm{X}^T \tilde{\mathcal{G}} d\mathrm{X}}{|d\mathrm{X}^T d\mathrm{X}|^2} = \hat{\mathrm{S}}^T \tilde{\mathcal{G}} \hat{\mathrm{S}}$$

and

$$\frac{d}{dt}\left(\lambda^2\right) = \frac{d}{dt}\left(\hat{\mathrm{S}}^T \tilde{\mathcal{G}} \hat{\mathrm{S}}\right) = \hat{\mathrm{S}}^T \dot{\tilde{\mathcal{G}}} \hat{\mathrm{S}}$$

- The *Measure of Deformation* defined as

$$\chi = \delta s^2 - dS^2$$

can be calculated as

$$\chi = 2(d\mathrm{X}^T \tilde{\mathcal{L}} d\mathrm{X})$$

and its time rate as

$$\dot{\chi} = d\mathrm{X}^T \dot{\tilde{\mathcal{G}}} d\mathrm{X}$$

In fact,

$$\begin{aligned} \delta s^2 - dS^2 &= \delta \mathrm{x}^T \delta \mathrm{x} - d\mathrm{X}^T d\mathrm{X} = d\mathrm{X}^T \tilde{\mathcal{G}} d\mathrm{X} - d\mathrm{X}^T \mathrm{I} d\mathrm{X} = \\ &= d\mathrm{X}^T \left(\tilde{\mathcal{G}} - \mathrm{I}\right) d\mathrm{X} = 2 d\mathrm{X}^T \tilde{\mathcal{L}} d\mathrm{X} \end{aligned}$$

and

$$\frac{d}{dt}\left(\delta s^2 - dS^2\right) = \frac{d}{dt}(\delta s^2) = d\mathrm{X}^T \dot{\tilde{\mathcal{G}}} d\mathrm{X}$$

- The *Unit Extension* defined as

$$\epsilon = \frac{\delta s - dS}{dS} = \lambda - 1$$

can be calculated as

$$\epsilon = \left(\hat{\mathrm{S}}^T \tilde{\mathcal{G}} \hat{\mathrm{S}}\right)^{1/2} - 1$$

and its time rate as

$$\dot{\epsilon} = \frac{1}{2}\left(\hat{\mathrm{S}}^T \dot{\tilde{\mathcal{G}}} \hat{\mathrm{S}}\right)\left(\hat{\mathrm{S}}^T \tilde{\mathcal{G}} \hat{\mathrm{S}}\right)^{-1/2}$$

In fact,

$$\epsilon = \lambda - 1 = \left(\hat{\mathrm{S}}^T \tilde{\mathcal{G}} \hat{\mathrm{S}}\right)^{1/2} - 1$$

and

$$\frac{d}{dt}\epsilon = \frac{d}{dt}\left[\left(\hat{\mathrm{S}}^T \tilde{\mathcal{G}} \hat{\mathrm{S}}\right)^{1/2} - 1\right] = \frac{1}{2}\left(\hat{\mathrm{S}}^T \dot{\tilde{\mathcal{G}}} \hat{\mathrm{S}}\right)\left(\hat{\mathrm{S}}^T \tilde{\mathcal{G}} \hat{\mathrm{S}}\right)^{-1/2}$$

- The *Angle Change* defined as, Fig. 6.2,

$$\gamma = \cos(\hat{\mathrm{s}}^{(q)}, \hat{\mathrm{s}}^{(r)})$$

can be calculated as

$$\gamma = \frac{\hat{\mathrm{S}}^{(q)T}\tilde{\mathcal{G}}\hat{\mathrm{S}}^{(r)}}{\lambda_q\lambda_r} = \frac{\tilde{\mathcal{G}}_{qr}}{\lambda_q\lambda_r}$$

and its time rate as

$$\dot{\gamma} = \hat{\mathrm{S}}^{(q)T}\widetilde{\mathcal{A}}\hat{\mathrm{S}}^{(r)}$$

where

$$\widetilde{\mathcal{A}} = \dot{\tilde{\mathcal{G}}} - \frac{1}{2}\tilde{\mathcal{G}}\left[\frac{\hat{\mathrm{S}}^{(q)T}\dot{\tilde{\mathcal{G}}}\hat{\mathrm{S}}^{(q)}}{\hat{\mathrm{S}}^{(q)T}\tilde{\mathcal{G}}\hat{\mathrm{S}}^{(q)}} + \frac{\hat{\mathrm{S}}^{(r)T}\dot{\tilde{\mathcal{G}}}\hat{\mathrm{S}}^{(r)}}{\hat{\mathrm{S}}^{(r)T}\tilde{\mathcal{G}}\hat{\mathrm{S}}^{(r)}}\right]$$

In fact,

$$\begin{aligned}
\gamma &= \cos\left(\hat{\mathrm{s}}^{(q)}, \hat{\mathrm{s}}^{(r)}\right) = \hat{\mathrm{s}}^{(q)T}\hat{\mathrm{s}}^{(r)} = \left(\frac{\delta\mathbf{x}^{(q)}}{\delta s^{(q)}}\right)^T\left(\frac{\delta\mathbf{x}^{(r)}}{\delta s^{(r)}}\right) = \\
&= \frac{\left(d\mathrm{X}^{(q)T}\widetilde{\mathcal{F}}^T\right)\left(\mathcal{F}\delta\mathrm{X}^{(r)}\right)}{\delta s^{(q)}\delta s^{(r)}} = \frac{\left(dS^{(q)}\hat{\mathrm{S}}^{(q)T}\widetilde{\mathcal{F}}^T\right)\left(\widetilde{\mathcal{F}}dS^{(r)}\hat{\mathrm{S}}^{(r)}\right)}{\delta s^{(q)}\delta s^{(r)}} = \\
&= \frac{\hat{\mathrm{S}}^{(q)T}\tilde{\mathcal{G}}\hat{\mathrm{S}}^{(r)}}{\lambda_q\lambda_r}
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}\gamma &= \frac{d}{dt}\left(\frac{\hat{\mathrm{S}}^{(q)T}\tilde{\mathcal{G}}\hat{\mathrm{S}}^{(r)}}{\lambda_q\lambda_r}\right) = \\
&= \frac{1}{\lambda_q\lambda_r}\left[\left(\hat{\mathrm{S}}^{(q)T}\dot{\tilde{\mathcal{G}}}\hat{\mathrm{S}}^{(r)}\right)\lambda_q\lambda_r - \hat{\mathrm{S}}^{(q)T}\tilde{\mathcal{G}}\hat{\mathrm{S}}^{(r)}\left(\dot{\lambda}_q\lambda_r + \lambda_q\dot{\lambda}_r\right)\right] = \\
&= \hat{\mathrm{S}}^{(q)T}\left[\dot{\tilde{\mathcal{G}}} - \tilde{\mathcal{G}}\left(\frac{\dot{\lambda}_q}{\lambda_q} + \frac{\dot{\lambda}_r}{\lambda_r}\right)\right]\hat{\mathrm{S}}^{(r)}
\end{aligned}$$

From the above reported relationships for λ^2 it results that

$$\begin{aligned}
\lambda &= \left(\hat{\mathrm{S}}^T\tilde{\mathcal{G}}\hat{\mathrm{S}}\right)^{1/2} \\
\dot{\lambda} &= \frac{1}{2}\left(\hat{\mathrm{S}}^T\dot{\tilde{\mathcal{G}}}\hat{\mathrm{S}}\right)\left(\hat{\mathrm{S}}^T\tilde{\mathcal{G}}\hat{\mathrm{S}}\right)^{-1/2}
\end{aligned}$$

from which

$$\frac{\dot{\lambda}}{\lambda} = \frac{1}{2}\left(\widehat{\mathrm{S}}^T \dot{\widetilde{\mathcal{G}}} \widehat{\mathrm{S}}\right)\left(\widehat{\mathrm{S}}^T \widetilde{\mathcal{G}} \widehat{\mathrm{S}}\right)^{1/2}$$

Consequently,

$$\dot{\gamma} = \widehat{\mathrm{S}}^{(q)T} \widetilde{\mathcal{A}} \widehat{\mathrm{S}}^{(r)}$$

6.3.3 Surface and volume

Let $d\mathcal{M}_o$ be an infinitesimal portion of material at time $t = 0$ defined by the three differential vectors $d\mathrm{X}^{(1)}, d\mathrm{X}^{(2)}$ and $d\mathrm{X}^{(3)}$, Fig. 6.3. According

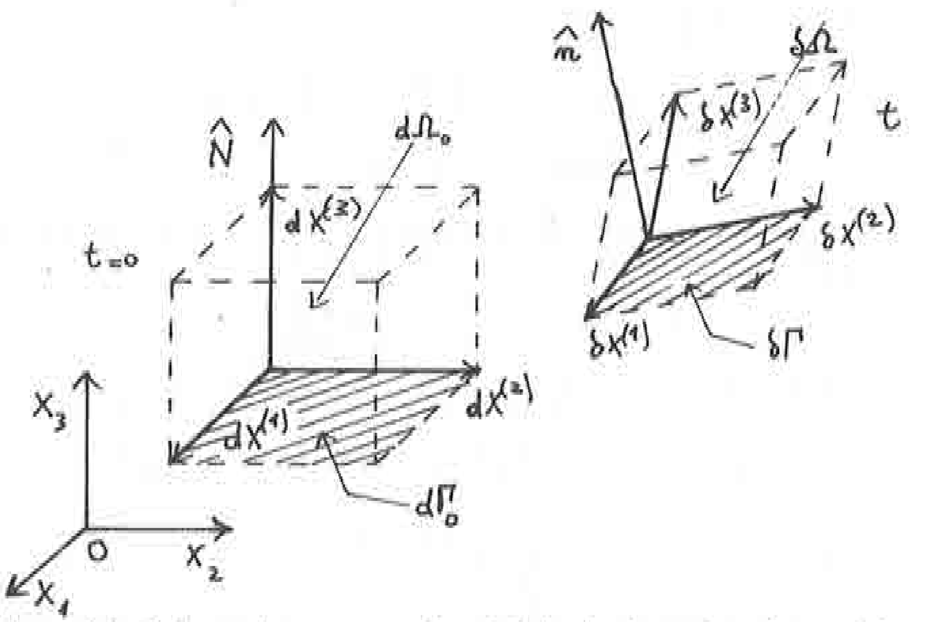


Figure 6.3: Deformation process of an infinitesimal portion of material

to Eqs. 2.24 and 2.25, the surface defined by the vectors $(d\mathrm{X}^{(1)}, d\mathrm{X}^{(2)})$ and the volume of $d\mathcal{M}_o$ can be calculated as

$$\begin{aligned} \widehat{\mathrm{N}} d\Gamma_o &= d\mathrm{X}^{(1)} \times d\mathrm{X}^{(2)} \\ d\Omega_o &= \left(d\mathrm{X}^{(1)} \times d\mathrm{X}^{(2)}\right)^T d\mathrm{X}^{(3)} \end{aligned}$$

Assume that at the current time t this portion of material reaches the configuration $d\mathcal{M}$ described by the three differential vectors $\delta\mathrm{x}^{(1)}, \delta\mathrm{x}^{(2)}$ and $\delta\mathrm{x}^{(3)}$.

The current value of the surface and the volume above defined is then given by

$$\begin{aligned}\hat{\mathbf{n}}\delta\Gamma &= \delta\mathbf{x}^{(1)} \times \delta\mathbf{x}^{(2)} \\ \delta\Omega &= \left(\delta\mathbf{x}^{(1)} \times \delta\mathbf{x}^{(2)}\right)^T \delta\mathbf{x}^{(3)}\end{aligned}$$

According to the Lagrangian formulation, these surfaces and volumes are related as

$$\hat{\mathbf{n}}\delta\Gamma = \tilde{J}\widetilde{\mathcal{F}}^{-T}\hat{\mathbf{N}}d\Gamma_o \tag{6.19}$$

$$\delta\Omega = \tilde{J}d\Omega_o \tag{6.20}$$

and their time rate can be calculated as

$$\frac{d}{dt}(\hat{\mathbf{n}}\delta\Gamma) = \left[\dot{\widetilde{J}}\widetilde{\mathcal{F}}^{-T} + \tilde{J}\dot{\widetilde{\mathcal{F}}}^{-T}\right]\hat{\mathbf{N}}d\Gamma_o \tag{6.21}$$

$$\frac{d}{dt}(\delta\Omega) = \text{div}\ \mathbf{v}\delta\Omega \tag{6.22}$$

In order to prove these statements, we preliminarily observe that according to Eq. 2.10,

$$\epsilon_{ijk}\frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s}\frac{\partial x_k}{\partial X_t} = \epsilon_{rst}\det\left[\frac{\partial x_m}{\partial X_n}\right] = \epsilon_{rst}\tilde{J} \tag{6.23}$$

Then, writing in extended form the area expression, we obtain

$$\begin{aligned}\hat{n}_i\delta\Gamma &= \epsilon_{ijk}\delta x_j^{(1)}\delta x_k^{(2)} = \epsilon_{ijk}\frac{\partial x_j}{\partial X_s}dX_s^{(1)}\frac{\partial x_k}{\partial X_t}dX_t^{(2)} = \\ &= \epsilon_{ijk}\frac{\partial x_j}{\partial X_s}\frac{\partial x_k}{\partial X_t}dX_s^{(1)}dX_t^{(2)}\end{aligned}$$

from which, premultiplying both sides by $x_{i,r}$ and taking into account Eq. 6.23,

$$\begin{aligned}\frac{\partial x_i}{\partial X_r}\hat{n}_i\delta\Gamma &= \epsilon_{ijk}\frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s}\frac{\partial x_k}{\partial X_t}dX_s^{(1)}dX_t^{(2)} = \epsilon_{rst}\tilde{J}dX_s^{(1)}dX_t^{(2)} \\ &= \tilde{J}\hat{N}_r d\Gamma_o\end{aligned}$$

Hence,

$$\begin{aligned}\hat{\mathbf{n}}^T\widetilde{\mathcal{F}}\delta\Gamma &= \tilde{J}\hat{\mathbf{N}}^T d\Gamma_o \\ \hat{\mathbf{n}}^T\delta\Gamma &= \tilde{J}\hat{\mathbf{N}}^T\widetilde{\mathcal{F}}^{-1}d\Gamma_o \\ \hat{\mathbf{n}}\delta\Gamma &= \tilde{J}\widetilde{\mathcal{F}}^{-T}\hat{\mathbf{N}}d\Gamma_o\end{aligned}$$

Writing in extended form the volume expression and taking into account Eq. 6.23, we obtain

$$\begin{aligned}
\delta\Omega &= \left(\delta\mathbf{x}^{(1)} \times \delta\mathbf{x}^{(2)}\right)^T \delta\mathbf{x}^{(3)} = \epsilon_{ijk}\delta x_i^{(1)}\delta x_j^{(2)}\delta x_k^{(3)} = \\
&= \epsilon_{ijk}\frac{\partial x_i}{\partial X_r}dX_r^{(1)}\frac{\partial x_j}{\partial X_s}dX_s^{(2)}\frac{\partial x_k}{\partial X_t}dX_t^{(3)} = \\
&= \epsilon_{ijk}\frac{\partial x_i}{\partial X_r}\frac{\partial x_j}{\partial X_s}\frac{\partial x_k}{\partial X_t}dX_r^{(1)}dX_s^{(2)}dX_t^{(3)} = \\
&= \epsilon_{rst}\tilde{J}dX_r^{(1)}dX_s^{(2)}dX_t^{(3)} = \\
&= \tilde{J}d\Omega_o
\end{aligned}$$

The time rate of change of the area can be found as

$$\frac{d}{dt}\hat{\mathbf{n}}\delta\Gamma = \frac{d}{dt}\left[\tilde{J}\widetilde{\mathcal{F}}^{-T}\hat{\mathbf{N}}d\Gamma_o\right] = \left[\dot{\tilde{J}}\widetilde{\mathcal{F}}^{-T} + \tilde{J}\dot{\widetilde{\mathcal{F}}}^{-T}\right]\hat{\mathbf{N}}d\Gamma_o$$

The time rate of change of the volume can be found as

$$\frac{d}{dt}(\delta\Omega) = \frac{d}{dt}\left(\tilde{J}d\Omega_o\right) = \dot{\tilde{J}}d\Omega_o$$

and, since it will be proved in Section 6.5.2 that

$$\dot{\tilde{J}} = \tilde{J}\,\mathrm{div}\,\mathbf{v}$$

we finally can write

$$\frac{d}{dt}\delta\Omega = \dot{\tilde{J}}d\Omega_o = \tilde{J}\,\mathrm{div}\,\mathbf{v}d\Omega_o = \mathrm{div}\,\mathbf{v}d\Omega$$

6.3.4 Remarks

The relationships presented in Section 6.3.2 indicate that the Green tensor $\widetilde{\mathcal{G}}$, or alternatively the Finite Lagrange Strain tensors $\widetilde{\mathcal{L}}$, governs the relationships between deformed and undeformed geometrical quantities.

In a CaORS, the Green tensor $\widetilde{\mathcal{G}}$ is a symmetric and positive definite tensor; in fact,

$$\begin{aligned}
\widetilde{\mathcal{G}} &= \widetilde{\mathcal{F}}^T\widetilde{\mathcal{F}} = \widetilde{\mathcal{G}}^T \\
\delta\mathbf{X}^T\widetilde{\mathcal{G}}\delta\mathbf{X} &= ds^2 > 0
\end{aligned}$$

Consequently, $\tilde{\mathcal{G}}$ has always three positive principal values

$$\tilde{\mathcal{G}}_1 \geq \tilde{\mathcal{G}}_2 \geq \tilde{\mathcal{G}}_3 > 0 \tag{6.24}$$

and three orthogonal principal directions $\tilde{\mathrm{n}}^{(i)}$, Theorem 7.3.1 in [38]. It is immediately verified that the elements $\tilde{\mathcal{G}}_{ij}$ of $\tilde{\mathcal{G}}$ are proportional to the deformation process of the vectors $\delta\mathbf{X}$ oriented as the Cartesian axes. In fact, let

$$\begin{aligned} \delta\mathbf{X}^{(1)} &= \{1,0,0\}^T \\ \delta\mathbf{X}^{(2)} &= \{0,1,0\}^T \end{aligned}$$

be two unit vectors oriented as the Cartesian axis x_1 and x_2. Then, according to the relationship reported in Section 6.3.2,

$$\begin{aligned} \lambda_1^2 &= \frac{|\delta\mathbf{x}^{(1)}|^2}{|d\mathbf{X}^{(1)}|^2} = \tilde{\mathcal{G}}_{11} \\ \lambda_2^2 &= \frac{|\delta\mathbf{x}^{(2)}|^2}{|d\mathbf{X}^{(2)}|^2} = \tilde{\mathcal{G}}_{22} \\ \gamma_{12} &= \cos(\hat{\mathrm{s}}^{(1)}, \hat{\mathrm{s}}^{(2)}) = \frac{\widehat{\mathrm{S}}^{(1)T}\tilde{\mathcal{G}}\widehat{\mathrm{S}}^{(2)}}{\lambda_1\lambda_2} = \frac{\tilde{\mathcal{G}}_{12}}{(\tilde{\mathcal{G}}_{11}\tilde{\mathcal{G}}_{22})^{1/2}} \end{aligned}$$

Since

$$\tilde{\mathcal{L}} = \frac{1}{2}(\tilde{\mathcal{G}} - \mathbf{I})$$

we can establish that, Theorem 5.6.4 in [38], the Lagrange finite strain tensor $\tilde{\mathcal{L}}$ is, in general, a symmetric matrix and its three principal values $\tilde{\mathcal{L}}_i$ are related with those of $\tilde{\mathcal{G}}$ as

$$\tilde{\mathcal{L}}_i = \frac{1}{2}(\tilde{\mathcal{G}}_i - 1) \tag{6.25}$$

while its principal directions coincide with those of $\tilde{\mathcal{G}}$.

In general, the Lagrange deformation gradient $\widetilde{\mathcal{F}}$ is not a symmetric matrix. Its polar decomposition is given by, Theorem 8.4.2 in [38],

$$\widetilde{\mathcal{F}} = \widetilde{\mathcal{R}}\widetilde{\mathcal{M}} = \widetilde{\mathcal{N}}\widetilde{\mathcal{R}} \tag{6.26}$$

where:

$$\widetilde{\mathcal{M}} = \left(\widetilde{\mathcal{F}}^T\widetilde{\mathcal{F}}\right)^{\frac{1}{2}} = \tilde{\mathcal{G}}^{\frac{1}{2}}$$

$$
\begin{aligned}
\widetilde{\mathcal{N}} &= \left(\widetilde{\mathcal{F}}\widetilde{\mathcal{F}}^T\right)^{\frac{1}{2}} = \tilde{\mathcal{C}}^{\frac{1}{2}} \\
\widetilde{\mathcal{R}} &= \tilde{\mathrm{N}}\widetilde{\mathrm{M}}^T \\
\widetilde{\mathrm{M}} &= [\widetilde{\mathrm{m}}^{(1)}, \widetilde{\mathrm{m}}^{(2)}, \widetilde{\mathrm{m}}^{(3)}] \\
\tilde{\mathrm{N}} &= [\tilde{\mathrm{n}}^{(1)}, \tilde{\mathrm{n}}^{(2)}, \tilde{\mathrm{n}}^{(3)}]
\end{aligned}
$$

and $\widetilde{\mathrm{m}}^{(i)}$ are the principal directions of $\widetilde{\mathcal{G}}$ while $\tilde{\mathrm{n}}^{(i)}$ are the principal directions of $\tilde{\mathcal{C}}$. We notice that:

- the tensor $\widetilde{\mathcal{R}}$, also known as the *Lagrange rotation tensor*, is represented by an orthogonal matrix. It may be viewed as the tensor which rotates the principal axis of $\widetilde{\mathcal{G}}$ into the direction of the principal axis of $\tilde{\mathcal{C}}$.
- The tensors $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$, also known as the right and left *Lagrange stretch tensors*, respectively, are both represented by symmetric and positive definite matrices.

Hence, the operation

$$\delta \mathrm{x} = \widetilde{\mathcal{F}} d\mathrm{X} = \widetilde{\mathcal{R}}\widetilde{\mathcal{M}} d\mathrm{X}$$

may be interpreted as a stretch by the operator $\widetilde{\mathcal{M}}$ and then a rigid rotation by the operator $\widetilde{\mathcal{R}}$. Alternatively, the operation

$$\delta \mathrm{x} = \widetilde{\mathcal{F}} d\mathrm{X} = \widetilde{\mathcal{N}}\widetilde{\mathcal{R}} d\mathrm{X}$$

may be interpreted as a rigid rotation by the operator $\widetilde{\mathcal{R}}$ and then a stretch by the operator $\widetilde{\mathcal{N}}$

Finally, notice that $\tilde{J}$ may be expressed in terms of the invariants of $\widetilde{\mathcal{G}}$ and of $\tilde{\mathcal{L}}$ as

$$\tilde{J} = \det \widetilde{\mathcal{F}} = \left(I_3^{(\widetilde{\mathcal{G}})}\right)^{\frac{1}{2}} = \left(1 + 2I_1^{(\tilde{\mathcal{L}})} + 4I_2^{(\tilde{\mathcal{L}})} + 8I_3^{(\tilde{\mathcal{L}})}\right)^{\frac{1}{2}} \qquad (6.27)$$

where $I_i^{(\widetilde{\mathcal{G}})}$ are the invariants of $\widetilde{\mathcal{G}}$ and $I_i^{(\tilde{\mathcal{L}})}$ are the invariants of $\tilde{\mathcal{L}}$. In fact, we know from linear algebra theory that, Theorems 3.5.1 and 3.7.1 in [38],

$$
\begin{aligned}
\det(\mathbf{AB}) &= \det \mathbf{A} \det \mathbf{B} \\
\det \mathbf{A}^T &= \det \mathbf{A}
\end{aligned}
$$

for any square matrices $\mathbf{A}$ and $\mathbf{B}$. Accordingly,

$$\det \widetilde{\mathcal{G}} = \det(\widetilde{\mathcal{F}}^T \widetilde{\mathcal{F}}) = \det \widetilde{\mathcal{F}}^T \det \widetilde{\mathcal{F}} = (\det \widetilde{\mathcal{F}})^2 = \tilde{J}^2$$

In the principal direction of $\tilde{\mathcal{G}}$ we have that

$$\tilde{J}^2 = \det\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_1\tilde{\mathcal{G}}_2\tilde{\mathcal{G}}_3 = I_3^{(\tilde{\mathcal{G}})}$$

and

$$\begin{aligned}
\tilde{J}^2 &= \tilde{\mathcal{G}}_1\tilde{\mathcal{G}}_2\tilde{\mathcal{G}}_3 = \left[(1+2\tilde{\mathcal{L}}_1)(1+2\tilde{\mathcal{L}}_2)(1+2\tilde{\mathcal{L}}_3)\right] = \\
&= \left[(1+2\tilde{\mathcal{L}}_2+2\tilde{\mathcal{L}}_1+4\tilde{\mathcal{L}}_1\tilde{\mathcal{L}}_2)(1+2\tilde{\mathcal{L}}_3)\right] = \\
&= \left[1+2\tilde{\mathcal{L}}_3+2\tilde{\mathcal{L}}_2+4\tilde{\mathcal{L}}_2\tilde{\mathcal{L}}_3+2\tilde{\mathcal{L}}_1+4\tilde{\mathcal{L}}_1\tilde{\mathcal{L}}_3+4\tilde{\mathcal{L}}_1\tilde{\mathcal{L}}_2+8\tilde{\mathcal{L}}_1\tilde{\mathcal{L}}_2\tilde{\mathcal{L}}_3\right] = \\
&= \left[1+2\left(\tilde{\mathcal{L}}_1+\tilde{\mathcal{L}}_2+\tilde{\mathcal{L}}_3\right)+4\left(\tilde{\mathcal{L}}_1\tilde{\mathcal{L}}_2+\tilde{\mathcal{L}}_2\tilde{\mathcal{L}}_3+\tilde{\mathcal{L}}_3\tilde{\mathcal{L}}_1\right)+8\tilde{\mathcal{L}}_1\tilde{\mathcal{L}}_2\tilde{\mathcal{L}}_3\right] = \\
&= \left[1+2I_1^{(\tilde{\mathcal{L}})}+4I_2^{(\tilde{\mathcal{L}})}+8I_3^{(\tilde{\mathcal{L}})}\right]
\end{aligned}$$

6.4 Spatial Description: Eulerian Formulation

The spatial description concentrates on a region of space of the continuum $\mathcal{M}$ and the motion of any particle P is expressed by equations of the form

$$\mathbf{X} = \mathbf{X}(\mathbf{x}, t) \tag{6.28}$$

where $\mathbf{x}$ is the location of P at the current time t while $\mathbf{X}$ is the location of P at $t = 0$.

This description, where the independent variables are the current location $\mathbf{x}$ and the current time t, may be viewed as one which provides a tracing to its original location of the particle that now occupies the location $\mathbf{x}$. It is assumed that such a tracing is one-to-one and continuous, with continuous partial derivatives to whatever order is required. According to Eq. 6.28, we have that

$$d\mathbf{X} = \widehat{\mathcal{F}}d\mathbf{x} + \frac{\partial\mathbf{X}}{\partial t}dt \tag{6.29}$$

where

$$\widehat{\mathcal{F}} = \left[\frac{\partial X_i}{\partial x_j}\right]$$

is known as the *Eulerian Deformation Gradient* tensor. The first term on the right, namely

$$\delta\mathbf{X} = \widehat{\mathcal{F}}d\mathbf{x} \tag{6.30}$$

traces back the current relative distance $d\mathbf{x}$ into the original one $\delta\mathbf{X}$ in the undeformed configuration. The condition of one-to-one mapping requires the invertibility of $\widehat{\mathcal{F}}$ so that the correspondence of any $\delta\mathbf{X}$ into the current configuration is unique, namely:

$$d\mathbf{x} = \widehat{\mathcal{F}}^{-1}\delta\mathbf{X} \tag{6.31}$$

A necessary and sufficient condition for the inverse of $\widehat{\mathcal{F}}$ to exist is that the determinant

$$\hat{J} = \det \widehat{\mathcal{F}} \tag{6.32}$$

should not vanish for any material point in the current configuration. Then, for each material point in $\mathcal{M}$, we have that, Section 6.2:

- The displacement of the particle is given by

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \mathbf{X}(\mathbf{x}, t) \tag{6.33}$$

- The relative displacement vector of the particle P with respect to a neighboring point is given by

$$d\mathbf{u}(\mathbf{x}, t) = \hat{\mathcal{U}} d\mathbf{x} + \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} dt \tag{6.34}$$

  where

$$\hat{\mathcal{U}} = \left[\frac{\partial u_i}{\partial x_j}\right]$$

  is known as the *Euler Displacement Gradient* tensor.

- The velocity of the particle at P is given by

$$\mathbf{v}(\mathbf{x}, t) = \hat{\mathcal{U}}\mathbf{v}(\mathbf{x}, t) + \frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t} \tag{6.35}$$

- The acceleration of the particle at P is given by

$$\mathbf{a}(\mathbf{x}, t) = \hat{\mathcal{V}}\mathbf{v}(\mathbf{x}, t) + \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} \tag{6.36}$$

  where

$$\hat{\mathcal{V}} = \left[\frac{\partial v_i}{\partial x_j}\right]$$

  is known as the *Euler Velocity Gradient*.

The derivations of the above formulas are postponed to Section 6.4.1. However, we notice that the velocity and the acceleration expressions consist of two terms:

- The second terms $\partial \mathbf{u}/\partial t$ and $\partial \mathbf{v}/\partial t$ give the rate of change at a particular location and, accordingly, they are called *local rate of change.*
- The first terms $\hat{\mathcal{U}}\mathbf{v}$ and $\hat{\mathcal{V}}\mathbf{v}$ are called *convective rate of change* since they express the contribution due to the motion of the particle in the variable field.

6.4.1 Material derivative operator

Let

$$f = f(\mathbf{x}, t) \tag{6.37}$$

be any continuous function of the independent variables $\mathbf{x}$ and t. Then, it is immediate to verify that

$$\begin{aligned} df &= f_{,i}dx_i + f_{,t}dt & (6.38) \\ \frac{df}{dt} &= f_{,i}v_i + f_{,t} & (6.39) \end{aligned}$$

where

$$\begin{aligned} f_{,i} &= \frac{\partial f}{\partial x_i} \\ f_{,t} &= \frac{\partial f}{\partial t} \\ v_i &= x_{i,t} = \frac{\partial x_i}{\partial t} \end{aligned}$$

Moreover,

$$\begin{aligned} \left(\frac{df}{dt}\right)_{,j} &= f_{,ij}v_i + f_{,i}v_{i,j} + f_{,tj} & (6.40) \\ d(f_{,j}) &= f_{,ji}dx_i + f_{,jt}dt & (6.41) \\ \frac{d}{dt}(f_{,j}) &= f_{,ji}v_i + f_{,jt} = & (6.42) \\ &= \left(\frac{df}{dt}\right)_{,j} - f_{,i}v_{i,j} & (6.43) \end{aligned}$$

where

$$f_{,ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$
$$f_{,it} = \frac{\partial^2 f}{\partial x_i \partial t}$$
$$v_{i,j} = \frac{\partial v_i}{\partial x_j}$$

The differential forms in Eqs. 6.39 and 6.42 suggest the introduction of the *material derivative operator*

$$\frac{d}{dt} = \frac{\partial}{\partial x_i} v_i + \frac{\partial}{\partial t} \tag{6.44}$$

The application of this operator to the displacement function $\mathbf{u}$ yields to the velocity expression in Eq. 6.35; then, the application of this operator to the velocity function yields to the acceleration expression in Eq. 6.36.

6.4.2 Eulerian Tensors

The spatial formulation leads to define the following tensors:

- The *Euler Displacement Gradient* tensor defined as

$$\hat{\mathcal{U}} = [u_{i,j}] = \left[\frac{\partial u_i}{\partial x_j}\right]$$

- The *Euler Deformation Gradient* tensor defined as

$$\widehat{\mathcal{F}} = [X_{i,j}] = \left[\frac{\partial X_i}{\partial x_j}\right]$$

  which can be alternatively expressed as

$$\widehat{\mathcal{F}} = (\mathbf{I} - \hat{\mathcal{U}}) = [\delta_{ij} - u_{i,j}]$$

- The *Inverse Cauchy Strain* tensor defined as

$$\hat{\mathcal{C}} = \widehat{\mathcal{F}}^T \widehat{\mathcal{F}} = [X_{k,i} X_{k,j}]$$

  which can be alternatively expressed as

$$\hat{\mathcal{C}} = \mathbf{I} - (\hat{\mathcal{U}} + \hat{\mathcal{U}}^T) + \hat{\mathcal{U}}^T \hat{\mathcal{U}} = [\delta_{ij} - u_{i,j} - u_{j,i} + u_{k,i} u_{k,j}]$$

- The *Finite Euler Strain* tensor defined as

$$\hat{\mathcal{E}} = \frac{1}{2}(\mathrm{I} - \hat{\mathcal{C}}) = \frac{1}{2}\left(\delta_{ij} - X_{k,i}X_{k,j}\right)$$

which can be alternatively expressed as

$$\hat{\mathcal{E}} = \frac{1}{2}\left(\hat{\mathcal{U}} + \hat{\mathcal{U}}^T - \hat{\mathcal{U}}^T\hat{\mathcal{U}}\right) = \frac{1}{2}\left[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}\right]$$

- The *Euler Velocity Gradient* tensor defined as

$$\hat{\mathcal{V}} = [v_{i,j}] = \left[\frac{\partial v_i}{\partial x_j}\right] = \left[\left(\frac{\partial u_i}{\partial t}\right)_{,j}\right]$$

which can be alternatively expressed as, Eq. 6.43,

$$\hat{\mathcal{V}} = \dot{\hat{\mathcal{U}}} + \hat{\mathcal{U}}\hat{\mathcal{V}} = \left[\frac{d}{dt}(u_{i,j}) + u_{i,k}v_{k,j}\right]$$

- The *Euler Strain Rate or (stretching)* tensor defined as

$$\widehat{\mathcal{D}} = \frac{1}{2}(\hat{\mathcal{V}} + \hat{\mathcal{V}}^T) = \frac{1}{2}\left[v_{i,j} + v_{j,i}\right]$$

which can be alternatively expressed as

$$\begin{aligned}\widehat{\mathcal{D}} &= \frac{1}{2}(\dot{\hat{\mathcal{U}}} + \dot{\hat{\mathcal{U}}}^T) + \frac{1}{2}(\hat{\mathcal{U}}\hat{\mathcal{V}} + \hat{\mathcal{V}}^T\hat{\mathcal{U}}^T) = \\ &= \frac{1}{2}\frac{d}{dt}\left[u_{i,j} + u_{j,i}\right] + \frac{1}{2}\left[u_{i,k}v_{k,j} + u_{j,k}v_{k,i}\right]\end{aligned}$$

- The *Euler Vorticity (or spin)* tensor defined as

$$\widehat{\mathcal{W}} = \frac{1}{2}(\hat{\mathcal{V}} - \hat{\mathcal{V}}^T) = \frac{1}{2}\left[v_{i,j} - v_{j,i}\right]$$

which can be alternatively expressed as

$$\begin{aligned}\widehat{\mathcal{W}} &= \frac{1}{2}(\dot{\hat{\mathcal{U}}} - \dot{\hat{\mathcal{U}}}^T) + \frac{1}{2}(\hat{\mathcal{U}}\hat{\mathcal{V}} - \hat{\mathcal{V}}^T\hat{\mathcal{U}}^T) = \\ &= \frac{1}{2}\frac{d}{dt}\left[u_{i,j} - u_{j,i}\right] + \frac{1}{2}\left[u_{i,k}v_{k,j} - u_{j,k}v_{k,i}\right]\end{aligned}$$

The time rate of the above tensors can be expressed as

$$\begin{aligned}
\dot{\hat{\mathcal{U}}} &= (\mathbf{I}-\hat{\mathcal{U}})\hat{\mathcal{V}} = \widehat{\mathcal{F}}\hat{\mathcal{V}} \\
\dot{\widehat{\mathcal{F}}} &= -\dot{\hat{\mathcal{U}}} \\
\dot{\hat{\mathcal{C}}} &= \dot{\widehat{\mathcal{F}}}^T\widehat{\mathcal{F}} + \widehat{\mathcal{F}}^T\dot{\widehat{\mathcal{F}}} = -(\hat{\mathcal{V}}^T\hat{\mathcal{C}} + \hat{\mathcal{C}}\hat{\mathcal{V}}) \\
\dot{\hat{\mathcal{E}}} &= -\frac{\dot{\hat{\mathcal{C}}}}{2} = \left[\widehat{\mathcal{D}} - \left(\hat{\mathcal{V}}^T\hat{\mathcal{E}} + \hat{\mathcal{E}}\hat{\mathcal{V}}\right)\right]
\end{aligned}$$

All the above alternative expressions are obtained taking into account that according to Eq. 6.33

$$X_{i,j} = [x_i - u_i]_{,j} = x_{i,j} - u_{i,j} = \delta_{ij} - u_{i,j}$$

The rate of change expression of the Euler displacement gradient $\dot{\hat{\mathcal{U}}}$ follows immediately from Eq. 6.43. Then,

$$\begin{aligned}
\dot{\hat{\mathcal{U}}} &= \hat{\mathcal{V}} - \hat{\mathcal{U}}\hat{\mathcal{V}} = \left(\mathbf{I}-\hat{\mathcal{U}}\right)\hat{\mathcal{V}} = \widehat{\mathcal{F}}\hat{\mathcal{V}} \\
\dot{\widehat{\mathcal{F}}} &= \frac{d}{dt}(\mathbf{I}-\hat{\mathcal{U}}) = -\dot{\hat{\mathcal{U}}} = -\widehat{\mathcal{F}}\hat{\mathcal{V}} \\
\dot{\hat{\mathcal{C}}} &= \frac{d}{dt}\left(\widehat{\mathcal{F}}^T\widehat{\mathcal{F}}\right) = \dot{\widehat{\mathcal{F}}}^T\widehat{\mathcal{F}} + \widehat{\mathcal{F}}^T\dot{\widehat{\mathcal{F}}} = -\left[\left(\widehat{\mathcal{F}}\hat{\mathcal{V}}\right)^T\widehat{\mathcal{F}} + \widehat{\mathcal{F}}^T\widehat{\mathcal{F}}\hat{\mathcal{V}}\right] = \\
&= -\left[\hat{\mathcal{V}}^T\widehat{\mathcal{F}}^T\widehat{\mathcal{F}} + \widehat{\mathcal{F}}^T\widehat{\mathcal{F}}\hat{\mathcal{V}}\right] = -\left[\hat{\mathcal{V}}^T\hat{\mathcal{C}} + \hat{\mathcal{C}}\hat{\mathcal{V}}\right] \\
\dot{\hat{\mathcal{E}}} &= \frac{d}{dt}\left[\frac{1}{2}(\mathbf{I}-\mathcal{C})\right] = -\frac{\dot{\hat{\mathcal{C}}}}{2}
\end{aligned}$$

In order to derive the alternative expression for $\hat{\mathcal{E}}$, we have to preliminarily observe that, according to Eq. 6.12,

$$\begin{aligned}
\frac{d}{dt}(\delta x_i) &= \frac{d}{dt}\left(\frac{\partial x_i}{\partial X_j}dX_j\right) = \frac{\partial v_i}{\partial X_j}dX_j = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}dX_j = \\
&= \frac{\partial v_i}{\partial x_k}\delta x_k = \hat{\mathcal{V}}_{ik}\delta x_k
\end{aligned}$$

that is

$$\frac{d}{dt}(d\mathbf{x}) = \hat{\mathcal{V}}d\mathbf{x} \tag{6.45}$$

Consequently,

$$\begin{aligned}
\frac{d}{dt}\left(d\mathbf{x}^{(1)T}d\mathbf{x}^{(2)}\right) &= \left(\frac{d}{dt}d\mathbf{x}^{(1)}\right)^T d\mathbf{x}^{(2)} + \left(d\mathbf{x}^{(1)T}\frac{d}{dt}d\mathbf{x}^{(2)}\right) = \\
&= \left(\widehat{\mathcal{V}}d\mathbf{x}^{(1)}\right)^T d\mathbf{x}^{(2)} + d\mathbf{x}^{(1)T}\widehat{\mathcal{V}}d\mathbf{x}^{(2)} = \\
&= d\mathbf{x}^{(1)T}\widehat{\mathcal{V}}^T d\mathbf{x}^{(2)} + d\mathbf{x}^{(1)T}\widehat{\mathcal{V}}d\mathbf{x}^{(2)} = \\
&= d\mathbf{x}^{(1)T}\left(\widehat{\mathcal{V}}^T + \widehat{\mathcal{V}}\right)d\mathbf{x}^{(2)}
\end{aligned}$$

Since $\widehat{\mathcal{V}}$ can be decomposed into, Eq. 3.3,

$$\widehat{\mathcal{V}} = \widehat{\mathcal{D}} + \widehat{\mathcal{W}}$$

where $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{W}}$ are a symmetric and a skewsymmetric, respectively, we have that

$$\begin{aligned}
\frac{d}{dt}\left(d\mathbf{x}^{(1)T}d\mathbf{x}^{(2)}\right) &= d\mathbf{x}^{(1)T}\left(\widehat{\mathcal{V}}^T + \widehat{\mathcal{V}}\right)d\mathbf{x}^{(2)} = \\
&= d\mathbf{x}^{(1)T}\left(\widehat{\mathcal{D}}^T + \widehat{\mathcal{W}}^T + \widehat{\mathcal{D}} + \widehat{\mathcal{W}}\right)d\mathbf{x}^{(2)}
\end{aligned}$$

and, being $\widehat{\mathcal{D}} = \widehat{\mathcal{D}}^T$ and $\widehat{\mathcal{W}} = -\widehat{\mathcal{W}}^T$, we can finally establish that

$$\frac{d}{dt}\left(d\mathbf{x}^{(1)T}d\mathbf{x}^{(2)}\right) = 2d\mathbf{x}^{(1)T}\widehat{\mathcal{D}}d\mathbf{x}^{(2)} \tag{6.46}$$

In particular, from this relationship we can write that

$$\frac{d}{dt}(ds^2) = \frac{d}{dt}\left(d\mathbf{x}^T d\mathbf{x}\right) = 2d\mathbf{x}^T\widehat{\mathcal{D}}d\mathbf{x} \tag{6.47}$$

On the other hand, we will prove in Section 6.4.3 that

$$ds^2 - \delta S^2 = 2d\mathbf{x}^T\mathcal{E}d\mathbf{x}$$

from which it follows that

$$\begin{aligned}
\frac{d}{dt}\left(ds^2\right) &= \frac{d}{dt}\left(2d\mathbf{x}^T\widehat{\mathcal{E}}d\mathbf{x}\right) \\
&= 2\left[\left(\frac{d}{dt}d\mathbf{x}\right)^T\widehat{\mathcal{E}}d\mathbf{x} + d\mathbf{x}^T\dot{\widehat{\mathcal{E}}}d\mathbf{x} + d\mathbf{x}^T\widehat{\mathcal{E}}\frac{d}{dt}d\mathbf{x}\right] = \\
&= 2\left[\left(\widehat{\mathcal{V}}d\mathbf{x}\right)^T\widehat{\mathcal{E}}d\mathbf{x} + d\mathbf{x}^T\dot{\widehat{\mathcal{E}}}d\mathbf{x} + d\mathbf{x}^T\widehat{\mathcal{E}}\widehat{\mathcal{V}}d\mathbf{x}\right] =
\end{aligned}$$

that is

$$\frac{d}{dt}\left(ds^2\right) = 2d\mathrm{x}^T\left[\hat{\mathcal{V}}^T\hat{\mathcal{E}} + \dot{\hat{\mathcal{E}}} + \hat{\mathcal{E}}\hat{\mathcal{V}}\right]d\mathrm{x} \tag{6.48}$$

Hence, equalizing the above two alternative expressions for ds^2/dt, we obtain

$$d\mathrm{x}^T\left[\widehat{\mathcal{D}} - \left(\hat{\mathcal{V}}^T\hat{\mathcal{E}} + \dot{\hat{\mathcal{E}}} + \hat{\mathcal{E}}\hat{\mathcal{V}}\right)\right]d\mathrm{x} = 0$$

for all $d\mathrm{x}$, from which we can establish that

$$\widehat{\mathcal{D}} = \left(\hat{\mathcal{V}}^T\hat{\mathcal{E}} + \dot{\hat{\mathcal{E}}} + \hat{\mathcal{E}}\hat{\mathcal{V}}\right)$$

or, alternatively,

$$\dot{\hat{\mathcal{E}}} = \widehat{\mathcal{D}} - \left(\hat{\mathcal{V}}^T\hat{\mathcal{E}} + \hat{\mathcal{E}}\hat{\mathcal{V}}\right)$$

6.4.3 Measure of Deformation

Let $\mathcal{M}$ be a continuum and P, Q and R be three neighboring material points in $\mathcal{M}$, Fig. 6.4. Let us indicate the distance between these points as:

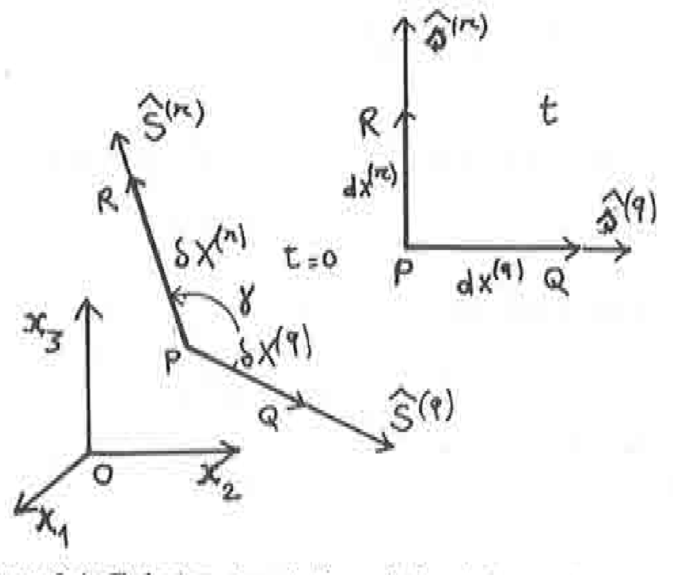


Figure 6.4: Eulerian description of the deformation process

- $\delta\mathbf{X} \equiv \delta\mathbf{X}^{(q)}$ and $d\mathbf{x} \equiv d\mathbf{x}^{(q)}$ the relative distance between P and Q at time $t = 0$ and t, respectively.
- $\delta\mathbf{X}^{(r)}$ and $d\mathbf{x}^{(r)}$ the relative distance between P and R at time $t = 0$ and t, respectively.
- and, for $i = q, r$,

$$\delta\widehat{\mathbf{S}}^{(i)} = \frac{\delta\mathbf{X}^{(i)}}{|\delta\mathbf{X}^{(i)}|}; \qquad d\widehat{\mathbf{s}}^{(i)} = \frac{d\mathbf{x}^{(i)}}{|d\mathbf{x}^{(i)}|}$$

  the unit vectors in the direction of $\delta\mathbf{X}^{(i)}$ and $d\mathbf{x}^{(i)}$, respectively.

Then, according to the Eulerian formulation, we can measure their deformation process as follows:

- The *Undeformed Length* defined as

$$\delta S^2 = \delta\mathbf{X}^T\delta\mathbf{X}$$

  can be calculated as

$$\delta S^2 = d\mathbf{x}^T\widehat{\mathcal{C}}d\mathbf{x}$$

  In fact,

$$\delta S^2 = \delta\mathbf{X}^T\delta\mathbf{X} = d\mathbf{x}^T\widehat{\mathcal{F}}^T\widehat{\mathcal{F}}d\mathbf{x} = d\mathbf{x}^T\widehat{\mathcal{C}}d\mathbf{x}$$

- The *Deformed Length* defined as

$$ds^2 = d\mathbf{x}^Td\mathbf{x}$$

  and its rate of deformation can be calculated as, Eq. 6.47,

$$\frac{d}{dt}(ds^2) = 2d\mathbf{x}^T\widehat{\mathcal{D}}d\mathbf{x}$$

- The *Stretch* defined as

$$\lambda^2 = \frac{ds^2}{\delta S^2} = \frac{d\mathbf{x}^Td\mathbf{x}}{\delta\mathbf{X}^T\delta\mathbf{X}}$$

  can be calculated as

$$\lambda^2 = (\widehat{\mathbf{s}}^T\widehat{\mathcal{C}}\widehat{\mathbf{s}})^{-1}$$

  In fact,

$$\lambda^2 = \frac{ds^2}{\delta S^2} = \frac{d\mathbf{x}^Td\mathbf{x}}{d\mathbf{X}^T\delta\mathbf{X}} = \frac{ds^2}{d\mathbf{x}^T\widehat{\mathcal{C}}d\mathbf{x}} = \frac{1}{\widehat{\mathbf{s}}^T\widehat{\mathcal{C}}\widehat{\mathbf{s}}} = \left(\widehat{\mathbf{s}}^T\widehat{\mathcal{C}}\widehat{\mathbf{s}}\right)^{-1}$$

- The *Measure of Deformation* defined as

$$\chi = ds^2 - dS^2$$

can be calculated as

$$\chi = 2d\mathbf{x}^T \widehat{\mathcal{E}} d\mathbf{x}$$

In fact,

$$\begin{aligned} ds^2 - \delta S^2 &= d\mathbf{x}^T d\mathbf{x} - \delta\mathbf{X}^T \delta\mathbf{X} = d\mathbf{x}^T \mathbf{I} d\mathbf{x} - d\mathbf{x}^T \widehat{\mathcal{C}} d\mathbf{x} = \\ &= d\mathbf{x}^T \left(\mathbf{I} - \widehat{\mathcal{C}}\right) d\mathbf{x} = 2d\mathbf{x}^T \widehat{\mathcal{E}} d\mathbf{x} \end{aligned}$$

- The *Unit Extension* defined as

$$\epsilon = \frac{ds - dS}{dS} = \lambda - 1$$

can be calculated as

$$\epsilon = \left(\widehat{\mathbf{s}}^T\, \widehat{\mathcal{C}}\, \widehat{\mathbf{s}}\right)^{-1/2} - 1$$

In fact,

$$\epsilon = \lambda - 1 = \left(\widehat{\mathbf{s}}^T \widehat{\mathcal{C}} \widehat{\mathbf{s}}\right)^{-1/2} - 1$$

- The *Angle Change* defined as, Fig. 6.4,

$$\gamma = \cos(\widehat{\mathbf{S}}^{(q)}, \widehat{\mathbf{S}}^{(r)})$$

can be calculated as

$$\gamma = \lambda_q \lambda_r \left(\widehat{\mathbf{s}}^{(q)T} \widehat{\mathcal{C}} \widehat{\mathbf{s}}^{(r)}\right) = \lambda_q \lambda_r \widehat{\mathcal{C}}_{qr}$$

In fact,

$$\begin{aligned} \gamma &= cos\left(\widehat{\mathbf{S}}^{(q)}, \widehat{\mathbf{S}}^{(r)}\right) = \widehat{\mathbf{S}}^{(q)T} \widehat{\mathbf{S}}^{(r)} = \left(\frac{\delta\mathbf{X}^{(q)}}{\delta S^{(q)}}\right)^T \frac{\delta\mathbf{X}^{(r)}}{\delta S^{(r)}} = \\ &= \frac{d\mathbf{x}^{(q)T} \widehat{\mathcal{F}}^T \widehat{\mathcal{F}} d\mathbf{x}^{(r)}}{\delta S^{(q)} \delta S^{(r)}} = \frac{ds^{(q)} \widehat{\mathbf{s}}^{(q)T} \widehat{\mathcal{C}} ds^{(r)} \widehat{\mathbf{s}}^{(r)}}{\delta S^{(q)} \delta S^{(r)}} = \\ &= \lambda_q \lambda_r \widehat{\mathbf{s}}^{(q)T} \widehat{\mathcal{C}} \widehat{\mathbf{s}}^{(r)} \end{aligned}$$

One way of deriving the $d(ds^2)/dt$ expression is that reported in the previous Section, Eq. 6.47. Alternatively, we may derive it by transforming the expression relative to the Lagrangian formulation. In fact, according to Eq. 6.12,

$$\delta \mathbf{x} = \widetilde{\mathcal{F}} d\mathbf{X}$$

and it will be proved in the next Section 6.5 that the rate of change of the Green tensor can be expressed as

$$\dot{\widetilde{\mathcal{G}}} = 2\widetilde{\mathcal{F}}^T \widetilde{\mathcal{D}} \widetilde{\mathcal{F}}$$

Consequently, the Lagrangian expression for $d(ds^2)/dt$, Section 6.3.2, can be transformed as

$$\begin{aligned} \frac{d}{dt}(\delta s^2) &= d\mathbf{X}^T \dot{\widetilde{\mathcal{G}}} d\mathbf{X} = \left(\widetilde{\mathcal{F}}^{-1} \delta \mathbf{x}\right)^T \dot{\widetilde{\mathcal{G}}} \left(\widetilde{\mathcal{F}}^{-1} \delta \mathbf{x}\right) = \\ &= \delta \mathbf{x}^T \left(\widetilde{\mathcal{F}}^{-T} \dot{\widetilde{\mathcal{G}}} \widetilde{\mathcal{F}}^{-1}\right) \delta \mathbf{x} = 2\delta \mathbf{x}^T \widetilde{\mathcal{D}} \delta \mathbf{x} \end{aligned}$$

that is

$$\frac{d}{dt}(ds^2) = \delta \mathbf{X}^T \dot{\widetilde{\mathcal{G}}} \delta \mathbf{X} = 2 d\mathbf{x}^T \widetilde{\mathcal{D}} d\mathbf{x}$$

Similarly, we can determine the other time rate expressions.

6.4.4 Surface and volume

Let $d\mathcal{M}_o$ be an infinitesimal portion of material at time $t = 0$ defined by the three differential vectors $\delta \mathbf{X}^{(1)}, \delta \mathbf{X}^{(2)}$ and $\delta \mathbf{X}^{(3)}$, Fig. 6.5. According to Eqs. 2.24 and 2.25, the surface defined by the vectors $(\delta \mathbf{X}^{(1)}, \delta \mathbf{X}^{(2)})$ and the volume of $d\mathcal{M}_o$ can be calculated as

$$\begin{aligned} \hat{\mathbf{N}} \delta \Gamma_o &= \delta \mathbf{X}^{(1)} \times \delta \mathbf{X}^{(2)} \\ \delta \Omega_o &= \left(\delta \mathbf{X}^{(1)} \times \delta \mathbf{X}^{(2)}\right)^T \delta \mathbf{X}^{(3)} \end{aligned}$$

Assume that at the current time t this portion of material reaches the configuration $d\mathcal{M}$ described by the three differential vectors $d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}$ and $d\mathbf{x}^{(3)}$. The current value of the surface and the volume above defined is then given by

$$\begin{aligned} \hat{\mathbf{n}} \delta \Gamma &= d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)} \\ \delta \Omega &= \left(d\mathbf{x}^{(1)} \times d\mathbf{x}^{(2)}\right)^T d\mathbf{x}^{(3)} \end{aligned}$$

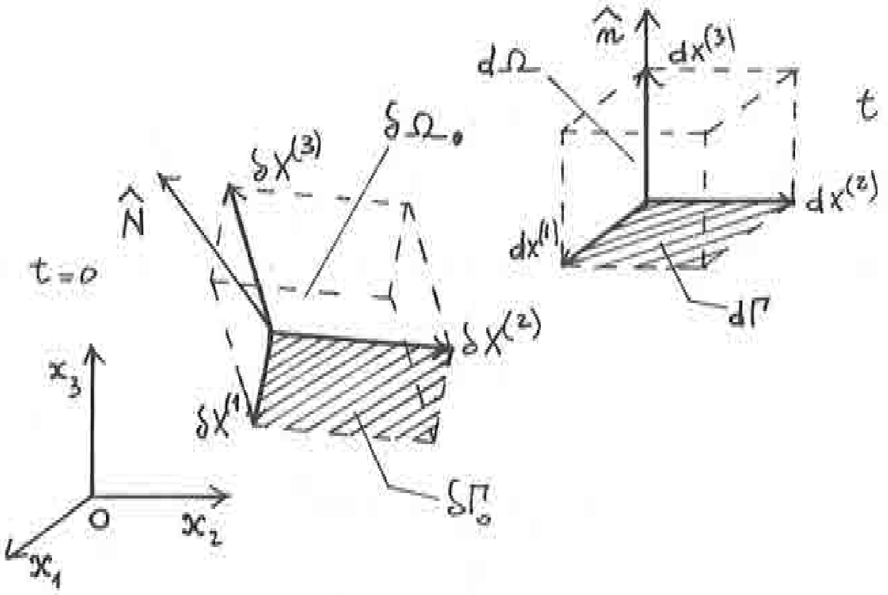


Figure 6.5: Deformation process of an infinitesimal portion of material

According to the Eulerian formulation, these surfaces and volumes are related as

$$\hat{\mathbf{N}}\delta\Gamma_o = \hat{J}\hat{\mathcal{F}}^{-T}\hat{\mathbf{n}}d\Gamma \tag{6.49}$$

$$\delta\Omega_o = \hat{J}d\Omega \tag{6.50}$$

In order to prove these statements, we preliminarily observe that according to Eq. 2.10,

$$\epsilon_{ijk}\frac{\partial X_i}{\partial x_r}\frac{\partial X_j}{\partial x_s}\frac{\partial X_k}{\partial x_t} = \epsilon_{rst}\det\left[\frac{\partial X_m}{\partial x_n}\right] = \epsilon_{rst}\hat{J} \tag{6.51}$$

Then, writing in extended form the area expression, we obtain

$$\begin{aligned}\hat{N}_i\delta\Gamma_o &= \epsilon_{ijk}\delta X_j^{(1)}\delta X_k^{(2)} = \epsilon_{ijk}\frac{\partial X_j}{\partial x_s}dx_s^{(1)}\frac{\partial X_k}{\partial x_t}dx_t^{(2)} = \\ &= \epsilon_{ijk}\frac{\partial X_j}{\partial x_s}\frac{\partial X_k}{\partial x_t}dx_s^{(1)}dx_t^{(2)}\end{aligned}$$

from which, premultiplying both sides by $X_{i,r}$ and taking into account Eq. 6.51,

$$\frac{\partial X_i}{\partial x_r}\hat{N}_i\delta\Gamma_o = \epsilon_{ijk}\frac{\partial X_i}{\partial x_r}\frac{\partial X_j}{\partial x_s}\frac{\partial X_k}{\partial x_t}dx_s^{(1)}dx_t^{(2)} = \epsilon_{rst}\hat{J}dx_s^{(1)}dx_t^{(2)}$$

$$= \hat{J}\hat{n}_r d\Gamma$$

Hence,

$$\begin{aligned}
\hat{\mathbf{N}}^T \hat{\mathcal{F}} \delta\Gamma_o &= \hat{J}\hat{\mathbf{n}}^T d\Gamma \\
\hat{\mathbf{N}}^T \delta\Gamma_o &= \hat{J}\hat{\mathbf{n}}^T \hat{\mathcal{F}}^{-1} d\Gamma \\
\hat{\mathbf{N}} \delta\Gamma_o &= \hat{J}\hat{\mathcal{F}}^{-T} \hat{\mathbf{n}} d\Gamma
\end{aligned}$$

Writing in extended form the volume expression and taking into account Eq. 6.51, we obtain

$$\begin{aligned}
\delta\Omega_o &= \left(\delta\mathbf{X}^{(1)} \times \delta\mathbf{X}^{(2)}\right)^T \delta\mathbf{X}^{(3)} = \epsilon_{ijk}\delta X_i^{(1)}\delta X_j^{(2)}\delta X_k^{(3)} = \\
&= \epsilon_{ijk}\frac{\partial X_i}{\partial x_r}dx_r^{(1)}\frac{\partial X_j}{\partial x_s}dx_s^{(2)}\frac{\partial X_k}{\partial x_t}dx_t^{(3)} = \\
&= \epsilon_{ijk}\frac{\partial X_i}{\partial x_r}\frac{\partial X_j}{\partial x_s}\frac{\partial X_k}{\partial x_t}dx_r^{(1)}dx_s^{(2)}dx_t^{(3)} = \\
&= \epsilon_{rst}\hat{J}dx_r^{(1)}dx_s^{(2)}dx_t^{(3)} = \\
&= \hat{J}d\Omega
\end{aligned}$$

6.4.5 Remarks

The relationships presented in Section 6.4.3 indicate that the (inverse) Cauchy tensor $\hat{\mathcal{C}}$, or alternatively the Finite Euler Strain tensors $\hat{\mathcal{E}}$, governs the relationships between deformed and undeformed geometrical quantities.

In a CaORS, the (inverse) Cauchy tensor $\hat{\mathcal{C}}$ is a symmetric and positive definite tensor; in fact

$$\begin{aligned}
\hat{\mathcal{C}} &= \hat{\mathcal{F}}^T\hat{\mathcal{F}} = \hat{\mathcal{C}}^T \\
d\mathbf{x}^T\hat{\mathcal{C}}d\mathbf{x} &= dS^2 > 0
\end{aligned}$$

Consequently, $\hat{\mathcal{C}}$ has always three positive principal values

$$\hat{C}_1 \geq \hat{C}_2 \geq \hat{C}_3 > 0 \tag{6.52}$$

and three orthogonal principal directions $\hat{\mathbf{n}}^{(i)}$, Theorem 7.3.1 in [38]. It is immediate to verify that the elements $\hat{C}_{ij}$ of $\hat{\mathcal{C}}$ are proportional to the deformation process of the vectors $d\mathbf{x}$ oriented as the Cartesian axes. In fact, let

$$\begin{aligned}
d\mathbf{x}^{(1)} &= \{1,0,0\}^T \\
d\mathbf{x}^{(2)} &= \{0,1,0\}^T
\end{aligned}$$

be two unit vectors oriented as the Cartesian axis x_1 and x_2. Then, according to the relationship reported in Section 6.4.3,

$$\begin{aligned}
\lambda_1^2 &= \frac{|d\mathbf{x}^{(1)}|^2}{|\delta\mathbf{X}^{(1)}|^2} = \frac{1}{\hat{C}_{11}} \\
\lambda_2^2 &= \frac{|d\mathbf{x}^{(2)}|^2}{|\delta\mathbf{X}^{(2)}|^2} = \frac{1}{\hat{C}_{22}} \\
\gamma_{12} &= \cos(\hat{\mathrm{S}}^{(1)}, \hat{\mathrm{S}}^{(2)}) = \lambda_1\lambda_2 \hat{\mathrm{s}}^{(1)T}\hat{\mathcal{C}}\hat{\mathrm{s}}^{(2)} = \frac{\hat{C}_{12}}{(\hat{C}_{11}\hat{C}_{22})^{1/2}}
\end{aligned}$$

Since

$$\hat{\mathcal{E}} = \frac{1}{2}(\mathbf{I} - \hat{\mathcal{C}})$$

we can establish that, Theorem 5.6.4 in [38], the Euler finite strain tensor $\hat{\mathcal{E}}$ is, in general, a symmetric matrix and its three principal values $\hat{\mathcal{E}}_i$ are related with those of $\hat{\mathcal{C}}$ as

$$\hat{\mathcal{E}}_i = \frac{1}{2}(1 - \hat{C}_i) \tag{6.53}$$

while its principal directions coincide with those of $\hat{\mathcal{C}}$.

In general, the Eulerian deformation gradient $\widehat{\mathcal{F}}$ is not a symmetric matrix. Its polar decomposition is given by, Theorem 8.4.2 in [38],

$$\widehat{\mathcal{F}} = \widehat{\mathcal{R}}\widehat{\mathcal{M}} = \widehat{\mathcal{N}}\widehat{\mathcal{R}} \tag{6.54}$$

where:

$$\begin{aligned}
\widehat{\mathcal{M}} &= \left(\widehat{\mathcal{F}}^T\widehat{\mathcal{F}}\right)^{\frac{1}{2}} = \hat{\mathcal{C}}^{\frac{1}{2}} \\
\widehat{\mathcal{N}} &= \left(\widehat{\mathcal{F}}\widehat{\mathcal{F}}^T\right)^{\frac{1}{2}} = \hat{\mathcal{G}}^{\frac{1}{2}} \\
\widehat{\mathcal{R}} &= \hat{\mathrm{N}}\widehat{\mathrm{M}}^T \\
\widehat{\mathrm{M}} &= [\widehat{\mathbf{m}}^{(1)}, \widehat{\mathbf{m}}^{(2)}, \widehat{\mathbf{m}}^{(3)}] \\
\hat{\mathrm{N}} &= [\hat{\mathbf{n}}^{(1)}, \hat{\mathbf{n}}^{(2)}, \hat{\mathbf{n}}^{(3)}]
\end{aligned}$$

and $\widehat{\mathbf{m}}^{(i)}$ are the principal directions of $\hat{\mathcal{C}}$ while $\hat{\mathbf{n}}^{(i)}$ are the principal directions of $\hat{\mathcal{G}}$. We notice that:

- the tensor $\widehat{\mathcal{R}}$, also known as the *Eulerian rotation tensor*, is represented by an orthogonal matrix. It may be viewed as the tensor which rotates the principal axes of $\hat{\mathcal{C}}$ into the direction of the principal axes of $\hat{\mathcal{G}}$.

- The tensors $\widehat{\mathcal{M}}$ and $\widehat{\mathcal{N}}$, also known as the right and left *Eulerian stretch tensors*, respectively, are both represented by symmetric and positive definite matrices.

Hence, the operation

$$\delta \mathbf{X} = \widehat{\mathcal{F}} d\mathbf{x} = \widehat{\mathcal{R}}\widehat{\mathcal{M}} d\mathbf{x}$$

may be interpreted as a stretch by the operator $\widehat{\mathcal{M}}$ and then a rigid rotation by the operator $\widehat{\mathcal{R}}$. Alternatively, the operation

$$\delta \mathbf{X} = \widehat{\mathcal{F}} d\mathbf{x} = \widehat{\mathcal{N}}\widehat{\mathcal{R}} d\mathbf{x}$$

may be interpreted as a rigid rotation by the operator $\widehat{\mathcal{R}}$ and then a stretch by the operator $\widehat{\mathcal{N}}$

Finally, notice that the $\widehat{J}$ we be expressed in terms the $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{E}}$ tensor invariants as

$$\widehat{J} = \det \widehat{\mathcal{F}} = \left(I_3^{(\widehat{\mathcal{C}})}\right)^{\frac{1}{2}} = \left(1 - 2I_1^{(\widehat{\mathcal{E}})} + 4I_2^{(\widehat{\mathcal{E}})} - 8I_3^{(\widehat{\mathcal{E}})}\right)^{\frac{1}{2}} \qquad (6.55)$$

where $I_i^{(\widehat{\mathcal{C}})}$ are the invariants of $\widehat{\mathcal{C}}$ and $I_i^{(\widehat{\mathcal{E}})}$ are the invariants of $\widehat{\mathcal{E}}$. The proof is analogous to that presented in Section 6.3.4 for proving Eq. 6.27.

6.4.6 The velocity field

The (current) relative velocity of two neighboring points at a distance $d\mathbf{x}$ is given by

$$\delta v_i = \frac{\partial v_i}{\partial x_j} dx_j$$

that is

$$\delta \mathbf{v} = \widehat{\mathcal{V}} d\mathbf{x} \qquad (6.56)$$

where $\widehat{\mathcal{V}}$ is the Eulerian velocity gradient tensor. This tensor $\widehat{\mathcal{V}}$ may always be decomposed into

$$\widehat{\mathcal{V}} = \widehat{\mathcal{D}} + \widehat{\mathcal{W}} \qquad (6.57)$$

where $\widehat{\mathcal{D}}$ is the symmetric strain rate (or stretching) tensor and $\widehat{\mathcal{W}}$ is the skewsymmetric vorticity (or spin) tensor, Section 6.4.2. Consequently, the relative velocity can be alternatively expressed as

$$\delta \mathbf{v} = \widehat{\mathcal{V}} d\mathbf{x} = \delta \mathbf{v}^{(s)} + \delta \mathbf{v}^{(r)} \qquad (6.58)$$

where

$$\begin{aligned}\delta \mathbf{v}^{(s)} &= \widehat{\mathcal{D}} d\mathbf{x} \\ \delta \mathbf{v}^{(r)} &= \widehat{\mathcal{W}} d\mathbf{x} = \hat{\mathbf{w}} \times d\mathbf{x}\end{aligned}$$

and $\hat{\mathbf{w}}$, known as the *vorticy vector*, has components defined as, Eq. 3.5,

$$\hat{w}_i = -\frac{1}{2}\epsilon_{ijk}\widehat{\mathcal{W}}_{jk} = -\frac{1}{4}\epsilon_{ijk}[v_{j,k} - v_{k,j}]$$

From the strain rate relationships reported in Section 6.4.3, we can verify that if $\widehat{\mathcal{D}}$ is equal to zero, there is no rate of deformation. Conversely, the rate of deformation is not influenced by $\widehat{\mathcal{W}}$. Thus, we can state that $\widehat{\mathcal{D}}$ is the responsible for the whole deformation process while $\widehat{\mathcal{W}}$ is the responsible of a rigid rotational velocity field.

If the rate of deformation tensor is identically zero ($\hat{\mathcal{D}}_{ij} = 0$), the motion in the neighborhood is a rigid rotation and, since in this case $v_{j,k} = -v_{k,j}$, it results

$$\hat{w}_i = \frac{1}{2}\epsilon_{ijk}v_{k,j} \tag{6.59}$$

If the tensor $\widehat{\mathcal{W}}$ vanishes everywhere within the field, the velocity field is said to be *irrotational*. Necessary and sufficient condition for a velocity field to be irrotational, is the existence of a *velocity potential* Φ so that

$$v_i = -\Phi_{,i} \tag{6.60}$$

In this case,

$$v_{i,j} = v_{j,i}$$

and, consequently, $\widehat{\mathcal{W}} \equiv \mathbf{O}$.

6.4.7 The acceleration field

In the Eulerian formulation the acceleration in Eq. 6.36 can be alternatively expressed as

$$\begin{aligned}\frac{d\mathbf{v}}{dt} &= \frac{\partial \mathbf{v}}{\partial t} + \mathcal{V}\mathbf{v} = && (6.61) \\ &= \frac{\partial \mathbf{v}}{\partial t} + 2\mathcal{W}\mathbf{v} + \nabla\frac{v^2}{2} = && (6.62) \\ &= \frac{\partial \mathbf{v}}{\partial t} + 2\mathbf{q} \times \mathbf{v} + \nabla\frac{v^2}{2} && (6.63)\end{aligned}$$

where

$$\begin{aligned}\mathcal{V} &= [v_{i,j}] \\ 2\mathcal{W} &= [v_{i,j} - v_{j,i}] \\ q_j &= -\frac{1}{4}\epsilon_{ijk}(v_{j,k} - v_{k,j}) = -\frac{1}{2}\epsilon_{ijk}\mathcal{W}_{j,k}\end{aligned}$$

In fact,

$$\begin{aligned}\frac{dv_i}{dt} &= \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x_k}v_k = \\ &= \frac{\partial v_i}{\partial t} + \left(\frac{\partial v_i}{\partial x_k} - \frac{\partial v_k}{\partial x_i}\right)v_k + \frac{\partial v_k}{\partial x_i}v_k = \\ &= \frac{\partial v_i}{\partial t} + 2\mathcal{W}_{ik}v_k + \frac{1}{2}\frac{\partial v_k v_k}{\partial x_i} = \\ &= \frac{\partial v_i}{\partial t} + 2\epsilon_{ijk}q_j v_k + \frac{1}{2}\frac{\partial v^2}{\partial x_i}\end{aligned}$$

since, Eq. 3.5,

$$\mathcal{W}_{ik}v_k = \epsilon_{ijk}q_j v_k$$

6.5 Relationships Between the Two Formulations

Both the Lagrangian and the Eulerian formulations require the condition of one-to-one mapping, so that according to Eq. 6.12, Lagrangian formulation,

$$\delta\mathbf{x} = \widetilde{\mathcal{F}}d\mathbf{X}; \qquad d\mathbf{X} = \widetilde{\mathcal{F}}^{-1}\delta\mathbf{x} \tag{6.64}$$

where

$$\widetilde{\mathcal{F}} = \left[\frac{\partial x_i}{\partial X_j}\right]$$

and according to Eq. 6.30, Eulerian formulation,

$$d\mathbf{X} = \widehat{\mathcal{F}}\delta\mathbf{x}; \qquad \delta\mathbf{x} = \widehat{\mathcal{F}}^{-1}d\mathbf{X} \tag{6.65}$$

where

$$\widehat{\mathcal{F}} = \left[\frac{\partial X_i}{\partial x_j}\right]$$

It follows,

$$\widetilde{\mathcal{F}} = \widehat{\mathcal{F}}^{-1} \tag{6.66}$$

and, consequently,

$$\tilde{J} = \hat{J}^{-1} \tag{6.67}$$

These relationships allow to relate any previously defined Lagrangian tensor to the Eulerian ones, and viceversa.

6.5.1 Relationships between the Lagrange and the Euler tensors

The Lagrange tensors in Section 6.3.1 can be related to the Euler tensors in Section 6.4.2 as

$$\tilde{\mathcal{U}} = \hat{\mathcal{U}}\widehat{\mathcal{F}}^{-1} \tag{6.68}$$

$$\widetilde{\mathcal{F}} = \widehat{\mathcal{F}}^{-1} \tag{6.69}$$

$$\tilde{\mathcal{G}} = \hat{\mathcal{G}}^{-1} = \left(\widehat{\mathcal{F}}\widehat{\mathcal{F}}^{T}\right)^{-1} = \widehat{\mathcal{F}}^{-T}\widehat{\mathcal{F}}^{-1} \tag{6.70}$$

$$\tilde{\mathcal{L}} = \widehat{\mathcal{F}}^{-T}\hat{\mathcal{E}}\widehat{\mathcal{F}}^{-1} \tag{6.71}$$

$$\tilde{\mathcal{V}} = \hat{\mathcal{V}}\widehat{\mathcal{F}}^{-1} \tag{6.72}$$

and their rate of change as

$$\dot{\tilde{\mathcal{U}}} = \tilde{\mathcal{V}} = \widehat{\mathcal{F}}^{-1}\dot{\hat{\mathcal{U}}}\widehat{\mathcal{F}}^{-1} \tag{6.73}$$

$$\dot{\widetilde{\mathcal{F}}} = \dot{\tilde{\mathcal{U}}} \tag{6.74}$$

$$\dot{\tilde{\mathcal{G}}} = 2\widehat{\mathcal{F}}^{-T}\widehat{\mathcal{D}}\widehat{\mathcal{F}}^{-1} \tag{6.75}$$

$$\dot{\tilde{\mathcal{L}}} = \frac{1}{2}\dot{\tilde{\mathcal{G}}} \tag{6.76}$$

Conversely, the Euler tensors can be related to the Lagrange tensors as

$$\hat{\mathcal{U}} = \tilde{\mathcal{U}}\widetilde{\mathcal{F}}^{-1} \tag{6.77}$$

$$\widehat{\mathcal{F}} = \widetilde{\mathcal{F}}^{-1} \tag{6.78}$$

$$\hat{\mathcal{C}} = \tilde{\mathcal{C}}^{-1} = \left(\widetilde{\mathcal{F}}\widetilde{\mathcal{F}}^{T}\right)^{-1} = \widetilde{\mathcal{F}}^{-T}\widetilde{\mathcal{F}}^{-1} \tag{6.79}$$

$$\hat{\mathcal{E}} = \widetilde{\mathcal{F}}^{-T}\tilde{\mathcal{L}}\widetilde{\mathcal{F}}^{-1} \tag{6.80}$$

$$\hat{\mathcal{V}} = \tilde{\mathcal{V}}\widetilde{\mathcal{F}}^{-1} \tag{6.81}$$

$$\widehat{\mathcal{D}} = \frac{1}{2}(\hat{\mathcal{V}} + \hat{\mathcal{V}}^{T}) = \widetilde{\mathcal{F}}^{-T}\dot{\tilde{\mathcal{L}}}\widetilde{\mathcal{F}}^{-1} \tag{6.82}$$

$$\widehat{\mathcal{W}} = \frac{1}{2}(\hat{\mathcal{V}} - \hat{\mathcal{V}}^{T}) \tag{6.83}$$

and their rate of change as

$$\dot{\tilde{\mathcal{U}}} = \widetilde{\mathcal{F}}^{-1}\dot{\hat{\mathcal{U}}}\widetilde{\mathcal{F}}^{-1} \tag{6.84}$$

$$\dot{\widehat{\mathcal{F}}} = -\dot{\hat{\mathcal{U}}} \tag{6.85}$$

$$\dot{\tilde{\mathcal{C}}} = -\widetilde{\mathcal{F}}^{-T}\left[\left(\widetilde{\mathcal{F}}^{-1}\tilde{\mathcal{V}}\right)^{T} + \left(\widetilde{\mathcal{F}}^{-1}\tilde{\mathcal{V}}\right)\right]\widetilde{\mathcal{F}}^{-1} \tag{6.86}$$

$$\dot{\tilde{\mathcal{E}}} = -\frac{1}{2}\dot{\tilde{\mathcal{C}}} \tag{6.87}$$

In fact, according to Eq. 6.66

$$\widetilde{\mathcal{F}} = \widehat{\mathcal{F}}^{-1}$$

then

$$\begin{aligned}
\tilde{\mathcal{U}} &= \frac{\partial u_i}{\partial X_j} = \frac{\partial u_i}{\partial x_k}\frac{\partial x_k}{\partial X_j} = \hat{\mathcal{U}}\widetilde{\mathcal{F}} = \hat{\mathcal{U}}\widehat{\mathcal{F}}^{-1} \\
\tilde{\mathcal{G}} &= \left(\widetilde{\mathcal{F}}^{T}\widetilde{\mathcal{F}}\right) = \left(\widehat{\mathcal{F}}^{-T}\widehat{\mathcal{F}}^{-1}\right) = \left(\widehat{\mathcal{F}}\widehat{\mathcal{F}}^{T}\right)^{-1} = \hat{\mathcal{G}}^{-1} \\
\tilde{\mathcal{L}} &= \frac{1}{2}\left(\tilde{\mathcal{G}} - \mathbf{I}\right) = \frac{1}{2}\left(\hat{\mathcal{G}}^{-1} - \mathbf{I}\right) = \frac{1}{2}\left[\left(\widehat{\mathcal{F}}\widehat{\mathcal{F}}^{T}\right)^{-1} - \mathbf{I}\right] = \\
&= \frac{1}{2}\left[\widehat{\mathcal{F}}^{-T}\widehat{\mathcal{F}}^{-1} - \mathbf{I}\right] = \frac{1}{2}\widehat{\mathcal{F}}^{-T}\left[\mathbf{I} - \widehat{\mathcal{F}}^{T}\widehat{\mathcal{F}}\right]\widehat{\mathcal{F}}^{-1} = \\
&= \frac{1}{2}\widehat{\mathcal{F}}^{-T}\left[\mathbf{I} - \hat{\mathcal{C}}\right]\widehat{\mathcal{F}}^{-1} = \\
&= \widehat{\mathcal{F}}^{-T}\hat{\mathcal{E}}\widehat{\mathcal{F}}^{-1} \\
\tilde{\mathcal{V}} &= \frac{\partial v_i}{\partial X_j} = \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j} = \hat{\mathcal{V}}\widetilde{\mathcal{F}} = \hat{\mathcal{V}}\widehat{\mathcal{F}}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\tilde{\mathcal{U}}} &= \tilde{\mathcal{V}} = \hat{\mathcal{V}}\widehat{\mathcal{F}}^{-1} = \widehat{\mathcal{F}}^{-1}\dot{\hat{\mathcal{U}}}\widehat{\mathcal{F}}^{-1} \\
\dot{\tilde{\mathcal{G}}} &= \tilde{\mathcal{V}}^{T}\widetilde{\mathcal{F}} + \widetilde{\mathcal{F}}^{T}\tilde{\mathcal{V}} = \widetilde{\mathcal{F}}^{T}\hat{\mathcal{V}}^{T}\widetilde{\mathcal{F}} + \widetilde{\mathcal{F}}^{T}\hat{\mathcal{V}}\widetilde{\mathcal{F}} = \\
&= \widetilde{\mathcal{F}}^{T}\left(\hat{\mathcal{V}}^{T} + \hat{\mathcal{V}}\right)\widetilde{\mathcal{F}} = 2\widehat{\mathcal{F}}^{-T}\widehat{\mathcal{D}}\widehat{\mathcal{F}}^{-1}
\end{aligned}$$

The converse relationships between Eulerian and Lagrangian tensor can be obtained either by inverting the above found relationship, or as follows. According to Eq. 6.66

$$\widehat{\mathcal{F}} = \widetilde{\mathcal{F}}^{-1}$$

then

$$\begin{aligned}
\widehat{\mathcal{U}} &= \frac{\partial u_i}{\partial x_j} = \frac{\partial u_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \widetilde{\mathcal{U}}\widehat{\mathcal{F}} = \widetilde{\mathcal{U}}\widetilde{\mathcal{F}}^{-1} \\
\widehat{\mathcal{C}} &= \widehat{\mathcal{F}}^T\widehat{\mathcal{F}} = \widetilde{\mathcal{F}}^{-T}\widetilde{\mathcal{F}}^{-1} = \left(\widetilde{\mathcal{F}}\widetilde{\mathcal{F}}^T\right)^{-1} = \widetilde{\mathcal{C}}^{-1} \\
\widehat{\mathcal{E}} &= \frac{1}{2}\left(\mathrm{I} - \widehat{\mathcal{C}}\right) = \frac{1}{2}\left(\mathrm{I} - \widetilde{\mathcal{C}}^{-1}\right) = \frac{1}{2}\left(\mathrm{I} - \widetilde{\mathcal{F}}^{-T}\widetilde{\mathcal{F}}^{-1}\right) = \\
&= \frac{1}{2}\widetilde{\mathcal{F}}^{-T}\left(\widetilde{\mathcal{F}}^T\widetilde{\mathcal{F}} - \mathrm{I}\right)\widetilde{\mathcal{F}}^{-1} = \frac{1}{2}\widetilde{\mathcal{F}}^{-T}\left(\widetilde{\mathcal{G}} - \mathrm{I}\right)\widetilde{\mathcal{F}}^{-1} = \widetilde{\mathcal{F}}^{-T}\widetilde{\mathcal{L}}\widetilde{\mathcal{F}}^{-1} \\
\widehat{\mathcal{V}} &= \frac{\partial v_i}{\partial x_j} = \frac{\partial v_i}{\partial X_k}\frac{\partial X_k}{\partial x_j} = \widetilde{\mathcal{V}}\widehat{\mathcal{F}} = \widetilde{\mathcal{V}}\widetilde{\mathcal{F}}^{-1} \\
\widehat{\mathcal{D}} &= \frac{1}{2}\left(\widehat{\mathcal{V}} + \widehat{\mathcal{V}}^T\right) = \frac{1}{2}\left(\widetilde{\mathcal{V}}\widetilde{\mathcal{F}}^{-1} + \widetilde{\mathcal{F}}^{-T}\widetilde{\mathcal{V}}^T\right) = \\
&= \frac{1}{2}\widetilde{\mathcal{F}}^{-T}\left(\widetilde{\mathcal{F}}^T\widetilde{\mathcal{V}} + \widetilde{\mathcal{V}}^T\widetilde{\mathcal{F}}\right)\widetilde{\mathcal{F}}^{-1} = \frac{1}{2}\widetilde{\mathcal{F}}^{-T}\dot{\widetilde{\mathcal{G}}}\widetilde{\mathcal{F}}^{-1} = \widetilde{\mathcal{F}}^{-T}\dot{\widetilde{\mathcal{L}}}\widetilde{\mathcal{F}}^{-1}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\widehat{\mathcal{U}}} &= \widehat{\mathcal{F}}\widehat{\mathcal{V}} = \widetilde{\mathcal{F}}^{-1}\widetilde{\mathcal{V}}\widetilde{\mathcal{F}}^{-1} = \widetilde{\mathcal{F}}^{-1}\dot{\widetilde{\mathcal{U}}}\widetilde{\mathcal{F}}^{-1} \\
\dot{\widehat{\mathcal{C}}} &= -\left(\widehat{\mathcal{V}}^T\widehat{\mathcal{C}} + \widehat{\mathcal{C}}\widehat{\mathcal{V}}\right) = \\
&= -\left[\left(\widetilde{\mathcal{V}}\widetilde{\mathcal{F}}^{-1}\right)^T\left(\widetilde{\mathcal{F}}\widetilde{\mathcal{F}}^T\right)^{-1} + \left(\widetilde{\mathcal{F}}\widetilde{\mathcal{F}}^T\right)^{-1}\left(\widetilde{\mathcal{V}}\widetilde{\mathcal{F}}^{-1}\right)\right] = \\
&= -\left[\widetilde{\mathcal{F}}^{-T}\left(\widetilde{\mathcal{F}}^{-1}\widetilde{\mathcal{V}}\right)^T\widetilde{\mathcal{F}}^{-1} + \widetilde{\mathcal{F}}^{-T}\left(\widetilde{\mathcal{F}}^{-1}\widetilde{\mathcal{V}}\right)\widetilde{\mathcal{F}}^{-1}\right] = \\
&= -\widetilde{\mathcal{F}}^{-T}\left[\left(\widetilde{\mathcal{F}}^{-1}\widetilde{\mathcal{V}}\right)^T + \left(\widetilde{\mathcal{F}}^{-1}\widetilde{\mathcal{V}}\right)\right]\widetilde{\mathcal{F}}^{-1}
\end{aligned}$$

6.5.2 Remarks

Interestingly, since

$$\dot{\widetilde{\mathcal{F}}} = \widehat{\mathcal{V}}\widetilde{\mathcal{F}} \tag{6.88}$$

it is possible to express the rate of change of the determinant of $\dot{\widetilde{\mathcal{F}}}$ as

$$\frac{d}{dt}(\det\widetilde{\mathcal{F}}) = \dot{\widetilde{J}} = \widetilde{J}\,\mathrm{div}\,\mathbf{v} \tag{6.89}$$

where

$$\begin{aligned}
\mathrm{div}\,\mathbf{v} &= \mathrm{tr}\,\widehat{\mathcal{V}} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \\
&= v_{1,1} + v_{2,2} + v_{3,3}
\end{aligned}$$

In fact

$$\begin{aligned}
\dot{\widetilde{\mathcal{F}}} &= \left[\frac{d}{dt}\frac{\partial x_i}{\partial X_J}\right] = [\dot{x}_{i,J}] = [v_{i,J}] \\
v_{i,J} &= \frac{\partial v_i(\mathbf{X},t)}{\partial X_J} = \frac{\partial v_i(\mathbf{x},t)}{\partial x_s}\frac{\partial x_s}{\partial X_J} = \widehat{\mathcal{V}}_{is}\widetilde{\mathcal{F}}_{sJ}
\end{aligned}$$

from which we can verify Eq. 6.88. Then

$$\begin{aligned}
\dot{\widetilde{J}} &= \frac{d}{dt}\left(\det\widetilde{\mathcal{F}}\right) = \frac{d}{dt}\left[\epsilon_{IJK}\frac{\partial x_1}{\partial X_I}\frac{\partial x_2}{\partial X_J}\frac{\partial x_3}{\partial X_K}\right] = \\
&= \frac{d}{dt}\left(\epsilon_{IJK}x_{1,I}x_{2,J}x_{3,K}\right) = \\
&= \epsilon_{IJK}\left(\dot{x}_{1,I}x_{2,J}x_{3,K} + x_{1,I}\dot{x}_{2,J}x_{3,K} + x_{1,I}x_{2,J}\dot{x}_{3,K}\right)
\end{aligned}$$

Since, according to Eq. 6.88,

$$\dot{x}_{1,I} = v_{1,s}x_{s,I}$$

the $1st$ term can be expressed as

$$\begin{aligned}
\epsilon_{IJK}\dot{x}_{1,I}x_{2,J}x_{3,K} &= \epsilon_{IJK}v_{1,s}x_{s,I}x_{2,J}x_{3,K} = \\
&= \epsilon_{IJK}\left(v_{1,1}x_{1,I}x_{2,J}x_{3,K} + v_{1,2}x_{2,I}x_{2,J}x_{3,K}\right. \\
&\quad \left. + v_{1,3}x_{3,I}x_{2,J}x_{3,K}\right)
\end{aligned}$$

but

$$\begin{aligned}
\epsilon_{IJK}v_{1,2}x_{2,I}x_{2,J}x_{3,K} &= v_{1,2}\left(x_{2,1}x_{2,2}x_{3,3} + x_{2,2}x_{2,3}x_{3,1} + x_{2,3}x_{2,1}x_{3,2} +\right. \\
&\quad \left. - x_{2,3}x_{2,2}x_{3,1} - x_{2,1}x_{2,3}x_{3,2} - x_{2,2}x_{2,1}x_{3,3}\right) = \\
&= 0
\end{aligned}$$

and, analogously,

$$\epsilon_{IJK}v_{1,3}x_{3,I}x_{2,J}x_{3,K} = 0$$

so that

$$\epsilon_{IJK}\dot{x}_{1,I}x_{2,J}x_{3,K} = \epsilon_{IJK}v_{1,1}x_{1,I}x_{2,J}x_{3,K} = v_{1,1}\det\widetilde{\mathcal{F}} = v_{1,1}\widetilde{J}$$

With an analogous procedure we can prove that

$$\begin{aligned}
\epsilon_{IJK}x_{1,I}\dot{x}_{2,J}x_{3,K} &= v_{2,2}\widetilde{J} \\
\epsilon_{IJK}x_{1,I}x_{2,J}\dot{x}_{3,K} &= v_{3,3}\widetilde{J}
\end{aligned}$$

and, consequently, we obtain that

$$\begin{aligned}
\dot{\widetilde{J}} &= \epsilon_{IJK}\left(\dot{x}_{1,I}x_{2,J}x_{3,K} + x_{1,I}\dot{x}_{2,J}x_{3,K} + x_{1,I}x_{2,J}\dot{x}_{3,K}\right) = \\
&= \left(v_{1,1} + v_{2,2} + v_{3,3}\right)\widetilde{J} = \widetilde{J}\,\mathrm{div}\,\mathbf{v}
\end{aligned}$$

6.6 Small Deformation Theory

Usually, engineers solve solid mechanical problems using the so-called *small deformation theory*. This theory assumes that:

1. the displacement gradient components are small compared to unity, that is

$$|u_{i,J}| = \left|\frac{\partial u_i(\mathbf{X},t)}{\partial X_J}\right| \ll 1; \qquad |u_{i,j}| = \left|\frac{\partial u_i(\mathbf{x},t)}{\partial x_j}\right| \ll 1 \tag{6.90}$$

2. the displacement field is sufficiently small so that there is very little difference between the undeformed and the deformed configurations and thus between the material and the spatial coordinates of a continuum particle, so that

$$u_{i,J} \approx u_{i,j} \tag{6.91}$$

Accordingly, if each of the displacement gradient components $u_{i,j}$ is small compared to unity, the product terms are negligible and may be dropped, that is

$$u_{i,J}u_{k,S} \approx u_{i,j}u_{k,s} \approx 0 \tag{6.92}$$

Thus, under the hypothesis of the small deformation gradient, we have that

$$\breve{\mathcal{F}}_{iJ} = \delta_{iJ} + u_{i,J} \approx \delta_{iJ}; \qquad \hat{\mathcal{F}}_{ij} = \delta_{ij} - u_{i,j} \approx \delta_{ij}$$

which implies that the Lagrange and the Euler deformation gradients are approximately equal to the identity so that we can set

$$\breve{\mathcal{F}} \approx \mathbf{I} \approx \hat{\mathcal{F}} \tag{6.93}$$

and, accordingly,

$$\det \breve{\mathcal{F}} = \breve{J} \approx 1 \approx \hat{J} = \det \hat{\mathcal{F}} \tag{6.94}$$

It is important to underline that the hypothesis of small deformation implies that the deformation process involves very little change of distance between two neighboring points, so that

$$\begin{aligned} \delta\mathbf{x} &= \breve{\mathcal{F}} d\mathbf{X} \simeq d\mathbf{X} \\ \delta\mathbf{X} &= \hat{\mathcal{F}} d\mathbf{x} \simeq d\mathbf{x} \end{aligned}$$

however, the relative displacement

$$d\mathbf{u} = d\mathbf{x} - d\mathbf{X} \tag{6.95}$$

although small is not null.

Usually, the small deformation theory is presented as a particular case of the Lagrangian formulation. Accordingly, we can say that the small deformation theory concentrates on a given body of matter of the continuum $\mathcal{M}$ and the motion of any particle P is expressed by equations of the form

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{6.96}$$

where $\mathbf{X}$ is the location of P at the initial time $t = 0$ while $\mathbf{x}$ is the location of P at the current time t.

This description, where the independent variables are the initial location $\mathbf{X}$ and the current time t, may be viewed as a mapping of P from its original position into the current configuration. It is assumed that such a tracing is one-to-one and continuous, with continuous partial derivatives to whatever order is required. According to Eq. 6.96 and the hypothesis 1, we have that

$$d\mathbf{x} = \mathcal{F} d\mathbf{X} + \frac{\partial \mathbf{x}}{\partial t} dt \approx d\mathbf{X} + \frac{\partial \mathbf{x}}{\partial t} dt \tag{6.97}$$

being

$$\mathcal{F} = \left[\frac{\partial x_i}{\partial X_j} \right] \approx [\delta_{ij}]$$

Then, for each material point in $\mathcal{M}$, we have that, Section 6.3:

- The displacement of the particle is given by

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \tag{6.98}$$

- The relative displacement vector of the particle P with respect to a neighboring point is given by

$$d\mathbf{u}(\mathbf{X}, t) = \mathcal{U} d\mathbf{X} + \frac{\partial \mathbf{u}(\mathbf{X}, t)}{\partial t} dt \tag{6.99}$$

  where

$$\mathcal{U} = \left[\frac{\partial u_i}{\partial X_j} \right]$$

  is known as the *Lagrange Displacement Gradient* tensor.

- The velocity of the particle at P is given by

$$\mathbf{v}(\mathbf{X},t) = \dot{\mathbf{u}}(\mathbf{X},t) = \frac{\partial \mathbf{u}(\mathbf{X},t)}{\partial t} \tag{6.100}$$

- The acceleration of the particle at P is given by

$$\mathbf{a}(\mathbf{X},t) = \ddot{\mathbf{u}}(\mathbf{X},t) = \frac{\partial \mathbf{v}(\mathbf{X},t)}{\partial t} \tag{6.101}$$

6.6.1 Linear Lagrangian tensors

The small deformation theory leads to define the following tensors:

- The *Lagrange Displacement Gradient* tensor defined as

$$\mathcal{U} = [u_{i,j}] = \left[\frac{\partial u_i}{\partial X_j}\right]$$

- The *Lagrange Deformation Gradient* tensor defined as

$$\mathcal{F} = [x_{i,j}] = \left[\frac{\partial x_i}{\partial X_j}\right] \approx \mathbf{I}$$

  which can be alternatively expressed as

$$\mathcal{F} = \mathbf{I} + \mathbf{U} = [\delta_{ij} + u_{i,j}]$$

- The *Linear Green Strain* tensor defined as

$$\mathcal{G} = \mathbf{I} + \mathcal{U} + \mathcal{U}^T = [\delta_{ij} + u_{i,j} + u_{j,i}] \approx \mathbf{I}$$

- The *Linear Lagrange Strain* tensor defined as

$$\mathcal{L} = \frac{1}{2}(\mathcal{G} - \mathbf{I}) = \frac{1}{2}(\mathcal{U} + \mathcal{U}^T) = \frac{1}{2}[u_{i,j} + u_{j,i}]$$

- The *Linear Lagrange Rotation* tensor defined as

$$\mathbf{\Omega} = \frac{1}{2}(\mathcal{U} - \mathcal{U}^T) = \frac{1}{2}[u_{i,j} - u_{j,i}]$$

- The *Lagrange Velocity Gradient* tensor defined as

$$\mathcal{V} = [v_{i,i}] = \left[\frac{\partial v_i}{\partial X_j}\right]$$

- The *Strain Rate (or stretching)* tensor defined as

$$\mathcal{D} = \frac{1}{2}(\mathcal{V} + \mathcal{V}^T) = \frac{1}{2}[v_{i,j} + v_{j,i}]$$

- The *Vorticity (or spin) Velocity* tensor defined as

$$\mathcal{W} = \frac{1}{2}(\mathcal{V} - \mathcal{V}^T) = \frac{1}{2}[v_{i,j} - v_{j,i}]$$

The time rate of the above tensors can be expressed as

$$\begin{aligned}
\dot{\mathcal{U}} &= \mathcal{V} \\
\dot{\mathcal{F}} &= \dot{\mathcal{U}} \\
\dot{\mathcal{G}} &= 2\mathcal{D} \\
\dot{\mathcal{L}} &= \mathcal{D}
\end{aligned}$$

It is immediate to verify that the above tensors are the results of the linearization of the Lagrangian tensors in Section 6.3.1 and of the Eulerian tensors in Section 6.4.2 due to the hypotheses of the small deformation theory. In fact,

$$\begin{aligned}
\tilde{\mathcal{U}} &= [u_{i,j}] = \left[\frac{\partial u_i}{\partial X_j}\right] \equiv \mathcal{U} \\
\tilde{\mathcal{F}} &= [x_{i,j}] = \left[\frac{\partial x_i}{\partial X_j}\right] \equiv \mathcal{F} \approx \mathbf{I} \\
\tilde{\mathcal{G}} &= \mathbf{I} + \tilde{\mathcal{U}} + \tilde{\mathcal{U}}^T + \tilde{\mathcal{U}}^T\tilde{\mathcal{U}} \approx \mathbf{I} + \tilde{\mathcal{U}} + \tilde{\mathcal{U}}^T = \mathcal{G} \\
\tilde{\mathcal{L}} &= \frac{1}{2}\left(\tilde{\mathcal{G}} - \mathbf{I}\right) = \frac{1}{2}\left(\tilde{\mathcal{U}} + \tilde{\mathcal{U}}^T + \tilde{\mathcal{U}}^T\tilde{\mathcal{U}}\right) \approx \frac{1}{2}\left(\tilde{\mathcal{U}} + \tilde{\mathcal{U}}^T\right) = \mathcal{L} \\
\tilde{\mathcal{V}} &= [v_{i,i}] = \left[\frac{\partial v_i}{\partial X_j}\right] \equiv \mathcal{V}
\end{aligned}$$

and

$$\dot{\tilde{\mathcal{U}}} = \tilde{\mathcal{V}} \equiv \dot{\mathcal{U}}$$

$$\begin{aligned}
\dot{\widetilde{\mathcal{F}}} &= \dot{\tilde{\mathcal{U}}} = \tilde{\mathcal{V}} \equiv \dot{\mathcal{U}} = \dot{\mathcal{F}} \\
\dot{\tilde{\mathcal{G}}} &= \tilde{\mathcal{V}}^T \widetilde{\mathcal{F}} + \widetilde{\mathcal{F}}^T \tilde{\mathcal{V}} \approx \mathcal{V}^T + \tilde{\mathcal{V}} = 2\mathcal{D} = \dot{\mathcal{G}} \\
\dot{\tilde{\mathcal{L}}} &= \frac{\dot{\tilde{\mathcal{G}}}}{2} \approx \dot{\mathcal{L}}
\end{aligned}$$

Moreover, being

$$\widetilde{\mathcal{F}} = \widehat{\mathcal{F}}^{-1} \approx \mathbf{I}$$

$$\begin{aligned}
\mathcal{U} &\equiv \tilde{\mathcal{U}} = \hat{\mathcal{U}} \widehat{\mathcal{F}}^{-1} \approx \hat{\mathcal{U}} \\
\mathcal{G} &\approx \tilde{\mathcal{G}} = \hat{\mathcal{G}}^{-1} = \left(\widehat{\mathcal{F}} \widehat{\mathcal{F}}^T \right)^{-1} \approx \mathbf{I} \\
\mathcal{L} &\approx \tilde{\mathcal{L}} = \widehat{\mathcal{F}}^{-T} \hat{\mathcal{E}} \widehat{\mathcal{F}}^{-1} \approx \hat{\mathcal{E}} \\
\mathcal{V} &\equiv \tilde{\mathcal{V}} = \hat{\mathcal{V}} \widehat{\mathcal{F}}^{-1} \approx \hat{\mathcal{V}}
\end{aligned}$$

and

$$\begin{aligned}
\dot{\mathcal{U}} &\equiv \dot{\tilde{\mathcal{U}}} = \widehat{\mathcal{F}}^{-1} \dot{\hat{\mathcal{U}}} \widehat{\mathcal{F}}^{-1} \approx \dot{\hat{\mathcal{U}}} \\
\mathcal{D} &\approx \frac{\dot{\tilde{\mathcal{G}}}}{2} = \widehat{\mathcal{F}}^{-T} \widehat{\mathcal{D}} \widehat{\mathcal{F}}^{-1} \approx \widehat{\mathcal{D}}
\end{aligned}$$

6.6.2 Measure of Deformation

Let $\mathcal{M}$ be a continuum and P, Q and R be three neighboring material points in $\mathcal{M}$, Fig. 6.6. Let indicate the distance between these points as:

- $d\mathrm{X} \equiv d\mathrm{X}^{(q)}$ and $\delta\mathrm{x} \equiv \delta\mathrm{x}^{(q)}$ the relative distance between P and Q at time $t = 0$ and t, respectively.
- $d\mathrm{X}^{(r)}$ and $\delta\mathrm{x}^{(r)}$ the relative distance between P and R at time $t = 0$ and t, respectively.
- and, for $i = q, r$,

  $$\widehat{\mathrm{S}}^{(i)} = \frac{d\mathrm{X}^{(i)}}{|d\mathrm{X}^{(i)}|}; \qquad \delta\widehat{\mathrm{s}}^{(i)} = \frac{\delta\mathrm{x}^{(i)}}{|\delta\mathrm{x}^{(i)}|}$$

  the unit vectors in the direction of $d\mathrm{X}^{(i)}$ and $\delta\mathrm{x}^{(i)}$, respectively.

Then, under the hypothesis of small deformation, we can measure their deformation process, Section 6.6, as follows:

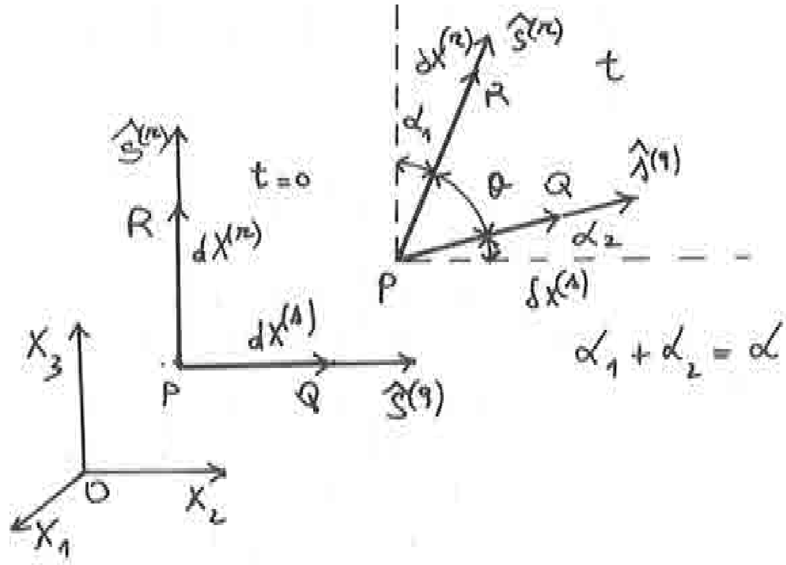


Figure 6.6: Material description of a *small* deformation process

- The *Undeformed Length* defined as

$$dS^2 = d\mathrm{X}^T d\mathrm{X}$$

- The *Deformed Length* defined as

$$\delta s^2 = \delta \mathrm{x}^T \delta \mathrm{x}$$

  results to be equal to

$$\delta s^2 \approx d\mathrm{X}^T \mathcal{G} d\mathrm{X}$$

  and its time rate to

$$\frac{d}{dt}(\delta s^2) \approx d\mathrm{X}^T \dot{\mathcal{G}} d\mathrm{X} = 2 d\mathrm{X}^T \dot{\mathcal{C}} d\mathrm{X}$$

- The *Stretch* defined as

$$\lambda^2 = \frac{\delta s^2}{dS^2}$$

  results to be equal to

$$\lambda^2 \approx \hat{\mathrm{S}}^T \mathcal{G} \hat{\mathrm{S}}$$

and its time rate to

$$\frac{d}{dt}(\lambda^2) \approx \hat{S}^T \dot{\mathcal{G}} \hat{S} = 2(\hat{S}^T \dot{\mathcal{L}} \hat{S})$$

- The *Measure of Deformation* defined as

$$\chi = \delta s^2 - dS^2$$

results to be equal to

$$\chi \approx 2(d\mathbf{X}^T \mathcal{L} d\mathbf{X})$$

and its time rate to

$$\dot{\chi} \approx d\mathbf{X}^T \dot{\mathcal{G}} d\mathbf{X} = 2 d\mathbf{X}^T \dot{\mathcal{L}} d\mathbf{X}$$

- The *Unit Extension* defined as

$$\epsilon = \frac{\delta s - dS}{dS} = \lambda - 1$$

can be calculated as

$$\epsilon \approx \hat{S}^T \mathcal{L} \hat{S}$$

and its time rate as

$$\dot{\epsilon} \approx \hat{S}^T \dot{\mathcal{L}} \hat{S}$$

In fact,

$$\begin{aligned} 2(d\mathbf{X}^T \mathcal{L} d\mathbf{X}) &\approx \chi = \delta s^2 - dS^2 = (\delta s + dS)(\delta s - dS) = \\ &\approx 2dS(\delta s - dS) = 2dS^2 \epsilon \end{aligned}$$

from which

$$\epsilon \approx \frac{d\mathbf{X}^T \mathcal{L} d\mathbf{X}}{dS^2} = \frac{d\mathbf{X}^T \mathcal{L} d\mathbf{X}}{d\mathbf{X}^T d\mathbf{X}} = \hat{S}^T \mathcal{L} \hat{S}$$

- The *Angle Change* defined as, Fig. 6.6,

$$\alpha = \frac{\pi}{2} - \theta$$

where

$$\theta = \cos^{-1}(\hat{s}^{(q)}, \hat{s}^{(r)})$$

can be calculated as

$$\alpha \approx 2\mathcal{L}_{qr}$$

and its time rate as

$$\dot{\alpha} \approx 2\dot{\mathcal{L}}_{qr}$$

In fact, according to the hypothesis of small deformation, we can assume that the angle change is small compared to unity, so that

$$\alpha = \frac{\pi}{2} - \theta \approx \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

and, Section 6.3,

$$\begin{aligned} \cos\theta &= \gamma = \frac{\tilde{\mathcal{G}}_{qr}}{\lambda_q \lambda_r} \\ &= \frac{2\tilde{\mathcal{L}}_{qr}}{(1+2\tilde{\mathcal{L}}_{qq})^{1/2}(1+2\tilde{\mathcal{L}}_{rr})^{1/2}} = \\ &\approx 2\mathcal{L}_{qr} \end{aligned}$$

6.6.3 Surface and volume

Let $d\mathcal{M}_o$ be an infinitesimal portion of material at time $t = 0$ defined by the three differential vectors $d\mathbf{X}^{(1)}, d\mathbf{X}^{(2)}$ and $d\mathbf{X}^{(3)}$, Fig. 6.3. Assume that at the current time t this portion of material reaches the configuration $d\mathcal{M}$ described by the three differential vectors $\delta\mathbf{x}^{(1)}, \delta\mathbf{x}^{(2)}$ and $\delta\mathbf{x}^{(3)}$. According to the (finite) Lagrangian deformation, Section 6.3.3, surfaces and volumes at time t and time $t = 0$ are related as

$$\begin{aligned} \hat{\mathbf{n}}\delta\Gamma &= \widetilde{J\mathcal{F}}^{-T}\hat{\mathbf{N}}d\Gamma_o \\ \delta\Omega &= \tilde{J}d\Omega_o \end{aligned}$$

and their time rate can be calculated as

$$\begin{aligned} \frac{d}{dt}(\hat{\mathbf{n}}\delta\Gamma) &= \left[\dot{\widetilde{J}}\widetilde{\mathcal{F}}^{-T} + \tilde{J}\dot{\widetilde{\mathcal{F}}}^{-T}\right]\hat{\mathbf{N}}d\Gamma_o \\ \frac{d}{dt}(\delta\Omega) &= \operatorname{div}\mathbf{v}\delta\Omega \end{aligned}$$

where

$$\begin{aligned} \hat{\mathbf{N}}d\Gamma_o &= d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} \\ d\Omega_o &= \left(d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)}\right)^T d\mathbf{X}^{(3)} \end{aligned}$$

and

$$\begin{aligned} \hat{\mathbf{n}}\delta\Gamma &= \delta\mathbf{x}^{(1)} \times \delta\mathbf{x}^{(2)} \\ \delta\Omega &= \left(\delta\mathbf{x}^{(1)} \times \delta\mathbf{x}^{(2)}\right)^T \delta\mathbf{x}^{(3)} \end{aligned}$$

Under the hypothesis of small deformation, the above expressions reduce to

$$\hat{\mathbf{n}}\delta\Gamma \approx \hat{\mathbf{N}}d\Gamma_o \tag{6.102}$$

$$\delta\Omega \approx d\Omega_o \tag{6.103}$$

and

$$\frac{d}{dt}(\hat{\mathbf{n}}\delta\Gamma) \approx \left[(\operatorname{div}\mathbf{v})\mathbf{I} + \mathcal{V}^{-T}\right]\hat{\mathbf{N}}d\Gamma_o \tag{6.104}$$

$$\frac{d}{dt}(\delta\Omega) = \operatorname{div}\mathbf{v}\delta\Omega \tag{6.105}$$

since

$$\begin{aligned} \tilde{J} &\approx 1 \\ \tilde{\mathcal{F}} &\approx \mathbf{I} \\ \dot{\tilde{J}} &= \tilde{J}\operatorname{div}\mathbf{v} \approx \operatorname{div}\mathbf{v} \\ \dot{\tilde{\mathcal{F}}} &= \mathcal{V} \end{aligned}$$

It is important to underline again that under the hypothesis of small deformation, although describing a deformation process where the initial and the current configurations are nearly equal, still the difference between any geometrical quantities of these two configurations is not null. In fact, it can be shown that the volumetric strain and its rate of change, respectively defined as

$$\epsilon_v = \frac{\delta\Omega - d\Omega_o}{d\Omega_o} \tag{6.106}$$

$$\dot{\epsilon}_v = \frac{d\epsilon_v}{dt} \tag{6.107}$$

result under the hypothesis of small deformation to be respectively equal to

$$\epsilon_v = \mathcal{L}_{11} + \mathcal{L}_{22} + \mathcal{L}_3 = I_1^{(\mathcal{L})} \tag{6.108}$$

$$\dot{\epsilon}_v = \dot{\mathcal{L}}_{11} + \dot{\mathcal{L}}_{22} + \dot{\mathcal{L}}_3 = I_1^{(\mathcal{V})} = \operatorname{div}\mathbf{v} \tag{6.109}$$

In fact, let us indicate

$$\begin{aligned} \delta v &= \frac{\delta\Omega}{d\Omega_o} \\ \epsilon_v &= \delta v - 1 \end{aligned}$$

According to Eqs. 6.20 and 6.27, we have that in finite deformation

$$\delta v = \widetilde{J} = \det \widetilde{\mathcal{F}} = \left(I_3^{(\widetilde{\mathcal{G}})}\right)^{\frac{1}{2}} = \left(1 + 2I_1^{(\widehat{\mathcal{L}})} + 4I_2^{(\widehat{\mathcal{L}})} + 8I_3^{(\widehat{\mathcal{L}})}\right)^{\frac{1}{2}}$$

from which we can establish that

$$(\delta v^2 - 1) = (\delta v + 1)(\delta v - 1) = 2I_1^{(\widehat{\mathcal{L}})} + 4I_2^{(\widehat{\mathcal{L}})} + 8I_3^{(\widehat{\mathcal{L}})}$$

In the hypothesis of small displacement, this expression reduces to

$$(\delta v - 1) = I_1^{(\mathcal{L})}$$

since $\delta v \approx 1$, $I_1^{(\widehat{\mathcal{L}})} \gg I_2^{(\widehat{\mathcal{L}})} \gg I_3^{(\widehat{\mathcal{L}})}$ and $I_1^{(\widehat{\mathcal{L}})} \approx I_1^{(\mathcal{L})}$.

6.6.4 Remarks

The relationships presented in Section 6.6.2 indicate that the linear Lagrange strain tensor $\mathcal{L}$ governs the relationships between deformed and undeformed geometrical quantities.

In a CaORS, the linear Lagrangian tensor $\mathcal{L}$ is a symmetric tensor and, consequently, it has always three real principal values

$$\mathcal{L}_1 \geq \mathcal{L}_2 \geq \mathcal{L}_3 \tag{6.110}$$

and three orthogonal principal directions $\mathbf{n}^{(i)}$, [38] Theorem 7.3.1. It is immediate to verify that the elements $\mathcal{L}_{ij}$ of $\mathcal{L}$ are proportial to the deformation process of the vectors $\delta\mathbf{X}$ oriented as the Cartesian axes. In fact, let

$$\begin{aligned} d\mathbf{X}^{(1)} &= \{1, 0, 0\}^T \\ d\mathbf{X}^{(2)} &= \{0, 1, 0\}^T \end{aligned}$$

be two unit vectors oriented as the Cartesian axis x_1 and x_2. Denote by $\delta\mathbf{x}^{(1)}$ and $\delta\mathbf{x}^{(2)}$ the respectively deformed vectors. Then, according to the relationships reported in Section 6.6.2,

$$\begin{aligned}
\epsilon_1 &= \frac{|\delta\mathbf{x}^{(1)}| - |d\mathbf{X}^{(1)}|}{|d\mathbf{X}^{(1)}|} = \mathcal{L}_{11} \\
\epsilon_2 &= \frac{|\delta\mathbf{x}^{(2)}| - |d\mathbf{X}^{(2)}|}{|d\mathbf{X}^{(2)}|} = \mathcal{L}_{22} \\
\gamma_{12} &= \text{angle}\,(\delta\mathbf{x}^{(1)}, \delta\mathbf{x}^{(1)}) = 2\mathcal{L}_{12}
\end{aligned}$$

According to Eq. 6.99, the (current) relative displacement of two neighboring points at a distance $d\mathbf{X}$ is given by

$$\delta\mathbf{u} = \mathcal{U}d\mathbf{X} \tag{6.111}$$

where $\mathcal{U}$ is the Lagrange displacement gradient tensor. This tensor $\mathcal{U}$ may always be decomposed into, Eq. 3.3,

$$\mathcal{U} = \mathcal{L} + \mathbf{\Omega} \tag{6.112}$$

where $\mathcal{L}$ is the symmetric linear Lagrange strain tensor and $\mathbf{\Omega}$ is the skew-symmetric linear Lagrange rotation tensor, Section 6.6.2. Consequently, the relative displacement can be alternatively expressed as

$$\delta\mathbf{u} = \mathcal{U}d\mathbf{x} = \delta\mathbf{u}^{(s)} + \delta\mathbf{u}^{(r)} \tag{6.113}$$

where

$$\begin{aligned}
\delta\mathbf{u}^{(s)} &= \mathcal{L}d\mathbf{x} \\
\delta\mathbf{u}^{(r)} &= \mathbf{\Omega}d\mathbf{x} = \boldsymbol{\omega} \times d\mathbf{x}
\end{aligned}$$

and $\boldsymbol{\omega}$, known as the *rigid rotation vector*, has components defined as, Eq. 3.5,

$$\omega_i = -\frac{1}{2}\epsilon_{ijk}\Omega_{jk}$$

From the strain relationships reported in Section 6.6.2, we can immediately verify that if $\mathcal{L}$ is equal to zero, there is no deformation. Conversely, the deformation is not influenced by $\mathbf{\Omega}$. Hence, we can state that $\mathcal{L}$ is the responsible of the whole deformation process while $\mathbf{\Omega}$ is the responsible of a rigid rotational displacement field.

The differentiation in time of Eq. 6.113 yields to

$$\delta\mathbf{v} = \mathcal{V}\mathbf{x} = \delta\mathbf{v}^{(s)} + \delta\mathbf{v}^{(r)} \tag{6.114}$$

where

$$\begin{aligned}\delta\mathbf{v}^{(s)} &= \mathcal{D}d\mathbf{x}\\ \delta\mathbf{v}^{(r)} &= \mathcal{W}d\mathbf{x} = \mathbf{w}\times d\mathbf{x}\end{aligned}$$

and $\mathbf{w}$, known as the *vorticity vector*, has components defined as

$$\omega_i = -\frac{1}{2}\epsilon_{ijk}\mathcal{W}_{jk}$$

We recognize that the velocity decomposition in Eq. 6.114 has the same definition and physical meaning of that in Eq. 6.58.

6.7 Compatibility Conditions for Strains

We have stated in Section 6.2 that the Continuum Mechanics Theory requires that the three displacement functions in $\mathbf{u}$ must be single-valued and continuous functions of the coordinates. We can rephrase this statement by saying that any arbitrary set of three displacement functions $\mathbf{u}$ can describe a continuous deformation process provided that $\mathbf{u}$ satisfy the requirements of being single-valued and continuous functions of the coordinate. Under these hypotheses the strain tensors are uniquely defined.

Suppose instead that the strain components are given explicitly as functions of the coordinates. Then, the question that may arise is the following: what are the requirements that an arbitrary set of strain component functions must satisfy in order to represent a continuous deformation process? In other terms, what are the requirements that these strain functions must satisfy so that there can exist a unique set of three single-valued and continuous displacement functions? The answer to these questions is provided by the so-called *compatibility conditions* for strain.

If the finite Lagrange strain tensor component $\tilde{\mathcal{L}}_{ij}$ is given explicitly as function of the coordinates, the search for the corresponding three displacement functions u_i requires the contemporaneous solution of the following six independent nonlinear partial differential equations

$$\tilde{\mathcal{L}}_{ij} = \frac{1}{2}(u_{i,J} + u_{j,I} + u_{k,I}u_{k,J}) \qquad (6.115)$$

The system is over-determined and will not, in general, possess a solution for an arbitrary choice of $\tilde{\mathcal{L}}_{ij}$ functions. Therefore, the existence of single-valued continuous solutions u_i must be conditioned by the existence of some relationship among the $\tilde{\mathcal{L}}_{ij}$ functions.

For our purpose it is sufficient to limit the investigation to the linear Lagrange strain tensor

$$\mathcal{L}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{6.116}$$

It can be proved that the compatibility conditions for $\mathcal{L}$ are given by

$$\nabla_x \times \mathcal{L} \times \nabla_x = \mathbf{o} \tag{6.117}$$

that is

$$\begin{aligned}
R_3 &= \mathcal{L}_{11,22} + \mathcal{L}_{22,11} - 2\mathcal{L}_{12,12} = 0 \\
R_1 &= \mathcal{L}_{22,33} + \mathcal{L}_{33,22} - 2\mathcal{L}_{23,23} = 0 \\
R_2 &= \mathcal{L}_{33,11} + \mathcal{L}_{11,33} - 2\mathcal{L}_{13,13} = 0 \\
S_1 &= -\mathcal{L}_{11,23} + (\mathcal{L}_{12,3} + \mathcal{L}_{13,2} - \mathcal{L}_{23,1})_{,1} = 0 \\
S_2 &= -\mathcal{L}_{22,13} + (\mathcal{L}_{12,3} - \mathcal{L}_{13,2} + \mathcal{L}_{23,1})_{,2} = 0 \\
S_3 &= -\mathcal{L}_{33,12} + (-\mathcal{L}_{12,3} + \mathcal{L}_{13,2} + \mathcal{L}_{23,1})_{,3} = 0
\end{aligned}$$

also known as the *St. Venant compatibility equations.* These six equations represent only three independent conditions, since the *incompatible components* $R_1, R_2, R_3, S_1, S_2, S_3$ (which vanish when the strain are compatible) satisfy the following identities,

$$\begin{aligned}
R_{1,1} + S_{3,2} + S_{2,3} &= 0 \\
S_{3,1} + R_{2,2} + S_{1,3} &= 0 \\
S_{2,1} + R_{1,2} + R_{3,3} &= 0
\end{aligned}$$

known as the *Bianchi formulas.*

The St. Venant compatibility conditions are determined by differentiation of Eq. 6.116. This yields to

$$\mathcal{L}_{ij,kl} = \frac{1}{2}(u_{i,jkl} + u_{j,ikl})$$

Interchanging the subscripts we have

$$\begin{aligned}
\mathcal{L}_{kl,ij} &= \frac{1}{2}(u_{k,lij} + u_{l,kij}) \\
\mathcal{L}_{jl,ik} &= \frac{1}{2}(u_{j,lki} + u_{l,jik}) \\
\mathcal{L}_{ik,jl} &= \frac{1}{2}(u_{i,kjl} + u_{k,ijl})
\end{aligned}$$

from which we can verify at once that

$$\mathcal{L}_{ij,kl} + \mathcal{L}_{kl,ij} - \mathcal{L}_{jl,ik} - \mathcal{L}_{ik,jl} = 0$$

Since there are four indeces, this system represents $3^4 = 81$ equations. However, only the six St. Venant's equations above reported are distinct while all the others are either identities or repetitions on account of the symmetry of $\mathcal{L}_{ij}$ and kl in $\mathcal{L}_{ij,kl}$.

It can be proved that, see for example [62] p. 188, the compatibility conditions are necessary and sufficient conditions for the *existence* of single-valued displacements in a simply connected body, but *they do not ensure uniqueness* of the displacement distribution for given strains. Indeed the displacements are not unique, since we can always superimpose a rigid-body motion, which changes the displacement and rotation but not the strains.

6.8 Change of Reference System

Let $F = Ox_1x_2x_3$ and $F = O'x'_1x'_2x'_3$ be two CaORS in relative motion and indicate their relative law of motion as, Section 4.3,

$$\mathbf{x}' = \mathbf{c} + \mathbf{R}\mathbf{x} \tag{6.118}$$

$$\dot{\mathbf{x}}' = \dot{\mathbf{c}} + \mathbf{W}(\mathbf{x}' - \mathbf{c}) + \mathbf{R}\dot{\mathbf{x}} = \dot{\mathbf{c}} + \dot{\mathbf{R}}\mathbf{x} + \mathbf{R}\dot{\mathbf{x}} \tag{6.119}$$

where

$$\mathbf{R} = \left[\frac{\partial x'_i}{\partial x_j}\right] \tag{6.120}$$

$$\mathbf{R}^T = \mathbf{R}^{-1} = \left[\frac{\partial x_i}{\partial x'_j}\right] \tag{6.121}$$

$$\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T \tag{6.122}$$

and all terms involved are function of time. Let us denote by

$$\mathbf{X}' = \mathbf{x}(t = t_o); \qquad \mathbf{X} = \mathbf{x}(t = t_o) \tag{6.123}$$

$$\mathbf{R}^{(o)} = \left[\frac{\partial X'_i}{\partial X_j}\right]; \qquad \mathbf{R}^{(o)T} = \left[\frac{\partial X_i}{\partial X'_j}\right] \tag{6.124}$$

the terms relative to some initial $t_o = t$.

Table 6.1: Strain tensor transformation under a change of reference frame

Tensors	Tensor transformation		
	General rule	For $\mathbf{R}^{(o)} = \mathbf{I}$	For $\mathbf{R}$ =const.= $\mathbf{R}^{(o)}$
Linear vector transformation $\mathbf{T}'$	$\mathbf{RTR}^T$	$\mathbf{RTR}^T$	$\mathbf{RTR}^T$
Lagrange deformation gradient $\tilde{\mathcal{F}}'$	$\mathbf{R}\tilde{\mathcal{F}}\mathbf{R}^{(o)T}$	$\mathbf{R}\tilde{\mathcal{F}}$	$\mathbf{R}\tilde{\mathcal{F}}\mathbf{R}^T$
Lagrange strain tensor $\tilde{\mathcal{L}}'$	$\mathbf{R}^{(o)}\tilde{\mathcal{L}}\mathbf{R}^{(o)T}$	$\tilde{\mathcal{L}}$	$\mathbf{R}\tilde{\mathcal{L}}\mathbf{R}^T$
Lagrange strain rate tensor $\dot{\tilde{\mathcal{L}}}'$	$\mathbf{R}^{(o)}\dot{\tilde{\mathcal{L}}}\mathbf{R}^{(o)T}$	$\dot{\tilde{\mathcal{L}}}$	$\mathbf{R}\dot{\tilde{\mathcal{L}}}\mathbf{R}^T$
Euler velocity gradient $\hat{\mathcal{V}}'$	$\mathbf{W} + \mathbf{R}\hat{\mathcal{V}}\mathbf{R}^T$	$\mathbf{W} + \mathbf{R}\hat{\mathcal{V}}\mathbf{R}^T$	$\mathbf{R}\hat{\mathcal{V}}\mathbf{R}^T$
Euler strain rate tensor $\hat{\mathcal{D}}'$	$\mathbf{R}\hat{\mathcal{D}}\mathbf{R}^T$	$\mathbf{R}\hat{\mathcal{D}}\mathbf{R}^T$	$\mathbf{R}\hat{\mathcal{D}}\mathbf{R}^T$
Euler vorticity tensor $\widehat{\mathcal{W}}'$	$\mathbf{W} + \mathbf{R}\widehat{\mathcal{W}}\mathbf{R}^T$	$\mathbf{W} + \mathbf{R}\widehat{\mathcal{W}}\mathbf{R}^T$	$\mathbf{R}\widehat{\mathcal{W}}\mathbf{R}^T$

N.B.: $\mathbf{R} = \mathbf{R}(t)$; $\mathbf{R}^{(o)} = \mathbf{R}(t=0)$ and $\mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T$

Then, it can be proved that the Lagrange and Euler tensors can be transformed from F to F' according to the rule reported in Table 6.1. In particular, we are interested in the two types of change of reference reported in the last two columns:

- The third column reports the specialization of the general rule to the case of transformations where at $t = t_o$ the CaORF coincide.
- The fourth column reports the specialization of the general rule to the case of transformations where the reference systems do not rotate one with respect to the other. These types of transformations include the case of inertial reference systems, Section 4.4.

The proof of these transformation rules are reported as follows. Consider the linear transformation of the type

$$\mathbf{b} = \mathbf{T}\mathbf{a} \tag{6.125}$$

where $\mathbf{a}$ and $\mathbf{b}$ are two vectors identified in F as

$$\begin{aligned} \mathbf{a} &= \mathbf{x}^{(2)} - \mathbf{x}^{(1)} \\ \mathbf{b} &= \mathbf{y}^{(2)} - \mathbf{y}^{(1)} \end{aligned}$$

In F' the same linear transformation takes on the form

$$\mathbf{b}' = \mathbf{T}'\mathbf{a}' \tag{6.126}$$

where

$$\begin{aligned} \mathbf{a}' &= \mathbf{x}'^{(2)} - \mathbf{x}'^{(1)} \\ \mathbf{b}' &= \mathbf{y}'^{(2)} - \mathbf{y}'^{(1)} \end{aligned}$$

According to Eq. 6.118,

$$\begin{aligned} \mathbf{a}' &= \mathbf{x}'^{(2)} - \mathbf{x}'^{(1)} = \mathbf{R}\left(\mathbf{x}^{(2)} - \mathbf{x}^{(1)}\right) = \mathbf{R}\mathbf{a} \\ \mathbf{b}' &= \mathbf{y}'^{(2)} - \mathbf{y}'^{(1)} = \mathbf{R}\left(\mathbf{y}^{(2)} - \mathbf{y}^{(1)}\right) = \mathbf{R}\mathbf{b} \end{aligned}$$

Then,

$$\mathbf{b}' = \mathbf{R}\mathbf{b} = \mathbf{R}\mathbf{T}\mathbf{a} = \mathbf{R}\mathbf{T}\mathbf{R}^T\mathbf{a}'$$

and, equalizing this expression for $\mathbf{b}'$ with that in Eq. 6.126, we obtain

$$\left(\mathbf{T}' - \mathbf{R}\mathbf{T}\mathbf{R}^T\right)\mathbf{a}' = \mathbf{o}$$

for all $\mathbf{a}'$. It follows

$$\mathbf{T}' = \mathbf{R}\mathbf{T}\mathbf{R}^T \tag{6.127}$$

The transformation rule for the Lagrange deformation gradient can be determined according to

$$\tilde{\mathcal{F}}'_{iJ} = \frac{\partial x'_i}{\partial X_J} = \frac{\partial x'_i}{\partial x_s}\frac{\partial x_s}{\partial X_T}\frac{\partial X_T}{\partial X'_J}$$

which, taking into account Eqs. 6.120 and 6.124, can be written as

$$\tilde{\mathcal{F}}' = \mathbf{R}\tilde{\mathcal{F}}\mathbf{R}^{(o)T} \tag{6.128}$$

Consequently, the transformation rule for the Green and finite Lagrange strain tensors can be established as

$$\begin{aligned}
\tilde{\mathcal{G}}' &= \tilde{\mathcal{F}}'^T\tilde{\mathcal{F}}' = (\mathbf{R}^{(o)}\tilde{\mathcal{F}}^T\mathbf{R}^T)(\mathbf{R}\tilde{\mathcal{F}}\mathbf{R}^{(o)T}) = \mathbf{R}^{(o)}\tilde{\mathcal{G}}\mathbf{R}^{(o)T} && (6.129)\\
\tilde{\mathcal{L}}' &= \frac{1}{2}(\mathcal{G}' - \mathbf{I}) = \frac{1}{2}(\mathbf{R}^{(o)}\tilde{\mathcal{G}}\mathbf{R}^{(o)T} - \mathbf{R}^{(o)}\mathbf{R}^{(o)T}) = \\
&= \frac{\mathbf{R}^{(o)}}{2}(\tilde{\mathcal{G}} - \mathbf{I})\mathbf{R}^{(o)T} = \mathbf{R}^{(o)}\tilde{\mathcal{L}}\mathbf{R}^{(o)T} && (6.130)\\
\dot{\tilde{\mathcal{L}}}' &= \mathbf{R}^{(o)}\dot{\tilde{\mathcal{L}}}\mathbf{R}^{(o)T} && (6.131)
\end{aligned}$$

The transformation rule for the Euler velocity gradient $\hat{\mathcal{V}}$ can be determined as, Eq. 6.119,

$$\hat{\mathcal{V}}'_{ij} = \frac{\partial \dot{x}'_i}{\partial x'_j} = \frac{\partial c_i}{\partial x'_j} + \dot{R}_{ik}\frac{\partial x_k}{\partial x'_j} + R_{ik}\frac{\partial \dot{x}_k}{\partial x_s}\frac{\partial x_s}{\partial x'_j}$$

which, taking into account Eqs. 6.121 and 6.122, and that

$$\frac{\partial \dot{c}_i}{\partial x'_j} = 0$$

being $\dot{c}_i$ independent on x'_j, it can be written as

$$\hat{\mathcal{V}}' = \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\hat{\mathcal{V}}\mathbf{R}^T = \mathbf{W} + \mathbf{R}\hat{\mathcal{V}}\mathbf{R}^T \tag{6.132}$$

The transformation rules for the Euler strain rate and vorticity tensors can be immediately proved taking into account that according Eq. 6.132

$$\begin{aligned}
\hat{\mathcal{V}}' &= \dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\hat{\mathcal{V}}\mathbf{R}^T\\
\hat{\mathcal{V}}'^T &= \mathbf{R}\dot{\mathbf{R}}^T + \mathbf{R}\hat{\mathcal{V}}^T\mathbf{R}^T
\end{aligned}$$

and

$$\dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T$$

Consequently,

$$\begin{aligned}
\widehat{\mathcal{D}}' &= \frac{1}{2}\left(\hat{\mathcal{V}}' + \hat{\mathcal{V}}'^T\right) = \frac{1}{2}\left[\left(\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\hat{\mathcal{V}}\mathbf{R}^T\right) + \left(\mathbf{R}\dot{\mathbf{R}}^T + \mathbf{R}\hat{\mathcal{V}}^T\mathbf{R}^T\right)\right] \\
&= \frac{1}{2}\mathbf{R}\left[\hat{\mathcal{V}} + \hat{\mathcal{V}}^T\right]\mathbf{R}^T = \mathbf{R}\widehat{\mathcal{D}}\mathbf{R}^T \qquad (6.133) \\
\widehat{\mathcal{W}}' &= \frac{1}{2}\left(\hat{\mathcal{V}}' - \hat{\mathcal{V}}'^T\right) = \frac{1}{2}\left[\left(\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\hat{\mathcal{V}}\mathbf{R}^T\right) - \left(\mathbf{R}\dot{\mathbf{R}}^T + \mathbf{R}\hat{\mathcal{V}}^T\mathbf{R}^T\right)\right] = \\
&= \frac{1}{2}\left[\dot{\mathbf{R}}\mathbf{R}^T - \mathbf{R}\dot{\mathbf{R}}^T\right] + \frac{1}{2}\mathbf{R}\left[\hat{\mathcal{V}} - \hat{\mathcal{V}}^T\right]\mathbf{R}^T = \\
&= \mathbf{W} + \mathbf{R}\widehat{\mathcal{W}}\mathbf{R}^T \qquad (6.134)
\end{aligned}$$

Finally, it is of interest to remark that from the vorticity transformation rule we can eventually find that

$$\dot{\mathbf{R}} = \widehat{\mathcal{W}}'\mathbf{R} - \mathbf{R}\widehat{\mathcal{W}} \qquad (6.135)$$

and also, being $\widehat{\mathcal{W}}'$ and $\widehat{\mathcal{W}}$ both skew-symmetric matrices,

$$\dot{\mathbf{R}}^T = -\mathbf{R}^T\widehat{\mathcal{W}}' + \widehat{\mathcal{W}}\mathbf{R}^T \qquad (6.136)$$

In fact, multiplying Eq. 6.134 by $\mathbf{R}$ on the right,

$$\widehat{\mathcal{W}}'\mathbf{R} = \dot{\mathbf{R}}\mathbf{R}^T\mathbf{R} + \mathbf{R}\widehat{\mathcal{W}}\mathbf{R}^T\mathbf{R} = \dot{\mathbf{R}} + \mathbf{R}\widehat{\mathcal{W}}$$

6.9 Strain Invariant Definitions

We have already observed that in a CaORS all the following strain tensors are represented by symmetric matrices:

- Strain tensors in the material description $\tilde{\mathcal{G}}, \tilde{\mathcal{L}}, \dot{\tilde{\mathcal{G}}}$ and $\dot{\tilde{\mathcal{L}}}$.
- Strain tensors in the spatial description $\hat{\mathcal{C}}, \hat{\mathcal{E}}, \widehat{\mathcal{D}}, \dot{\hat{\mathcal{C}}}$ and $\dot{\hat{\mathcal{E}}}$.
- Strain tensors in the small deformation theory $\mathcal{G}, \mathcal{L}, \mathcal{D}, \dot{\mathcal{G}}$ and $\dot{\mathcal{L}}$.

Then, each of these tensors has three real principal values and three orthogonal principal directions. The three principal values may be determined according to the analytical solution reported in Section 3.7 and their geometrical interpretation may be found in Section 3.8.

In particular, let us indicate any one of the above listed strain tensors as

$$\mathbf{L} = [L_{ij}] \tag{6.137}$$

and the relative characteristic equations as, Section 3.5,

$$l^3 - I_1 l^2 + I_2 l - I_3 = 0 \tag{6.138}$$

where I_i and l_i, for $i = 1, 2, 3$, are the strain invariants and principal values, respectively. Associated with $\mathbf{L}$ we can define the *deviatoric strain tensor*, Section 3.6,

$$\mathbf{E} = \mathbf{L} - \mathbf{I}\frac{I_1}{3} \tag{6.139}$$

The relative characteristic equations is given by, Section 3.6,

$$e^3 - J_2 e - J_3 = 0 \tag{6.140}$$

where J_i and e_i, for $i = 1, 2, 3$, are the deviatoric strain invariants and principal values, respectively.

An alternative definition of the strain invariants is given by

$$\epsilon_v = I_1 \tag{6.141}$$

$$\epsilon_s = \frac{2}{3}(3J_2)^{1/2} \tag{6.142}$$

$$\epsilon_\theta = \frac{1}{3}\arcsin\left(-\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right) \tag{6.143}$$

which, clearly, are strictly related to the octahedral components of $\mathbf{L}$, Section 3.8. Usually, ϵ_v is called *volumetric strain invariant* and ϵ_s is called *shear strain invariant*. Strictly speaking, these names find a physical justification only in the case of the small deformation in which $\mathbf{L} \equiv \boldsymbol{\mathcal{E}}$ or $\mathbf{L} \equiv \dot{\boldsymbol{\mathcal{E}}}$.

6.10 Strain Vector Definition

For each of the symmetric strain tensors listed in Section 6.9, we can define a *strain vector* and a relative *deviatoric strain vector* as

$$\boldsymbol{\epsilon} = \{L_{11}, L_{22}, L_{33}, L_{12}, L_{13}, L_{21}, L_{23}, L_{31}, L_{32}\}^T \tag{6.144}$$

$$\mathbf{e} = \{E_{11}, E_{22}, E_{12}, E_{12}, E_{13}, E_{21}, E_{23}, E_{31}, E_{32}\}^T \tag{6.145}$$

According to this definition, we have that

$$\boldsymbol{\epsilon} = \mathbf{e} + \widehat{\mathbf{m}}\frac{\epsilon_v}{3} \tag{6.146}$$

where

$$\widehat{\mathbf{m}} = \{1, 1, 1, 0, 0, 0, 0, 0, 0\}^T$$

and the strain invariants can be indicated as the results of the following vector operations

$$I_1 = \widehat{\mathbf{m}}^T \boldsymbol{\epsilon} \tag{6.147}$$

$$J_2 = \frac{1}{2}\mathbf{e}^T\mathbf{e} \tag{6.148}$$

and

$$\epsilon_v = \widehat{\mathbf{m}}^T \boldsymbol{\epsilon} \tag{6.149}$$

$$\epsilon_s = \left(\frac{2}{3}\mathbf{e}^T\mathbf{e}\right)^{\frac{1}{2}} \tag{6.150}$$

Moreover, the tensor operations

$$\epsilon^2 = \mathbf{L} : \mathbf{L}$$

can be indicated as

$$\epsilon^2 = \boldsymbol{\epsilon}^T\boldsymbol{\epsilon} = \mathbf{e}^T\mathbf{e} + \frac{1}{3}\epsilon_v^2 = \frac{3}{2}\epsilon_s^2 + \frac{1}{3}\epsilon_v^2 \tag{6.151}$$

Chapter 7

Force and Stress in a Continuum

7.1 Introduction

In Continuum Mechanics the term *free body* denotes a portion of continuum instantaneously bounded by an arbitrary closed surface. This closed surface can consist of part of an actual bounding surface of the medium, but it may be wholly or in part an imaginary surface within the medium. According to this definition a *particle* is the smallest free body in a continuum.

Let $\Delta\mathcal{M}$ be a free body of $\mathcal{M}$ of volume $\Delta\Omega$ and mass Δm. The quantity

$$\varrho_{av} = \frac{\Delta m}{\Delta\Omega} \tag{7.1}$$

is defined as the *average density* of the free body $\Delta\mathcal{M}$. Accordingly, the density of a particle results to be given by

$$\varrho(\mathbf{x}, t) = \lim_{\Delta\Omega \to 0} \frac{\Delta m}{\Delta\Omega} = \frac{dm}{d\Omega} \tag{7.2}$$

where $d\Omega$ is the infinitesimal volume of the particle currently located at $\mathbf{x}$, Fig. 7.1.

Hence, in Continuum Mechanics the mass of a particle dP of a continuum $\mathcal{M}$ results to be equal to

$$dm = \varrho(\mathbf{x}, t)\, d\Omega \tag{7.3}$$

where the *mass density*

$$\varrho = \varrho(\mathbf{x}, t)$$

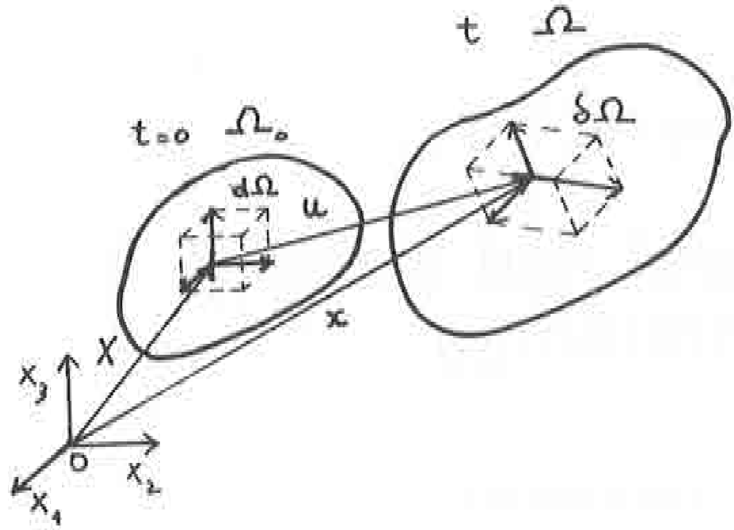


Figure 7.1: Spatial domain at two distinct times

is a continuous function on the spatial and time domain of $\mathcal{M}$. Each particle dP of $\mathcal{M}$ is assumed to obey Newton's laws of motion, Section 4.5. Consequently, by paraphrasing all the mathematical procedures reported in Chapter 5 for the case of a system of interacting particles, we can eventually establish the mechanical principles governing a continuum.

7.2 Continuity Equation

In general, we can state that the mass of a material point dP in a continuum $\mathcal{M}$ at time $t = 0$ is equal to

$$dm_o = \varrho(\mathbf{X}, t = 0)\, d\Omega_o = \varrho_o\, d\Omega_o \tag{7.4}$$

and at time t is equal to

$$dm = \varrho(\mathbf{x}, t)\, d\Omega = \varrho\, d\Omega \tag{7.5}$$

where, Eq. 6.20,

$$d\Omega = \bar{J}\, d\Omega_o$$

One of the fundamental assumptions of the Continuum Mechanics Theory is the so-called *Principle of Conservation of Mass* according to which

$$dm = dm_o \tag{7.6}$$

that is

$$dm = \varrho\, d\Omega = \varrho_o\, d\Omega_o \tag{7.7}$$

From this principle it follows the *Continuity Equation*

$$\frac{dm}{dt} = \frac{d}{dt}(\varrho\, d\Omega) = 0 \tag{7.8}$$

and the following its equivalent forms:

- The *Lagrangian form* of the continuity equation

  $$\varrho_o = \varrho \tilde{J} \tag{7.9}$$

  or equivalently

  $$\frac{d}{dt}(\varrho \tilde{J}) = 0 \tag{7.10}$$

  which can be immediately derived from Eq. 7.7.

- The *Eulerian form* of the continuity equation

  $$\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = 0 \tag{7.11}$$

  or equivalently

  $$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{v}) = 0 \tag{7.12}$$

  which for homogeneous material may be specialized into

  $$\frac{\partial \varrho}{\partial t} + \varrho \operatorname{div} \mathbf{v} = 0 \tag{7.13}$$

  These forms can be obtained from Eq. 7.8 recalling that, Eq. 6.22

  $$\frac{1}{d\Omega}\frac{d}{dt}(d\Omega) = \operatorname{div} \mathbf{v}$$

  In fact,

  $$\begin{aligned} 0 &= \frac{dm}{dt} = \frac{d}{dt}(\varrho\, d\Omega) = \dot{\varrho}\, d\Omega + \varrho \frac{d}{dt}\, d\Omega = \\ &= \dot{\varrho}\, d\Omega + \varrho \operatorname{div} \mathbf{v}\, d\Omega = \left[\frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v}\right] d\Omega \end{aligned}$$

and

$$\begin{aligned} 0 &= \frac{d\varrho}{dt} + \varrho \operatorname{div} \mathbf{v} = \frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial x_s}\frac{\partial x_s}{\partial t} + \varrho \frac{\partial v_s}{\partial x_s} = \\ &= \frac{\partial \varrho}{\partial t} + \frac{\partial \varrho}{\partial x_s} v_s + \varrho \frac{\partial v_s}{\partial x_s} = \frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x_s}(\varrho \mathbf{v}) \end{aligned}$$

An important mathematical consequence of the continuity equation is given by

$$\frac{d}{dt}\left[\varrho T_{ij\ldots}(\mathbf{x},t)\, d\Omega\right] = \varrho \left[\dot{T}_{ij\ldots}(\mathbf{x},t)\right] d\Omega \tag{7.14}$$

where $T_{ij\ldots}(\mathbf{x},t)$ represents any physical quantity (per unit mass). In fact,

$$\begin{aligned} \frac{d}{dt}\left[\varrho T_{ij}\ldots(\mathbf{x},t)\, d\Omega\right] &= \dot{\varrho} T_{ij}\ldots d\Omega + \varrho \dot{T}_{ij}\ldots d\Omega + \varrho T_{ij}\ldots \frac{d}{dt} d\Omega = \\ &= \dot{\varrho} T_{ij}\ldots d\Omega + \varrho \dot{T}_{ij}\ldots d\Omega + \varrho T_{ij}\ldots \operatorname{div} \mathbf{v}\, d\Omega = \\ &= \left[\dot{\varrho} + \varrho \operatorname{div} \mathbf{v}\right] T_{ij}\ldots d\Omega + \varrho \dot{T}_{ij}\ldots d\Omega = \\ &= \varrho \frac{d}{dt}\left[T_{ij}\ldots(\mathbf{x},t)\right] d\Omega \end{aligned}$$

being

$$\dot{\varrho} + \varrho \operatorname{div} \mathbf{v} = 0$$

Finally, we notice that in an incompressible material

$$\frac{d\varrho}{dt} = 0 \tag{7.15}$$

from which we can establish that in an incompressible material

$$\frac{1}{d\Omega}\frac{d}{dt}(\,d\Omega) = \operatorname{div} \mathbf{v} = 0 \tag{7.16}$$

7.3 Kinetic Quantities

Let $\mathcal{M}$ be a continuum and dP be a particle of $\mathcal{M}$ of mass

$$dm = \varrho\, d\Omega$$

Analogously to the case of a single moving particle, Section 7.3, the kinetic quantities which can be associated with dP are, Fig. 7.2:

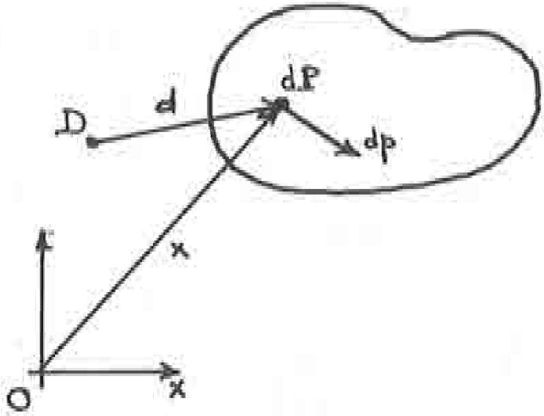


Figure 7.2: Kinetic quantities associated to a particle in a continuum

- The *momentum* defined as product between the mass and the velocity $\dot{\mathbf{x}}$ of P, namely
$$d\mathbf{p} = dm\,\dot{\mathbf{x}} = \varrho\dot{\mathbf{x}}\,d\Omega$$
- The *angular momentum* about any given point D defined as the moment of momentum of P,
$$d\gamma = \mathbf{d} \times d\mathbf{p} = \varrho(\mathbf{d} \times \dot{\mathbf{x}})\,d\Omega$$
- The *time rate of momentum* defined as the time derivative of the momentum, namely
$$\frac{d}{dt}(d\mathbf{p}) = \frac{d}{dt}(\varrho\dot{\mathbf{x}}\,d\Omega)$$
which, according to Eq. 7.14, can be alternatively expressed as
$$\frac{d}{dt}(d\mathbf{p}) = \varrho\ddot{\mathbf{x}}\,d\Omega$$
- The *time rate of angular momentum* about a given point D defined as the time derivative of the angular momentum, namely
$$\frac{d}{dt}(d\gamma) = \frac{d}{dt}[\varrho(\mathbf{d} \times \dot{\mathbf{x}})\,d\Omega]$$

which, according to Eq. 7.14, can be alternatively expressed as

$$\frac{d}{dt}(d\gamma) = \varrho\left[\frac{d}{dt}(\mathbf{d} \times \dot{\mathbf{x}})\right] d\Omega = \varrho(\dot{\mathbf{d}} \times \dot{\mathbf{x}} + \mathbf{d} \times \ddot{\mathbf{x}})\, d\Omega$$

- The *kinetic energy* defined as the scalar component of $\mathbf{p}$ in the direction of the motion $\dot{\mathbf{x}}$, namely

$$de^{(k)} = \frac{1}{2} d\mathbf{p}^T \dot{\mathbf{x}} = \frac{1}{2}\varrho \dot{x}^2\, d\Omega$$

Integrating these elemental quantities over the spatial domain Ω of the continuum $\mathcal{M}$ we obtain the *total mass*

$$m = \int_\Omega dm = \int_\Omega \varrho\, d\Omega \tag{7.17}$$

and the following total kinetic quantities:

- The *total momentum* defined as

$$\mathbf{p} = \int d\mathbf{p} = \int_\Omega \varrho \dot{\mathbf{x}}\, d\Omega$$

- The *total angular momentum* about a given point D defined as

$$\boldsymbol{\gamma} = \int d\boldsymbol{\gamma} = \int_\Omega \varrho(\mathbf{d} \times \dot{\mathbf{x}})\, d\Omega$$

- The *total time rate of momentum* defined as

$$\dot{\mathbf{p}} = \frac{d}{dt}\int d\mathbf{p} = \frac{d}{dt}\int_\Omega \varrho \dot{\mathbf{x}}^{(i)}\, d\Omega$$

which, according to Eq. 7.14, can be alternatively expressed as

$$\dot{\mathbf{p}} = \int_\Omega \varrho \ddot{\mathbf{x}}^{(i)}\, d\Omega$$

- The *total time rate of angular momentum* about a given point D defined as

$$\dot{\boldsymbol{\gamma}} = \frac{d}{dt}\int d\boldsymbol{\gamma} = \frac{d}{dt}\int_\Omega \varrho(\mathbf{d} \times \dot{\mathbf{x}})\, d\Omega$$

which, according to Eq. 7.14, can be alternatively expressed as

$$\dot{\boldsymbol{\gamma}} = \int_\Omega \varrho \frac{d}{dt}(\mathbf{d} \times \dot{\mathbf{x}})\, d\Omega = \int_\Omega \varrho(\dot{\mathbf{d}} \times \dot{\mathbf{x}} + \mathbf{d} \times \ddot{\mathbf{x}})\, d\Omega$$

- The *total kinetic energy* defined as

$$e^{(k)} = \int de^{(k)} = \frac{1}{2}\int_\Omega \varrho \dot{x}^2\, d\Omega$$

7.4 Center of Mass

Analogously to the case of a system of discrete interacting particles, the *center of mass* or *centroid* of a continuum is defined as that point C having position vector,

$$\mathbf{c} = \frac{\int_\Omega \varrho \mathbf{x} \, d\Omega}{\int_\Omega \varrho \, d\Omega} \tag{7.18}$$

It is possible to verify, with a mathematical procedure analogous to that reported in Section 5.3, that if $\boldsymbol{\xi}$ is the relative position of the particles in the continuum with respect to the center of mass C, so that

$$\mathbf{x} = \mathbf{c} + \boldsymbol{\xi}$$

then the following quantities are null

$$\int_\Omega \varrho \boldsymbol{\xi} \, d\Omega = \int_\Omega \varrho \dot{\boldsymbol{\xi}} \, d\Omega = \int_\Omega \varrho \ddot{\boldsymbol{\xi}} \, d\Omega = 0$$

Consequently, all the total kinetic quantities associated with a continuum defined in the previous Section 7.3, can be expressed as follows:

- Total momentum

$$\mathbf{p} = \int d\mathbf{p} = m\dot{\mathbf{c}}$$

- Total angular momentum about a given point D

$$\boldsymbol{\gamma} = \int d\boldsymbol{\gamma} = m(\mathbf{d} \times \dot{\mathbf{c}}) + \int_\Omega \varrho(\boldsymbol{\xi} \times \dot{\boldsymbol{\xi}}) \, d\Omega$$

- Total time rate of momentum

$$\dot{\mathbf{p}} = \frac{d}{dt} \int d\mathbf{p} = m\ddot{\mathbf{c}}$$

- Total time rate of angular momentum about a given point D defined as

$$\dot{\boldsymbol{\gamma}} = \frac{d}{dt} \int d\boldsymbol{\gamma} = m(\mathbf{d} \times \ddot{\mathbf{c}}) + m(\dot{\mathbf{d}} \times \dot{\mathbf{c}}) + \int_\Omega \varrho(\boldsymbol{\xi} \times \ddot{\boldsymbol{\xi}}) \, d\Omega$$

- Total kinetic energy as

$$e^{(k)} = \int de_t^k = \frac{1}{2} m\dot{c}^2 + \frac{1}{2} \int_\Omega \varrho \dot{\xi}^2 \, d\Omega$$

We remark that all the above relationships contain two distinct terms:

- the first one concerning the motion of the center of mass in which the total mass m of the continuum is supposed to be concentrated;
- the second one concerning the relative motion of the continuum with respect to the center of mass.

7.5 Mechanical Quantities

In the Continuum Mechanics Theory, forces may be classified as *external forces* acting on a free body or as *internal forces* acting between two parts of the free body. The external forces are then classified into two kinds, Fig. 7.3:

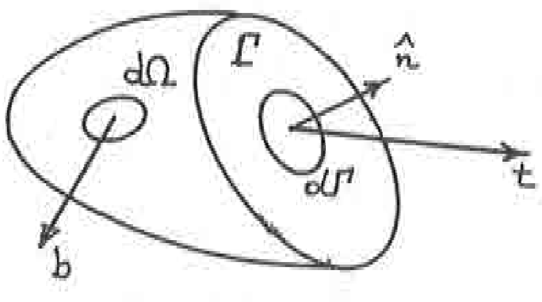


Figure 7.3: External forces acting on a free body.

- *surface forces* which are contact forces acting on the free body at its bounding surface; these forces are usually reckoned per unit area of the surface across which they act, namely

$$d\mathbf{f} = \mathbf{t}\, d\Gamma$$

where $\mathbf{t}$ is known as the *traction force* per unit area or simply *stress*.

- *body forces* which are action-at-a-distance forces acting on the elementary particles inside the free body; these forces are usually reckoned per unit mass, namely

$$d\mathbf{f} = \varrho \mathbf{b}\, d\Omega$$

where $\mathbf{b}$ is known as the *body force* per unit volume.

Thus, on a particle dP of a continuum $\mathcal{M}$ may act an elemental force and torque respectively equal to

$$d\mathbf{f} = \mathbf{t}\, d\Gamma + \varrho \mathbf{b}\, d\Omega \tag{7.19}$$

$$d\boldsymbol{\lambda} = \mathbf{d} \times d\mathbf{f} = (\mathbf{d} \times \mathbf{t})\, d\Gamma + \varrho(\mathbf{d} \times \mathbf{b})\, d\Omega \tag{7.20}$$

Internal forces have been defined as the forces acting between two parts of the free body. However, each of any two parts of the free body is a free body itself. Then, the internal force may be still considered as an external force which one of the parts of the free body exerts on the other part.

In Continuum Mechanics it is postulated that internal forces vary continuously in the medium. Moreover, it is also postulated the validity of the Newton's laws. Thus, if at every action corresponds an equal and opposite reaction, then, as for the case of a system of interacting particles in Section 5.5, the resultant of all internal forces and moments are null. Accordingly, the resultant of the force and torque fields acting on a continuum are respectively equal to the resultant of the external force and torque fields, so that

$$\mathbf{f} = \int d\mathbf{f} = \int_\Gamma \mathbf{t}\, d\Gamma + \int_\Omega \varrho \mathbf{b}\, d\Omega \tag{7.21}$$

$$\boldsymbol{\lambda} = \int d\boldsymbol{\lambda} = \int_\Gamma (\mathbf{d} \times \mathbf{t})\, d\Gamma + \int_\Omega \varrho(\mathbf{d} \times \mathbf{b})\, d\Omega \tag{7.22}$$

Moreover, since according to the 2nd Newton's law

$$d\mathbf{f} = \frac{d}{dt}(d\mathbf{p}) \tag{7.23}$$

it follows that the above defined total mechanical quantities are related to total kinetic quantities defined in the previous Sections 7.3 and 7.4 as follows

$$\mathbf{f} = \dot{\mathbf{p}} \tag{7.24}$$

$$\boldsymbol{\lambda} = \dot{\boldsymbol{\gamma}} - \int_\Omega \varrho(\dot{\mathbf{d}} \times \dot{\mathbf{x}})\, d\Omega \tag{7.25}$$

Equalizing the above alternative expressions for $\mathbf{f}$ and $\boldsymbol{\lambda}$ we obtain

$$\int_\Gamma \mathbf{t}\, d\Gamma + \int_\Omega \varrho \mathbf{b}\, d\Omega = \dot{\mathbf{p}} \tag{7.26}$$

$$\int_\Gamma (\mathbf{d} \times \mathbf{t})\, d\Gamma + \int_\Omega \varrho(\mathbf{d} \times \mathbf{b})\, d\Omega = \dot{\gamma} - \int_\Omega \varrho(\dot{\mathbf{d}} \times \dot{\mathbf{x}})\, d\Omega \tag{7.27}$$

where $\dot{\mathbf{p}}$ and $\dot{\gamma}$ can be equivalently expressed as

$$\begin{aligned}\dot{\mathbf{p}} &= \int_\Omega \varrho \ddot{\mathbf{x}}\, d\Omega = m\frac{d}{dt}(\dot{\mathbf{c}}) \\ \dot{\gamma} &= \int_\Omega \varrho(\dot{\mathbf{d}} \times \dot{\mathbf{x}} + \mathbf{d} \times \ddot{\mathbf{x}})\, d\Omega = \\ &= m(\mathbf{d} \times \ddot{\mathbf{c}}) + m(\dot{\mathbf{d}} \times \dot{\mathbf{c}}) + \int \varrho(\boldsymbol{\xi} \times \ddot{\boldsymbol{\xi}})\, d\Omega\end{aligned}$$

7.6 Mechanical Principles

From the relationships between the mechanical and kinetic quantities reported in Eqs. 7.26 and 7.26, we can establish the following principles governing the motion of a continuum.

Theorem 7.6.1 (Linear Momentum Principle). *The resultant external force acting on a continuum free body size equals the time rate of change of the total momentum of this continuum, that is*

$$\int_\Gamma \mathbf{t}\, d\Gamma + \int_\Omega \varrho \mathbf{b}\, d\Omega = \frac{d}{dt}\int_\Omega \varrho \dot{\mathbf{x}}\, d\Omega$$

or, equivalently,

$$\int_\Gamma \mathbf{t}\, d\Gamma + \int_\Omega \varrho \mathbf{b}\, d\Omega = \frac{d}{dt}(m\dot{\mathbf{c}})$$

where $\mathbf{c}$ *represents the location of the center of mass in which we may suppose to concentrate the total mass*

$$m = \int_\Omega \varrho\, d\Omega$$

Theorem 7.6.2 (Principle of Conservation of Momentum). *If the resultant external force acting on a continuum free body is zero, then the total momentum remains constant, that is*

$$\int_\Omega \varrho \dot{\mathbf{x}}\, d\Omega = const.$$

or, equivalently,

$$m\dot{\mathbf{c}} = const.$$

where $\mathbf{c}$ *represents the location of the center of mass in which we may suppose to concentrate the total mass*

$$m = \int_\Omega \varrho \, d\Omega$$

In such case, therefore, the center of mass preserves its state of rest or motion with constant velocity.

Theorem 7.6.3 (Angular Momentum Principle). *The total external torque acting on a continuum free body about the origin of the CaORS, or about the center of mass, equals the time rate of change of the angular moment, that is*

$$\int_\Gamma (\mathbf{x} \times \mathbf{t}) \, d\Gamma + \int_\Omega \varrho(\mathbf{x} \times \mathbf{b}) \, d\Omega = \frac{d}{dt} \int_\Omega \varrho(\mathbf{x} \times \dot{\mathbf{x}}) \, d\Omega$$

Theorem 7.6.4 (Principle of conservation of angular momentum). *If the total external torque acting on a given continuum free body calculated about the origin of the CaORS, or about the center of mass, is equal to zero, then the angular momentum remains constant, that is*

$$\int_\Omega \varrho(\mathbf{x} \times \dot{\mathbf{x}}) \, d\Omega = const$$

7.7 Cauchy Stress Tensor

In Section 7.3 we have defined the quantity

$$d\mathbf{f} = \mathbf{t} \, d\Gamma$$

as the force acting on an elemental surface $d\Gamma$ of a free body, Fig. 7.3. The traction force $\mathbf{t}$ has to be interpreted as a continuous quantity in the continuum whose value depends on the position and on the orientation of the surface $d\Gamma$. Then, consider a material point dP and let be:

- $\bar{\mathbf{e}}^{(i)} \, d\Gamma_i$, for $i = 1, 2, 3$ the three mutually orthogonal elemental areas at dP parallel to the Cartesian coordinate plane.

- $\mathbf{t}^{(i)}$ the traction force of vector components

$$\mathbf{t}^{(i)} = \{\sigma_{i1}, \sigma_{i2}, \sigma_{i3}\}^T$$

acting on the area $\hat{\mathbf{e}}^{(i)}\, d\Gamma_i$.

It is possible to prove that the traction force $\mathbf{t}$ acting at dP on a surface $\hat{\mathbf{n}}\, d\Gamma$ can be calculated as

$$\mathbf{t} = \mathcal{T}^T \hat{\mathbf{n}} \tag{7.28}$$

where

$$\mathcal{T} = \left[\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \mathbf{t}^{(3)}\right]^T = [\sigma_{ij}]$$

is known as the *Cauchy stress* tensor. In fact, consider the so-called Cauchy tetrahedron shown in Fig. 7.4. If $\hat{\mathbf{n}}\Delta\Gamma$ is the area of the base of this

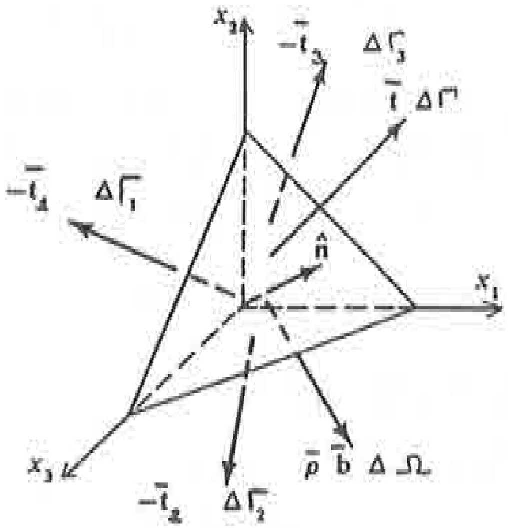


Figure 7.4: The Cauchy tetrahedron.

tetrahedron, where $\hat{\mathbf{n}}$ is the unit normal to this base, then:

- the area of the other three faces perpendicular to the coordinate axis can be calculated as

$$\Delta\Gamma_i = \hat{\mathbf{n}}^T \hat{\mathbf{e}}^{(i)} \Delta\Gamma = \hat{n}_i \Delta\Gamma \tag{7.29}$$

- The volume of the tetrahedron can be calculated as

$$\Delta\Omega = \frac{h}{3}\Delta\Gamma \tag{7.30}$$

where h is the distance of the vertex P from the base $\Delta\Gamma$.

According to the Linear Momentum Principle, Theorem 7.6.1,

$$\int_{\Gamma} \mathbf{t}\, d\Gamma = m\ddot{\mathbf{c}} - \int_{\Omega} \varrho \mathbf{b}\, d\Omega$$

from which, applying the Mean-value Theorem of the Integral Calculus to the integrals over the face and the volume of the tetrahedron, we obtain

$$\overline{\mathbf{t}}\Delta\Gamma - \sum_{i=1}^{3} \overline{\mathbf{t}}^{(i)}\Delta\Gamma_i = \overline{\varrho}(\ddot{\mathbf{c}} - \overline{\mathbf{b}})\Delta\Omega \tag{7.31}$$

where

- $\overline{\mathbf{t}}$ and $\overline{\mathbf{t}}^{(i)}$ are the average values of the traction forces over the base and over the other three faces of the tetrahedron, respectively.
- $\overline{\varrho}$ and $\overline{\mathbf{b}}$ are the average values of the mass density and of the body forces over the volume of the tetrahedron, respectively.

Expressing in $\Delta\Gamma_i$ and $\Delta\Omega$ in Eq. 7.31 as in Eqs. 7.29 and 7.30 and, then, dividing the resulting expressions by $\Delta\Gamma$, we obtain

$$\overline{\mathbf{t}} - \sum_{i=1}^{3} \overline{\mathbf{t}}^{(i)} n_i = \overline{\varrho}(\ddot{\mathbf{c}} - \overline{\mathbf{b}})\frac{h}{3} \tag{7.32}$$

The above equation is valid for any size of the tetrahedron and, in particular, for $h \to 0$ the tetrahedron reduces to the material point dP. In this case we have that

$$\mathbf{t} = \mathbf{t}^{(1)}\bar{n}_1 + \mathbf{t}^{(2)}\bar{n}_2 + \mathbf{t}^{(3)}\bar{n}_3 \tag{7.33}$$

proving therefore the tensor relationship in Eq. 7.28.

7.8 Piola-Kirchhoff Stress Tensors

We have defined the contact force as, Section 7.5,

$$d\mathbf{f} = \mathbf{t}\, d\Gamma \tag{7.34}$$

where the traction $\mathbf{t}$ may be calculated as, Eq. 7.28,

$$\mathbf{t} = \mathcal{T}^T \hat{\mathbf{n}}$$

and $\mathcal{T}$ is the Cauchy tensor while $\hat{\mathbf{n}}$ is the unit vector orthogonal to the area $d\Gamma$ defined in the current configuration. Alternatively, we can define $d\mathbf{f}$ in terms of undeformed area $\hat{\mathbf{N}}\, d\Gamma_o$ as follows:

- 1st Piola-Kirchhoff definition

$$d\mathbf{f} = \mathbf{t}^{(o)}\, d\Gamma_o \tag{7.35}$$

  where the traction $\mathbf{t}^{(o)}$ is defined as, Fig. 7.5,

$$\mathbf{t}^{(o)} = \mathcal{T}^{(o)T} \hat{\mathbf{N}}$$

  and $\mathcal{T}^{(o)}$ is the *1st Piola-Kirchhoff stress tensor* defined as

$$\mathcal{T}^{(o)} = \tilde{J} \tilde{\mathcal{F}}^{-1} \mathcal{T} = \frac{\varrho_o}{\varrho} \tilde{\mathcal{F}}^{-1} \mathcal{T}$$

- 2nd Piola-Kirchhoff definition

$$d\mathbf{f} = \tilde{\mathcal{F}} d\tilde{\mathbf{f}} = \tilde{\mathcal{F}} \tilde{\mathbf{t}}\, d\Gamma_o \tag{7.36}$$

  where the traction $\tilde{\mathbf{t}}$ is defined as, Fig. 7.5,

$$\tilde{\mathbf{t}} = \tilde{\mathcal{T}}^T \hat{\mathbf{N}}$$

  and $\tilde{\mathcal{T}}$ is the *2nd Piola-Kirchhoff stress tensor* defined as

$$\tilde{\mathcal{T}} = \tilde{J} \tilde{\mathcal{F}}^{-1} \mathcal{T} \tilde{\mathcal{F}}^{-T} = \frac{\varrho_o}{\varrho} \tilde{\mathcal{F}}^{-1} \mathcal{T} \tilde{\mathcal{F}}^{-T}$$

where $\tilde{\mathcal{F}}$ is Lagrange Deformation Gradient tensor, Section 6.3.1. Notice that for $\mathcal{T}$ symmetric tensor, $\mathcal{T}^{(o)}$ results to be a nonsymmetric tensor while $\tilde{\mathcal{T}}$ results to be a symmetric tensor. From the above definitions it follows that the Cauchy tensor can be alternatively expressed as

$$\mathcal{T} = \frac{\varrho}{\varrho_o} \tilde{\mathcal{F}} \mathcal{T}^{(o)} = \tilde{J}\tilde{\mathcal{F}}^{-1} \mathcal{T}^{(o)} \tag{7.37}$$

$$\mathcal{T} = \frac{\varrho}{\varrho_o} \tilde{\mathcal{F}} \tilde{\mathcal{T}} \tilde{\mathcal{F}}^T = \tilde{J}\tilde{\mathcal{F}}^{-1} \tilde{\mathcal{T}} \tilde{\mathcal{F}}^{-T} \tag{7.38}$$

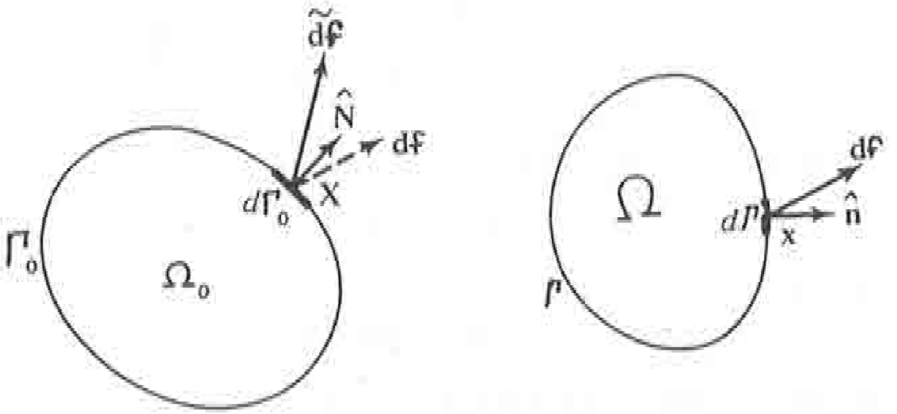


Figure 7.5: Force vectors for Piola-Kirchhoff stress definition

and the two Piola-Kirchhoff tensors are related as

$$\mathcal{T}^{(o)} = \tilde{\mathcal{T}}\tilde{\mathcal{F}}^{T} = \tilde{\mathcal{T}}\hat{\mathcal{F}}^{-T} \tag{7.39}$$

$$\tilde{\mathcal{T}} = \mathcal{T}^{(o)}\tilde{\mathcal{F}}^{-T} = \mathcal{T}^{(o)}\hat{\mathcal{F}}^{T} \tag{7.40}$$

where $\hat{\mathcal{F}}$ is Euler Deformation Gradient tensor, Section 6.4.1. We notice that under the hypothesis of small deformation, Section 6.5, where

$$\tilde{J} \approx 1 \approx \hat{J}$$
$$\tilde{\mathcal{F}} \approx \mathrm{I} \approx \hat{\mathcal{F}}$$

the 1st and 2nd Piola-Kirchhoff stress tensors are approximately equal to the Cauchy stress tensor $\mathcal{T}$.

In order to prove the relationships of the Piola-Kirchhoff stress tensors with the Cauchy stress tensor, we have to recall that deformed and undeformed areas are related as, Eq. 6.19,

$$\hat{\mathrm{n}}\, d\Gamma = \tilde{J}\tilde{\mathcal{F}}^{-T}\hat{\mathrm{N}}\, d\Gamma_o$$

Then, according to the 1st Piola-Kirchhoff definition,

$$\mathrm{t}\, d\Gamma = \mathrm{t}^{(o)}\, d\Gamma_o = \mathcal{T}^{(o)T}\hat{\mathrm{N}}\, d\Gamma_o$$

from which

$$\begin{aligned} \mathbf{t}\,d\Gamma - \mathbf{t}^{(o)}\,d\Gamma_o &= \mathcal{T}^T\hat{\mathbf{n}}\,d\Gamma - \mathcal{T}^{(o)T}\hat{\mathbf{N}}\,d\Gamma_o = \\ &= \mathcal{T}^T\tilde{J}\tilde{\mathcal{F}}^{-T}\hat{\mathbf{N}}\,d\Gamma_o - \mathcal{T}^{(o)T}\hat{\mathbf{N}}\,d\Gamma_o = \\ &= \left[\tilde{J}\mathcal{T}^T\tilde{\mathcal{F}}^{-T} - \mathcal{T}^{(o)T}\right]\hat{\mathbf{N}}\,d\Gamma_o = 0 \end{aligned}$$

for all $\hat{\mathbf{N}}\,d\Gamma_o$. It follows that

$$\mathcal{T}^{(o)T} = \tilde{J}\mathcal{T}^T\tilde{\mathcal{F}}^{-T}$$

that is

$$\mathcal{T}^{(o)} = \tilde{J}\tilde{\mathcal{F}}^{-1}\mathcal{T}$$

According to the 2nd Piola-Kirchhoff definition,

$$\mathbf{t}\,d\Gamma = \tilde{\mathcal{F}}\tilde{\mathbf{t}}\,d\Gamma_o = \tilde{\mathcal{F}}\tilde{\mathcal{T}}^T\hat{\mathbf{N}}\,d\Gamma_o$$

from which

$$\begin{aligned} \mathbf{t}\,d\Gamma - \tilde{\mathcal{F}}\tilde{\mathbf{t}}\,d\Gamma_o &= \mathcal{T}^T\hat{\mathbf{n}}\,d\Gamma - \tilde{\mathcal{F}}\tilde{\mathcal{T}}^T\hat{\mathbf{N}}\,d\Gamma_o = \\ &= \mathcal{T}^T\tilde{J}\tilde{\mathcal{F}}^{-T}\hat{\mathbf{N}}\,d\Gamma_o - \tilde{\mathcal{F}}\tilde{\mathcal{T}}^T\hat{\mathbf{N}}\,d\Gamma_o = \\ &= \left[\tilde{J}\mathcal{T}^T\tilde{\mathcal{F}}^{-T} - \tilde{\mathcal{F}}\tilde{\mathcal{T}}^T\right]\hat{\mathbf{N}}\,d\Gamma_o = 0 \end{aligned}$$

for all $\hat{\mathbf{N}}\,d\Gamma_o$. It follows that

$$\tilde{\mathcal{F}}\tilde{\mathcal{T}}^T = \tilde{J}\mathcal{T}^T\tilde{\mathcal{F}}^{-T}$$

that is

$$\tilde{\mathcal{T}} = \tilde{J}\tilde{\mathcal{F}}^{-1}\mathcal{T}\tilde{\mathcal{F}}^{-T}$$

7.9 Co-rotational and Convected Stress Tensors

Let $F = Ox_1x_2x_3$ and $F = O'x_1'x_2'x_3'$ be two CaORS in relative motion and indicate their relative law of motion as, Section 4.3,

$$\mathbf{x}' = \mathbf{c} + \mathbf{R}\mathbf{x} \tag{7.41}$$

$$\dot{\mathbf{x}}' = \dot{\mathbf{c}} + \mathbf{W}(\mathbf{x}' - \mathbf{c}) + \mathbf{R}\dot{\mathbf{x}} = \dot{\mathbf{c}} + \dot{\mathbf{R}}\mathbf{x} + \mathbf{R}\dot{\mathbf{x}} \tag{7.42}$$

where

$$\begin{aligned}
\mathbf{R} &= \left[\frac{\partial x'_i}{\partial x_j}\right] \\
\mathbf{R}^T &= \mathbf{R}^{-1} = \left[\frac{\partial x_i}{\partial x'_j}\right] \\
\mathbf{W} &= \dot{\mathbf{R}}\mathbf{R}^T = -\mathbf{R}\dot{\mathbf{R}}^T
\end{aligned}$$

and all terms involved are function of time.

The Cauchy stress tensor $\mathcal{T}$, being a linear operator relating $\hat{\mathbf{n}}$ to $\mathbf{t}$, Eq. 7.28, transforms according to the rule, Table 6.1,

$$\mathcal{T}' = \mathbf{R}\mathcal{T}\mathbf{R}^T \tag{7.43}$$

Consequently, its rate of change transforms according to this rule

$$\dot{\mathcal{T}}' = \mathbf{R}\dot{\mathcal{T}}\mathbf{R}^T + \dot{\mathbf{R}}\mathcal{T}\mathbf{R}^T + \mathbf{R}\mathcal{T}\dot{\mathbf{R}}^T \tag{7.44}$$

In Continuum Mechanics it is interesting to define stress rate tensors which under a change of CORF transforms according to the rule

$$\dot{\mathbf{T}}' = \mathbf{R}\dot{\mathbf{T}}\mathbf{R}^T \tag{7.45}$$

This need has led to the definition of a class of stress tensors among which the best known are:

- the *co-rotational stress tensor* $\overset{\circ}{\mathcal{T}}$ defined as

$$\overset{\circ}{\mathcal{T}} = \dot{\mathcal{T}} + \mathcal{T}\widehat{\mathcal{W}} - \widehat{\mathcal{W}}\mathcal{T} \tag{7.46}$$

- the *convected stress tensor* $\overset{\circ}{\mathcal{T}}$ defined as

$$\overset{\circ}{\mathcal{T}} = \dot{\mathcal{T}} + \mathcal{T}\hat{\mathcal{V}} + \hat{\mathcal{V}}^T\mathcal{T} \tag{7.47}$$

where

$$\begin{aligned}
\hat{\mathcal{V}} &= \left[\frac{\partial v_i}{\partial x_j}\right] = \widehat{\mathcal{D}} + \widehat{\mathcal{W}} \\
\widehat{\mathcal{D}} &= \frac{1}{2}(\hat{\mathcal{V}} + \hat{\mathcal{V}}^T) \\
\widehat{\mathcal{W}} &= \frac{1}{2}(\hat{\mathcal{V}} - \hat{\mathcal{V}}^T)
\end{aligned}$$

are respectively the Euler Velocity Gradient, the Stretching and the Spin tensors, Section 6.4.2. It can be proved that these two stress rate tensors are related as

$$\overset{\circ}{\mathcal{T}} = \overset{\circ}{\mathcal{T}} + (\widehat{\mathcal{D}}\mathcal{T} + \mathcal{T}\widehat{\mathcal{D}}) \tag{7.48}$$

The co-rotational stress rate tensor $\overset{\circ}{\mathcal{T}}$ is found by noticing that:

- Since $\widehat{\mathcal{D}}$ and $\widehat{\mathcal{W}}$ are respectively symmetric and skew-symmetric matrices,

$$\begin{aligned} \widehat{\mathcal{W}} &= \hat{\mathcal{V}} - \widehat{\mathcal{D}} \\ -\widehat{\mathcal{W}} &= \widehat{\mathcal{W}}^T = \hat{\mathcal{V}}^T - \widehat{\mathcal{D}} \end{aligned}$$

- According to Eqs 6.135 and 6.136,

$$\begin{aligned} \dot{\mathbf{R}} &= \widehat{\mathcal{W}}'\mathbf{R} - \mathbf{R}\widehat{\mathcal{W}} \\ \dot{\mathbf{R}}^T &= \widehat{\mathcal{W}}\mathbf{R}^T - \mathbf{R}^T\widehat{\mathcal{W}}' \end{aligned}$$

  where $\widehat{\mathcal{W}}'$ is the Spin defined in F'.

Then,

$$\begin{aligned} \dot{\mathcal{T}}' &= \dot{\mathbf{R}}\mathcal{T}\mathbf{R}^T + \mathbf{R}\dot{\mathcal{T}}\mathbf{R}^T + \mathbf{R}\mathcal{T}\dot{\mathbf{R}}^T = \\ &= \left(\widehat{\mathcal{W}}'\mathbf{R} - \mathbf{R}\widehat{\mathcal{W}}\right)\mathcal{T}\mathbf{R}^T + \mathbf{R}\dot{\mathcal{T}}\mathbf{R}^T + \mathbf{R}\mathcal{T}\left(\widehat{\mathcal{W}}\mathbf{R}^T - \mathbf{R}^T\widehat{\mathcal{W}}'\right) = \\ &= \widehat{\mathcal{W}}'\mathbf{R}\mathcal{T}\mathbf{R}^T - \mathbf{R}\widehat{\mathcal{W}}\mathcal{T}\mathbf{R}^T + \mathbf{R}\dot{\mathcal{T}}\mathbf{R}^T + \mathbf{R}\mathcal{T}\widehat{\mathcal{W}}\mathbf{R}^T - \mathbf{R}\mathcal{T}\mathbf{R}^T\widehat{\mathcal{W}}' = \\ &= \widehat{\mathcal{W}}'\mathcal{T}' + \mathbf{R}\left(-\widehat{\mathcal{W}}\mathcal{T} + \dot{\mathcal{T}} + \mathcal{T}\widehat{\mathcal{W}}\right)\mathbf{R}^T - \mathcal{T}'\widehat{\mathcal{W}}' \end{aligned}$$

from which

$$\dot{\mathcal{T}}' - \widehat{\mathcal{W}}'\mathcal{T}' + \mathcal{T}'\widehat{\mathcal{W}}' = \mathbf{R}\left(\dot{\mathcal{T}} - \widehat{\mathcal{W}}\mathcal{T} + \mathcal{T}\widehat{\mathcal{W}}\right)\mathbf{R}^T$$

that is

$$\overset{\circ}{\mathcal{T}}' = \mathbf{R}\,\overset{\circ}{\mathcal{T}}\,\mathbf{R}^T \tag{7.49}$$

The convected stress rate tensor $\overset{\circ}{\mathcal{T}}$ may be then obtained noticing that

$$\begin{aligned} \overset{\circ}{\mathcal{T}} &= \dot{\mathcal{T}} - \widehat{\mathcal{W}}\mathcal{T} + \mathcal{T}\widehat{\mathcal{W}} = \dot{\mathcal{T}} + \widehat{\mathcal{W}}^T\mathcal{T} + \mathcal{T}\widehat{\mathcal{W}} = \\ &= \dot{\mathcal{T}} + (\hat{\mathcal{V}}^T - \widehat{\mathcal{D}}^T)\mathcal{T} + \mathcal{T}(\hat{\mathcal{V}} - \widehat{\mathcal{D}}) = \\ &= (\dot{\mathcal{T}} + \hat{\mathcal{V}}^T\mathcal{T} + \mathcal{T}\hat{\mathcal{V}}) - (\widehat{\mathcal{D}}\mathcal{T} + \mathcal{T}\widehat{\mathcal{D}}) = \\ &= \overset{\circ}{\mathcal{T}} - (\widehat{\mathcal{D}}\mathcal{T} + \mathcal{T}\widehat{\mathcal{D}}) \end{aligned}$$

and that the quantities $\widehat{\mathcal{D}}\mathcal{T}$ and $\mathcal{T}\widehat{\mathcal{D}}$ transform according to the rule

$$\begin{aligned}\widehat{\mathcal{D}}'\mathcal{T}' &= (\mathbf{R}\widehat{\mathcal{D}}\mathbf{R}^T)(\mathbf{R}\mathcal{T}\mathbf{R}^T) = \mathbf{R}\widehat{\mathcal{D}}\mathcal{T}\mathbf{R}^T \\ \mathcal{T}'\widehat{\mathcal{D}}' &= (\mathbf{R}\mathcal{T}\mathbf{R}^T)(\mathbf{R}\widehat{\mathcal{D}}\mathbf{R}^T) = \mathbf{R}\mathcal{T}\widehat{\mathcal{D}}\mathbf{R}^T\end{aligned}$$

Hence,

$$\overset{\circ}{\mathcal{T}}{}' = \mathbf{R}\,\overset{\circ}{\mathcal{T}}\,\mathbf{R}^T \tag{7.50}$$

7.10 Cauchy Equations of Motion and Equilibrium

The linear momentum and angular momentum principles state that in a free body Theorems 7.6.1 and 7.6.3,

$$\int_\Gamma \mathbf{t}\,d\Gamma + \int_\Omega \varrho\mathbf{b}\,d\Omega - \frac{d}{dt}\int_\Omega \varrho\dot{\mathbf{x}}\,d\Omega = 0 \tag{7.51}$$

$$\int_\Gamma (\mathbf{x}\times\mathbf{t})\,d\Gamma + \int_\Omega \varrho(\mathbf{x}\times\mathbf{b})\,d\Omega - \frac{d}{dt}\int_\Omega \varrho(\mathbf{x}\times\dot{\mathbf{x}})\,d\Omega = 0 \tag{7.52}$$

Accordingly, we can establish that in any material point of the free body, *Cauchy Equations of Motion and Equilibrium*,

$$\nabla_x^T\mathcal{T} + \varrho\mathbf{f} = 0 \tag{7.53}$$

$$\mathcal{T} = \mathcal{T}^T \tag{7.54}$$

where

$$\mathbf{f} = \mathbf{b} - \ddot{\mathbf{x}}$$

that is

$$\frac{\partial\sigma_{ji}}{\partial x_j} + \varrho f_i = 0 \tag{7.55}$$

$$\sigma_{ji} = \sigma_{ij} \tag{7.56}$$

with

$$f_i = b_i - \ddot{x}_i$$

These equations are obtained recalling that

$$t_i = n_j\sigma_{ji} \tag{7.57}$$

$$\int_\Gamma n_i\mathcal{A}_{ij\ldots}\,d\Gamma = \int_\Gamma \mathcal{A}_{ij\ldots,i}\,d\Omega \tag{7.58}$$

$$\frac{d}{dt}\int \varrho\mathcal{A}_{ij\ldots}\,d\Omega = \int \varrho\dot{\mathcal{A}}_{ij\ldots}\,d\Omega \tag{7.59}$$

as established by Eqs. 7.28, 2.41 and Eq. 7.14, respectively. Then, according to Eq. 7.51,

$$\begin{aligned}
0 &= \int_\Gamma t_i \, d\Gamma + \int_\Omega \varrho b_i \, d\Omega - \frac{d}{dt}\int_\Omega \varrho \dot{x}_i \, d\Omega = \\
&= \int_\Gamma n_j \sigma_{ji} \, d\Gamma + \int_\Omega \varrho b_i \, d\Omega - \int_\Omega \varrho \ddot{x}_i \, d\Omega = \\
&= \int_\Omega [\sigma_{ji,j} + \varrho(b_i - \ddot{x}_i)] \, d\Omega
\end{aligned}$$

for all Ω; this implies that

$$\sigma_{ji,j} + \varrho(b_i - \ddot{x}_i) = 0$$

On the other hand the terms of Eq. 7.52 can be explicited as

$$\begin{aligned}
\int_\Gamma (\mathbf{x} \times \mathbf{t}) \, d\Gamma &= \int_\Gamma \epsilon_{ijk} x_j t_k \, d\Gamma = \int_\Gamma \epsilon_{ijk} x_j n_s \sigma_{sk} \, d\Gamma = \\
&= \int_\Omega (\epsilon_{ijk} x_j \sigma_{sk})_{,s} \, d\Omega = \int_\Omega \epsilon_{ijk}(x_{j,s}\sigma_{sk} + x_j \sigma_{sk,s}) \, d\Omega = \\
&= \int_\Omega \epsilon_{ijk}(\delta_{js}\sigma_{sk} + x_j \sigma_{sk,s}) \, d\Omega = \\
&= \int_\Omega \epsilon_{ijk}(\sigma_{jk} + x_j \sigma_{sk,s}) \, d\Omega \\
\int_\Omega \varrho(\mathbf{x} \times \mathbf{b}) \, d\Omega &= \int_\Omega \varrho \epsilon_{ijk} x_j b_k \, d\Omega \\
\frac{d}{dt}\int_\Omega \varrho(\mathbf{x} \times \dot{\mathbf{x}}) \, d\Omega &= \frac{d}{dt}\int_\Omega \varrho \epsilon_{ijk} x_j \dot{x}_k \, d\Omega = \int_\Omega \varrho \frac{d}{dt}(\epsilon_{ijk} x_j \dot{x}_k) \, d\Omega = \\
&= \int_\Omega \varrho \epsilon_{ijk}(\dot{x}_j \dot{x}_k + x_j \ddot{x}_k) \, d\Omega = \int_\Omega \varrho \epsilon_{ijk} x_j \ddot{x}_k \, d\Omega
\end{aligned}$$

so that

$$\begin{aligned}
0 &= \int_\Gamma \epsilon_{ijk} x_j t_k \, d\Gamma + \int_\Omega \varrho \epsilon_{ijk} x_j b_k \, d\Omega - \frac{d}{dt}\int_\Omega \varrho \epsilon_{ijk} x_j \dot{x}_k \, d\Omega = \\
&= \int_\Omega \epsilon_{ijk}(\sigma_{jk} + x_j \sigma_{sk,s}) \, d\Omega + \int_\Omega \varrho \epsilon_{ijk} x_j b_k \, d\Omega - \int_\Omega \varrho \epsilon_{ijk} x_j \ddot{x}_k \, d\Omega = \\
&= \int_\Omega \epsilon_{ijk} \left\{\sigma_{jk} + x_j \left[\sigma_{sk,s} + \varrho(b_k - \ddot{x}_k)\right]\right\} \, d\Omega = \\
&= \int_\Omega \epsilon_{ijk} \sigma_{jk} \, d\Omega
\end{aligned}$$

for all Ω; this implies that

$$\epsilon_{ijk}\sigma_{jk} = 0$$

that is

$$\sigma_{jk} = \sigma_{kj}$$

The Linear Momentum Equation in Eq. 7.51 and the relative Cauchy Equation of Equilibrium in Eq. 7.53 are given in terms of current quantities, Eulerian formulation. The corresponding ones in the undeformed configuration, Lagrangian formulation, may be obtained setting

$$\begin{aligned} \mathbf{x} &= \mathbf{x}(\mathbf{X}) \\ \mathbf{b}^{(o)} &= \mathbf{b}\left[\mathbf{x}(\mathbf{X})\right] \end{aligned}$$

and recalling that

$$\begin{aligned} \mathbf{t}\, d\Gamma &= \mathbf{t}^{(o)}\, d\Gamma_o = \mathcal{T}^{(o)T}\hat{\mathbf{N}}\, d\Gamma_o \\ \varrho\, d\Omega &= \varrho_o\, d\Omega_o \end{aligned}$$

as established by Eq. 7.36 and Eq. 7.7, respectively. Thus, Eq. 7.51 can be transformed as

$$\int_\Omega \mathbf{t}^{(o)}\, d\Gamma_o + \int_\Omega \varrho_o \mathbf{b}^{(o)}\, d\Omega_o - \frac{d}{dt}\int_\Omega \varrho_o \dot{\mathbf{x}}\, d\Omega_o = 0 \tag{7.60}$$

and, following a mathematical procedure analogous to that above reported, we can eventually establish that

$$\nabla_X^T \mathcal{T}^{(o)} + \varrho_o \mathbf{f}^{(o)} = \mathbf{o} \tag{7.61}$$

where

$$\mathbf{f}^{(o)} = \mathbf{b}^{(o)} - \ddot{\mathbf{x}}$$

that is

$$\frac{\partial \sigma_{JI}^{(o)}}{\partial X_J} + \varrho_o f_I^{(o)} = 0 \tag{7.62}$$

where

$$f_I^{(o)} = \left(b_I^{(o)} - \ddot{x}_I\right)$$

7.11 Stress Invariant Definitions

We have proved in Section 7.10 that in a CaORS the Cauchy tensor $\mathcal{T}$ is represented by a symmetric matrix. Consequently, also the following stress tensors result to be represented by symmetric matrices:

- The 2nd Piola-Kirchhoff stress tensor $\mathcal{T}$.
- The rate stress tensors $\dot{\mathcal{T}}$, $\overset{\circ}{\mathcal{T}}$ and $\overset{\circ}{\dot{\mathcal{T}}}$.

Then, each of these tensors has three real principal values and three orthogonal principal directions. The three principal values may be determined according to the analytical solution reported in Section 3.7 and their geometrical interpretation may be found in Section 3.8.

In particular, let us indicate any of these two stress tensor as

$$\mathbf{T} = [T_{ij}] \tag{7.63}$$

and the relative characteristic equation as, Section 3.5,

$$t^3 - I_1 t^2 + I_2 t - I_3 = 0 \tag{7.64}$$

where I_i and t_i, for $i = 1, 2, 3$, are the stress invariants and principal values, respectively. Associated with $\mathbf{T}$ we can define the *deviatoric stress tensor*, Section 3.6,

$$\mathbf{S} = \mathbf{T} - \mathbf{I}\frac{I_1}{3} \tag{7.65}$$

The relative characteristic equation is given by, Section 3.6,

$$s^3 - J_2 s - J_3 = 0 \tag{7.66}$$

where J_i and s_i, for $i = 1, 2, 3$, are the deviatoric stress invariants and principal values, respectively.

An alternative definition of the stress invariants is given by

$$p = \frac{I_1}{3} \tag{7.67}$$

$$q = (3J_2)^{1/2} \tag{7.68}$$

$$\theta = \frac{1}{3}\arcsin\left(-\frac{3\sqrt{3}}{2}\frac{J_3}{J_2^{3/2}}\right) \tag{7.69}$$

which, clearly, are strictly related to the octahedral components of $\mathbf{T}$, Section 3.8. Usually, p is called *mean pressure* and q is called *shear stress invariant*.

7.12 Stress Vector Definition

For each of the symmetric stress tensors listed in Section 7.11, we can define a *stress vector* and a relative *deviatoric stress vector* as

$$\boldsymbol{\sigma} = \{T_{11}, T_{22}, T_{33}, T_{12}, T_{13}, T_{21}, T_{23}, T_{31}, T_{32}\}^T \tag{7.70}$$

$$\mathbf{s} = \{S_{11}, S_{22}, S_{33}, S_{12}, S_{13}, S_{21}, S_{23}, S_{31}, S_{32}\}^T \tag{7.71}$$

According to this definition, we have that

$$\boldsymbol{\sigma} = \mathbf{s} + \widehat{\mathbf{m}}p \tag{7.72}$$

where

$$\widehat{\mathbf{m}} = \{1, 1, 1, 0, 0, 0, 0, 0, 0\}^T$$

and the stress invariants can be indicated as the results of the following vector operations

$$I_1 = \widehat{\mathbf{m}}^T \boldsymbol{\sigma} \tag{7.73}$$

$$J_2 = \frac{1}{2}\mathbf{s}^T\mathbf{s} \tag{7.74}$$

and

$$p = \frac{1}{3}\widehat{\mathbf{m}}^T \boldsymbol{\sigma} \tag{7.75}$$

$$q = \left(\frac{3}{2}\mathbf{s}^T\mathbf{s}\right)^{\frac{1}{2}} \tag{7.76}$$

Moreover, the tensor operation

$$\sigma^2 = \mathbf{T} : \mathbf{T}$$

can be indicated as

$$\sigma^2 = \boldsymbol{\sigma}^T\boldsymbol{\sigma} = \mathbf{s}^T\mathbf{s} + 3p^2 = \frac{2}{3}q^2 + 3p^2 \tag{7.77}$$

Chapter 8

Mechanical Energy Principles

8.1 Introduction

This Chapter presents the concepts of work and power in a continuum subjected only to mechanical quantities. The basis for all the mathematical developments is given by the *Theorem of Stress Mean*, Section 8.2. Two of the possible specializations of this theorem yield to the so-called *Principle of Virtual Work* and *Principle of Virtual Power*, Sections 8.3-8.4. These principles are not actually energy principles because they relate fictitious (virtual) work and power made of not necessarily real quantities such as kinematical admissible displacement (velocity) field and dynamically admissible stress field defined as follows:

- A *dynamically (statically) admissible stress field* is one satisfying the equation of motion (equilibrium), Section 7.10, in the interior of the body, and boundary conditions $\boldsymbol{T}^T\hat{\mathbf{n}} = \mathbf{t}$ (or $\boldsymbol{T}^{(o)T}\hat{\mathbf{N}} = \mathbf{t}^{(o)}$) wherever any boundary tractions $\mathbf{t}$ (or $\mathbf{t}^{(o)}$) are prescribed.

- A *kinematically admissible displacement (velocity) field* is one satisfying any prescribed displacement (velocity) boundary conditions and possessing continuous first partial derivatives in the interior of the body.

However, these principles are the fundamental variational principles and play a prominent role in Continuum Mechanics. For example, the Principle of Virtual Work finds extensive applications in *linear structural analysis* as as in the *limit analysis theory* of rigid perfectly-plastic solid materials, [17]. Moreover, it may be used as the fundamental variational principle from which derive the numerical formulation for the complete solution of nonlinear solid medium problem, [40].

Finally, when these principles are specialized for the actual displacement and velocity fields they yield to the (actual) work and power principles for a continuum subjected only to mechanical quantities, Sections 8.5-8.7.

8.2 Theorem of Stress Mean

Let $\mathcal{M}$ be a continuum of volume Ω bounded by a surface

$$\Gamma = \Gamma_t + \Gamma_u$$

and

$$\begin{aligned} \mathbf{t} &= \mathcal{T}\hat{\mathbf{n}} && (8.1) \\ \mathbf{f} &= \mathbf{b} - \ddot{\mathbf{x}} && (8.2) \end{aligned}$$

are a given set of traction and body forces acting on Γ_t and Ω, respectively. Moreover, let be:

- $\varphi = \varphi(\mathbf{x})$ a continuous function in Ω.
- φ possesses 1-st order derivative in Ω.
- $\varphi = 0$ on Γ_u where the traction forces are not prescribed.

If

$$\mathcal{T} = [\sigma_{ij}]$$

is a dynamically (statically) admissible Cauchy stress tensor for $\mathcal{M}$, then

$$\int_\Omega \varrho\varphi f_i \, d\Omega + \int_\Gamma \varphi t_i \, d\Gamma = \int_\Omega \varphi_{,j}\sigma_{ji} \, d\Omega \qquad (8.3)$$

for any arbitrary function φ. Conversely, if the identity in Eq. 8.3 is verified for all possible choices of function φ respecting the three above listed conditions, then $\mathcal{T}$ is a dynamically (statically) admissible Cauchy stress field.

In fact, *direct proposition*, if $\mathcal{T}$ is a dynamically (statically) admissible stress field, then it satisfies the Cauchy Equation of Motion (Equilibrium) in Eq. 7.53 and, therefore,

$$\int_\Omega \varphi \left[\sigma_{ji,j} + \varrho f_i\right] d\Omega = 0$$

for any arbitrary function φ. Since the first term can also be expressed as

$$\varphi\sigma_{ji,j} = (\varphi\sigma_{ji})_{,j} - \varphi_{,j}\sigma_{ji}$$

the above integral may be written as

$$\int_\Omega (\varphi_{,j}\sigma_{ji} - \varrho\varphi f_i)\, d\Omega = \int_\Omega (\varphi\sigma_{ji})_{,j}\, d\Omega \tag{8.4}$$

The term on the right-hand-side can be transformed into an integral over the bounding surface Γ through the Divergence Theorem, Eq. 2.40, namely

$$\int_\Omega \varphi\left(\sigma_{ji}\right)_{,j}\, d\Omega = \int_\Gamma \varphi\hat{n}_j\sigma_{ji}\, d\Gamma$$

and, according to Eq. 8.1, we also have

$$\int_\Gamma \varphi\hat{n}_j\sigma_{ji}\, d\Gamma = \int_\Gamma \varphi t_i\, d\Gamma = \int_{\Gamma_t} \varphi t_i\, d\Gamma$$

so that

$$\int_\Omega \varphi\left(\sigma_{ji}\right)_{,j}\, d\Omega = \int_{\Gamma_t} \varphi t_i\, d\Gamma \tag{8.5}$$

Substituting this expression in Eq. 8.4, we obtain the identity in Eq. 8.3.

Conversely, *inverse proposition*, assume that $\mathcal{T}$ is a (symmetric) stress tensor field respecting the identity in Eq. 8.3 for all possible choices of function φ respecting the above listed conditions. Then, by adding and subtracting $\varphi\hat{n}_j\sigma_{ji}$ in the surface integral we obtain

$$\int_\Omega \varrho\varphi f_i\, d\Omega + \int_{\Gamma_t} \varphi(t_i - \hat{n}_j\sigma_{ji})\, d\Gamma + \int_{\Gamma_t} \varphi\hat{n}_j\sigma_{ji}\, d\Gamma = \int_\Omega \varphi_{,j}\sigma_{ji}\, d\Omega \tag{8.6}$$

The second surface integral on the left-hand-side can be transformed into a volume integral through the Divergence Theorem, Eq. 2.40, namely

$$\begin{aligned}\int_{\Gamma_t} \varphi\hat{n}_j\sigma_{ji}\, d\Gamma &= \int_\Gamma \varphi\hat{n}_j\sigma_{ji}\, d\Gamma = \int_\Omega (\varphi\sigma_{ji})_{,j}\, d\Omega = \\ &= \int_\Omega (\varphi_{,j}\sigma_{ji} + \varphi\sigma_{ji,j})\, d\Omega\end{aligned}$$

Substituting this expression in Eq. 8.6 and eliminating the common terms, we eventually obtain

$$\int_{\Gamma_t} \varphi(t_i - \hat{n}_j\sigma_{ji})\, d\Gamma + \int_{\Omega} \varphi(\sigma_{ji,j} + \varrho f_i)\, d\Omega = 0 \tag{8.7}$$

If this condition is verified for all possible choices of function φ respecting the three above defined conditions, then the term in parenthesis are independently null. That is, $\mathcal{T}$ is a dynamically (statically) admissible stress field.

The result in Eq. 8.3 is known as the *Theorem of Stress Mean* and the first formulation may be traced back to the original works of Castigliano, later extended by Signorini, [87]. In terms of material coordinates, Lagrangian formulation, the Theorem of the Stress Mean takes on the form

$$\int_{\Omega_o} \varrho_o\varphi_o f_I^{(o)}\, d\Omega_o + \int_{\Gamma_o} \varphi_o t_I^{(o)}\, d\Gamma_o = \int_{\Omega_o} \varphi_{o,J} T_{JI}^{(o)}\, d\Omega_o \tag{8.8}$$

where

$$\varphi_o = \varphi[\mathbf{x}(\mathbf{X})]$$

and

$$\begin{aligned} \mathbf{t}^{(o)} &= \mathcal{T}^{(o)T}\hat{\mathbf{N}} \\ \mathbf{f}^{(o)} &= \mathbf{b}^{(o)} - \ddot{\mathbf{x}} \end{aligned}$$

and $\mathcal{T}^{(o)}$ is a dynamically (statically) admissible 1-st Piola-Kirchhoff stress field, Section 7.8. The mathematical proof of this identity is analogous to that above reported.

8.3 The Principle of Virtual Work

In general, the *Principle of Virtual Work* establishes the identity

$$\delta\mathcal{W}^{(E)} = \delta\mathcal{W}^{(I)} + \delta\mathcal{E}^{(K)} \tag{8.9}$$

where

- $\delta\mathcal{W}^{(E)}$ is the virtual work done by the external forces;
- $\delta\mathcal{W}^{(I)}$ is the virtual work done by the internal forces;
- $\delta\mathcal{E}^{(I)}$ is the virtual variation of the kinetic energy.

The explicit expression for these terms depends on the choice of the reference state, deformed or undeformed configuration.

8.3.1 Deformed configuration

The Theorem of Stress Mean in Eq. 8.3 specialized for $\varphi = \delta u_i$, an arbitrary kinematical admissible virtual displacement field, leads to the following form of the Principle of Virtual Work

$$\delta\mathcal{W}^{(E)} = \delta\mathcal{W}^{(I)} + \delta\mathcal{E}^{(K)} \tag{8.10}$$

where

$$\begin{aligned}
\delta\mathcal{W}^{(E)} &= \int_\Omega \varrho\mathbf{b}^T\delta\mathbf{u}\,d\Omega + \int_\Gamma \mathbf{t}^T\delta\mathbf{u}\,d\Gamma \\
\delta\mathcal{W}^{(I)} &= \int_\Omega \mathcal{T} : \delta\mathcal{L}\,d\Omega \\
\delta\mathcal{E}^{(K)} &= \int_\Omega \varrho\ddot{\mathbf{x}}^T\delta\mathbf{u}\,d\Omega
\end{aligned}$$

and

$$\delta\mathcal{L} = \left[\frac{1}{2}\left(\frac{\partial(\delta u_i)}{\partial x_j} + \frac{\partial(\delta u_j)}{\partial x_i}\right)\right]$$

is the virtual variation of the symmetric Linear Lagrange Strain tensor.

The identity in Eq. 8.10 is verified for any dynamically (statically) admissible stress field $\mathcal{T}$ and for any kinematically admissible displacement field $\delta\mathbf{u}$. Conversely, if the identity in Eq. 8.10 is verified for all kinematically admissible displacement field $\delta\mathbf{u}$, then the stress field $\mathcal{T}$ is dynamically (statically) admissible.

It is important to remark that:

- The stress and the displacement fields need not to be the actual ones occurring in the real material and they may be independently prescribed. The only requirements for $\mathcal{T}$ is the respect of the equilibrium and force boundary conditions while the only requirement for $\delta\mathbf{u}$ is to satisfy the prescribed boundary conditions and to possess continuous first partial derivatives in the interior of the body.

- The virtual displacement $\delta\mathbf{u}$ is to be considered as an additional displacement from the equilibrium configuration. Thus a virtual displacement component must be zero whenever the actual displacement is prescribed.

The identity in Eq. 8.10 is easly obtained from Eq. 8.3 setting $\varphi = \delta u_i$ so that

$$\int_\Omega \varrho\delta u_i f_i\,d\Omega + \int_\Gamma \delta u_i t_i\,d\Gamma = \int_\Omega \delta u_{i,j}\sigma_{ji}\,d\Omega$$

that is

$$\int_{\Omega} \varrho \delta \mathbf{u}^T \mathbf{f}\, d\Omega + \int_{\Gamma} \delta \mathbf{u}^T \mathbf{t}\, d\Gamma = \int_{\Omega} \delta \boldsymbol{\mathcal{U}} \cdot \cdot \boldsymbol{\mathcal{T}}\, d\Omega \tag{8.11}$$

where

$$\mathbf{f} = \mathbf{b} - \ddot{\mathbf{x}} \tag{8.12}$$

Splitting the Lagrange Displacement Gradient $\boldsymbol{\mathcal{U}}$ as

$$\delta \boldsymbol{\mathcal{U}} = \delta \boldsymbol{\mathcal{L}} + \delta \boldsymbol{\Omega}$$

where $\boldsymbol{\mathcal{L}}$ is the symmetric Linear Lagrange Strain Tensor and $\boldsymbol{\Omega}$ is the skew-symmetric Lagrange Rotation Tensor, we obtain

$$\delta \boldsymbol{\mathcal{U}} \cdot \cdot \boldsymbol{\mathcal{T}} = \delta \boldsymbol{\mathcal{L}} : \boldsymbol{\mathcal{T}} \tag{8.13}$$

being $\boldsymbol{\mathcal{T}}$ a symmetric tensor and, Eq. 3.3,

$$\delta \boldsymbol{\Omega} \cdot \cdot \boldsymbol{\mathcal{T}} = \delta \boldsymbol{\Omega} : \boldsymbol{\mathcal{T}} = \mathbf{O}$$

Substituting Eqs. 8.12 and 8.13 into Eq. 8.11 we obtain the identity in Eq. 8.10.

8.3.2 Undeformed configuration

The Theorem of Stress Mean in Eq. 8.8 specialized for $\varphi_o = \delta x_i = \delta u_i$, an arbitrary kinematical admissible virtual displacement field, leads to the following form of the Principle of Virtual Work

$$\delta \mathcal{W}^{(E)} = \delta \mathcal{W}^{(I)} + \delta \mathcal{E}^{(K)} \tag{8.14}$$

where

$$\begin{aligned}
\delta \mathcal{W}^{(E)} &= \int_{\Omega_o} \varrho_o \mathbf{b}^{(o)T} \delta \mathbf{u}\, d\Omega_o + \int_{\Gamma_o} \mathbf{t}^{(o)T} \delta \mathbf{u}\, d\Gamma_o \\
\delta \mathcal{W}^{(I)} &= \int_{\Omega_o} \boldsymbol{\mathcal{T}}^{(o)} \cdot \cdot \delta \widetilde{\boldsymbol{\mathcal{F}}}\, d\Omega_o = \frac{1}{2} \int_{\Omega_o} \widetilde{\boldsymbol{\mathcal{T}}} : \delta \widetilde{\boldsymbol{\mathcal{G}}}\, d\Omega_o = \int_{\Omega_o} \widetilde{\boldsymbol{\mathcal{T}}} : \delta \widetilde{\boldsymbol{\mathcal{L}}}\, d\Omega_o \\
\delta \mathcal{E}^{(K)} &= \int_{\Omega_o} \varrho_o \ddot{\mathbf{x}}^T \delta \mathbf{u}\, d\Omega_o
\end{aligned}$$

and, Sections 6.3.1 and 7.8,

- $\mathbf{t}^{(o)} = \boldsymbol{\mathcal{T}}^{(o)T} \widehat{\mathbf{N}}$ is the prescribed traction force;

- $\widetilde{\mathcal{F}}$ is the Lagrange Deformation Gradient;
- $\mathcal{T}^{(o)}$ is the 1-st Piola-Kirchhoff (not symmetric) stress tensor;
- $\tilde{\mathcal{G}}$ is the Green Strain Tensor;
- $\tilde{\mathcal{L}}$ is the (finite) Lagrange Strain Tensor;
- $\tilde{\mathcal{T}}$ is the 2-nd Piola-Kirchhoff (symmetric) stress tensor.

Notice that in the case of small deformation, Section 6.6, where

$$\tilde{\mathcal{T}} \approx \mathcal{T}, \; \tilde{\mathcal{L}} \approx \mathcal{L}, \; d\Omega_o \approx d\Omega$$

the identity in Eq. 8.10 can be approximated to that in Eq. 8.14.

The identity in Eq. 8.10 can be obtained from Eq. 8.8 setting $\varphi = \delta x_i = \delta u_i$ so that

$$\int_{\Omega_o} \varrho_o \delta u_i f_i^{(o)}\, d\Omega_o + \int_{\Gamma_o} \delta u_i t_i^{(o)}\, d\Gamma_o = \int_{\Omega_o} \delta x_{i,j} \sigma_{ji}^{(o)}\, d\Omega_o$$

that is

$$\int_{\Omega_o} \varrho_o \delta \mathbf{u}^T \mathbf{f}^{(o)}\, d\Omega_o + \int_{\Gamma_o} \delta \mathbf{u}^T \mathbf{t}^{(o)}\, d\Gamma_o = \int_{\Omega_o} \delta \widetilde{\mathcal{F}} \cdot\cdot \mathcal{T}^{(o)}\, d\Omega_o \tag{8.15}$$

where

$$\mathbf{f}^{(o)} = \mathbf{b}^{(o)} - \ddot{\mathbf{x}} \tag{8.16}$$

The alternative forms for the virtual internal work can be obtained as follows. According to Eq. 7.39,

$$\mathcal{T}^{(o)} = \tilde{\mathcal{T}} \widetilde{\mathcal{F}}^T$$

so that

$$w = \delta \widetilde{\mathcal{F}} \cdot\cdot \mathcal{T}^{(o)} = \delta \tilde{\mathcal{F}}_{is} \mathcal{T}_{si}^{(o)} = \delta \tilde{\mathcal{F}}_{is} \tilde{\mathcal{T}}_{sk} \tilde{\mathcal{F}}_{ki}^T = \delta \widetilde{\mathcal{F}} \tilde{\mathcal{T}} \widetilde{\mathcal{F}}^T$$

On the other hand, Section 6.3.1,

$$\begin{aligned} \tilde{\mathcal{G}} &= \widetilde{\mathcal{F}}^T \widetilde{\mathcal{F}} \\ \delta \tilde{\mathcal{G}} &= \delta \widetilde{\mathcal{F}}^T \widetilde{\mathcal{F}} + \widetilde{\mathcal{F}}^T \delta \widetilde{\mathcal{F}} \end{aligned}$$

and, accordingly,

$$\begin{aligned} \tilde{\mathcal{T}} : \delta \tilde{\mathcal{G}} &= \tilde{\mathcal{T}}_{is} \delta \tilde{\mathcal{G}}_{is} = \tilde{\mathcal{T}}_{is} (\delta \tilde{\mathcal{F}}_{ik}^T \tilde{\mathcal{F}}_{ks} + \tilde{\mathcal{F}}_{ik}^T \delta \tilde{\mathcal{F}}_{ks}) = \\ &= \tilde{\mathcal{T}}_{is} (\delta \tilde{\mathcal{F}}_{ki} \tilde{\mathcal{F}}_{sk}^T + \delta \tilde{\mathcal{F}}_{ks} \tilde{\mathcal{F}}_{ik}^T) = \\ &= \delta \tilde{\mathcal{F}}_{ki} \tilde{\mathcal{T}}_{is} \tilde{\mathcal{F}}_{sk}^T + \delta \tilde{\mathcal{F}}_{ks} \tilde{\mathcal{T}}_{si}^T \tilde{\mathcal{F}}_{ik}^T = \\ &= \delta \widetilde{\mathcal{F}} \tilde{\mathcal{T}} \widetilde{\mathcal{F}}^T + \delta \widetilde{\mathcal{F}} \tilde{\mathcal{T}}^T \widetilde{\mathcal{F}}^T = 2 \delta \widetilde{\mathcal{F}} \tilde{\mathcal{T}} \widetilde{\mathcal{F}}^T = 2w \end{aligned}$$

being $\tilde{\mathcal{T}} = \tilde{\mathcal{T}}^T$. Equalizing the above two alternative equations for w we obtain

$$w = \delta\widetilde{\mathcal{F}} \cdot \mathcal{T}^{(o)} = \frac{1}{2}\delta\tilde{\mathcal{G}} : \tilde{\mathcal{T}} = \delta\tilde{\mathcal{L}} : \tilde{\mathcal{T}} \tag{8.17}$$

Substituting Eqs. 8.16 and 8.17 into Eq. 8.15 we obtain the identity in Eq. 8.14.

8.4 The Principle of Virtual Power

The Theorem of Stress Mean specialized for $\varphi = \delta v_i$, an arbitrary kinematical admissible virtual velocity field, leads to the following *Principle of Virtual Power*

$$\delta\dot{\mathcal{W}}^{(E)} = \delta\dot{\mathcal{W}}^{(I)} + \delta\dot{\mathcal{E}}^{(K)} \tag{8.18}$$

where

$$\begin{aligned}
\delta\dot{\mathcal{W}}^{(E)} &= \int_\Omega \varrho \mathbf{b}^T \delta\mathbf{v}\, d\Omega + \int_\Gamma \mathbf{t}^T \delta\mathbf{v}\, d\Gamma \\
\delta\dot{\mathcal{W}}^{(I)} &= \int_\Omega \mathcal{T} : \delta\widehat{\mathcal{D}}\, d\Omega \\
\delta\dot{\mathcal{E}}^{(K)} &= \int_\Omega \varrho \ddot{\mathbf{x}}^T \delta\mathbf{v}\, d\Omega
\end{aligned}$$

and

$$\delta\widehat{\mathcal{D}} = \left[\frac{1}{2}\left(\frac{\partial(\delta v_i)}{\partial x_j} + \frac{\partial(\delta v_j)}{\partial x_i}\right)\right]$$

is the Euler Strain Rate tensor. The different terms in Eq. 8.18 are usually called as:

- $\delta\dot{\mathcal{W}}^{(E)}$ is the virtual input power;
- $\delta\dot{\mathcal{W}}^{(I)}$ is the virtual stress power;
- $\delta\dot{\mathcal{E}}^{(I)}$ is the virtual rate change of the kinetic energy.

The identity in Eq. 8.18 is verified for any dynamically (statically) admissible stress field $\mathcal{T}$ and for any kinematically admissible velocity field $\mathbf{v}$. Conversely, if the identity in Eq. 8.18 is verified for all kinematically admissible velocity field $\delta\mathbf{v}$, then the stress field $\mathcal{T}$ is dynamically (statically) admissible.

Analogously to the case of the virtual work, we remark that:

- The stress and the velocity fields do not need to be the actual ones occurring in the real material and they may be independently prescribed. The only requirements for $\mathcal{T}$ is the respect of the equilibrium and force boundary conditions while the only requirement for $\delta\mathbf{v}$ is to satisfy the prescribed boundary conditions and to possess continuous first partial derivatives in the interior of the body.
- The virtual velocity $\delta\mathbf{v}$ is to be considered as an additional velocity from the equilibrium configuration. Thus a virtual velocity component must be zero whenever the actual velocity is prescribed.

Finally, in the case of small deformation, Section 6.6, where

$$\widehat{\mathcal{D}} \approx \dot{\mathcal{L}}, \quad d\Omega_o \approx d\Omega$$

the internal virtual power can be approximated as

$$\delta\dot{\mathcal{W}}^{(I)} = \int_{\Omega_o} \mathcal{T} : \delta\dot{\mathcal{L}}\, d\Omega_o \tag{8.19}$$

The procedure for establishing the identity in Eq. 8.18 is analogous to that followed for establishing the identity in Eq. 8.10 In fact, setting $\varphi = \delta v_i$ in Eq. 8.3, we obtain

$$\int_\Omega \varrho \delta v_i f_i \, d\Omega + \int_\Gamma \delta v_i t_i \, d\Gamma = \int_\Omega \delta v_{i,j} \sigma_{ji} \, d\Omega$$

that is

$$\int_\Omega \varrho \delta\mathbf{v}^T \mathbf{f}\, d\Omega + \int_\Gamma \delta\mathbf{v}^T \mathbf{t}\, d\Gamma = \int_\Omega \delta\widehat{\mathcal{V}} \cdot\cdot \mathcal{T}\, d\Omega \tag{8.20}$$

where

$$\mathbf{f} = \mathbf{b} - \ddot{\mathbf{x}} \tag{8.21}$$

Splitting the Euler Velocity Gradient $\mathcal{V}$ as

$$\delta\widehat{\mathcal{V}} = \delta\widehat{\mathcal{D}} + \delta\widehat{\mathcal{W}}$$

where $\widehat{\mathcal{W}}$ is the skew-symmetric Euler Vorticity tensor, we obtain

$$\delta\widehat{\mathcal{V}} \cdot\cdot \mathcal{T} = \delta\widehat{\mathcal{D}} : \mathcal{T} \tag{8.22}$$

being $\mathcal{T}$ a symmetric tensor and, Eq. 3.3,

$$\delta\widehat{\mathcal{W}} \cdot\cdot \mathcal{T} = \delta\widehat{\mathcal{W}} : \mathcal{T} = \mathbf{O}$$

Substituting Eqs. 8.21 and 8.22 into Eq. 8.20 we obtain the identity in Eq. 8.18.

8.5 Work and Power

The Principle of Virtual Work presented in Section 8.3.2 specialized for the real displacement increment $d\mathbf{u} = d\mathbf{x}$ and stress $\mathcal{T}$ fields, establishes the relationship between the work done by the external and the internal forces, namely

$$d\mathcal{W}^{(E)} = d\mathcal{W}^{(I)} + d\mathcal{E}^{(K)} \tag{8.23}$$

where

$$\begin{aligned}
d\mathcal{W}^{(E)} &= \int_{\Omega_o} \varrho_o \mathbf{b}^{(o)T} d\mathbf{u}\, d\Omega_o + \int_{\Gamma_o} \mathbf{t}^{(o)T} d\mathbf{u}\, d\Gamma_o \\
d\mathcal{W}^{(I)} &= \int_{\Omega_o} \mathcal{T}^{(o)} \cdot\cdot d\mathcal{F}\, d\Omega_o = \frac{1}{2}\int_{\Omega_o} \tilde{\mathcal{T}} : d\tilde{\mathcal{G}}\, d\Omega_o = \int_{\Omega_o} \tilde{\mathcal{T}} : d\tilde{\mathcal{L}}\, d\Omega_o \\
d\mathcal{E}^{(K)} &= \int_{\Omega_o} \varrho_o \ddot{\mathbf{x}}^T d\mathbf{u}\, d\Omega_o
\end{aligned}$$

The different terms in Eq. 8.23 are usually called as:

- $d\mathcal{W}^{(E)}$, the work done by the external forces;
- $d\mathcal{W}^{(I)}$, the work done by the internal forces;
- $d\mathcal{E}^{(I)}$, the change of kinetic energy.

The Principle of Virtual Power presented in Section 8.4 specialized for the real velocity increment $d\mathbf{v}$ and stress $\mathcal{T}$ fields establishes the relationship between the power input by the external and the rate of change of the internal work, namely

$$\dot{\mathcal{W}}^{(E)} = \dot{\mathcal{W}}^{(I)} + \dot{\mathcal{E}}^{(K)} \tag{8.24}$$

where

$$\begin{aligned}
\dot{\mathcal{W}}^{(E)} &= \int_{\Omega} \mathbf{b}^T \mathbf{v}\, d\Omega + \int_{\Gamma} \mathbf{t}^T \mathbf{v}\, d\Gamma \\
\dot{\mathcal{W}}^{(I)} &= \int_{\Omega} \mathcal{T} : \overline{\mathcal{D}}\, d\Omega = \int_{\Omega_o} \mathcal{T}^{(o)} \cdot\cdot \dot{\mathcal{F}}\, d\Omega_o = \int_{\Omega_o} \tilde{\mathcal{T}} : \dot{\tilde{\mathcal{L}}}\, d\Omega_o \\
\dot{\mathcal{E}}^{(K)} &= \int_{\Omega} \varrho \ddot{\mathbf{x}}^T \mathbf{v}\, d\Omega = \frac{d}{dt}\int_{\Omega} \varrho \frac{v^2}{2}\, d\Omega
\end{aligned}$$

The different terms in Eq. 8.24 are usually called as:

- $\dot{\mathcal{W}}^{(E)}$, the power input by the external forces;

- $\dot{\mathcal{W}}^{(I)}$, the stress power;
- $\dot{\mathcal{E}}^{(I)}$, the rate of change of the kinetic energy.

Notice that in the case of small deformation, Section 6.6, where

$$\tilde{\mathcal{T}} \approx \mathcal{T},\ \tilde{\mathcal{L}} \approx \mathcal{L},\ \widehat{\mathcal{D}} \approx \dot{\mathcal{L}},\ d\Omega_o \approx d\Omega$$

the change of internal energy can be approximated to

$$\begin{aligned} d\mathcal{W}^{(I)} &= \int_\Omega \mathcal{T} : d\mathcal{L}\, d\Omega \\ \dot{\mathcal{W}}^{(E)} &= \int_\Omega \mathcal{T} : \dot{\mathcal{L}}\, d\Omega \end{aligned}$$

The alternative expressions for the stress power can be found as follows. Similarly to the case of the virtual power, we can establish that

$$\dot{\mathcal{W}}^{(I)} = \int_\Omega \mathcal{T} \cdot\cdot \hat{\mathcal{V}}\, d\Omega = \int_\Omega \mathcal{T} : \widehat{\mathcal{D}}\, d\Omega$$

Then, according to Eqs. 7.37, 6.88 and 6.50,

$$\begin{aligned} \mathcal{T} &= \hat{J}\tilde{\mathcal{F}}\mathcal{T}^{(o)} \\ \dot{\tilde{\mathcal{F}}} &= \hat{\mathcal{V}}\tilde{\mathcal{F}} \\ d\Omega_o &= \hat{J}\, d\Omega \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{T} \cdot\cdot \hat{\mathcal{V}} &= \mathcal{T}_{si}\hat{\mathcal{V}}_{is} = \hat{J}\tilde{\mathcal{F}}_{sj}\tilde{\mathcal{T}}^{(o)}_{ji}\hat{\mathcal{V}}_{is} = \hat{J}\tilde{\mathcal{T}}^{(o)}_{ji}\hat{\mathcal{V}}_{is}\tilde{\mathcal{F}}_{sj} = \\ &= \hat{J}\tilde{\mathcal{T}}^{(o)}_{ji}\dot{\tilde{\mathcal{F}}}_{ij} = \hat{J}\mathcal{T}^{(o)} \cdot\cdot \dot{\tilde{\mathcal{F}}} \end{aligned}$$

whence

$$\mathcal{T} \cdot\cdot \hat{\mathcal{V}}\, d\Omega = \hat{J}\mathcal{T}^{(o)} \cdot\cdot \dot{\tilde{\mathcal{F}}}\, d\Omega = \mathcal{T}^{(o)} \cdot\cdot \dot{\tilde{\mathcal{F}}}\, d\Omega_o \tag{8.25}$$

On the other hand, according to Eqs. 7.38 and 6.76,

$$\begin{aligned} \mathcal{T} &= \widetilde{J\mathcal{F}\mathcal{T}\mathcal{F}}^T \\ \dot{\tilde{\mathcal{L}}} &= \widetilde{\mathcal{F}}^{-T}\widetilde{\mathcal{D}\mathcal{F}} = \widetilde{\mathcal{F}}^T\widetilde{\mathcal{D}\mathcal{F}}^{-1} \end{aligned}$$

Thus

$$\begin{aligned} \mathcal{T} : \widehat{\mathcal{D}} &= \mathcal{T}_{is}\hat{\mathcal{D}}_{is} = \hat{J}\tilde{\mathcal{F}}_{ij}\tilde{\mathcal{T}}_{jk}\tilde{\mathcal{F}}^T_{ks}\hat{\mathcal{D}}_{is} = \hat{J}\tilde{\mathcal{T}}_{jk}\tilde{\mathcal{F}}^T_{ji}\hat{\mathcal{D}}_{is}\tilde{\mathcal{F}}_{sk} = \\ &= \hat{J}\tilde{\mathcal{T}}_{jk}\dot{\tilde{\mathcal{L}}}_{jk} = \hat{J}\tilde{\mathcal{T}} : \dot{\tilde{\mathcal{L}}} \end{aligned}$$

whence

$$\mathcal{T} : \widehat{\mathcal{D}}\, d\Omega = \hat{J}\tilde{\mathcal{T}} : \dot{\tilde{\mathcal{L}}}\, d\Omega = \tilde{\mathcal{T}} : \dot{\tilde{\mathcal{L}}}\, d\Omega_o \tag{8.26}$$

The alternative form for the rate of change of the kinetic energy is found noticing that

$$\ddot{\mathbf{x}}^T \mathbf{v} = \dot{\mathbf{v}}^T \mathbf{v} = \frac{1}{2}\frac{d}{dt}(\mathbf{v}^T \mathbf{v}) = \frac{d}{dt}\frac{v^2}{2}$$

and, consequently, Eq. 7.14,

$$\dot{\mathcal{E}}^{(K)} = \int_\Omega \varrho \ddot{\mathbf{x}}^T \mathbf{v}\, d\Omega = \int_\Omega \varrho \frac{d}{dt}\frac{v^2}{2}\, d\Omega = \frac{d}{dt}\int_\Omega \varrho \frac{v^2}{2}\, d\Omega$$

8.6 Stress Energy and Power

The most common forms for expressing the mechanical (virtual) work and energy of the internal forces are respectively

$$d\mathcal{W}^{(I)} = \int_{\Omega_o} \tilde{\mathcal{T}} : d\tilde{\mathcal{L}}\, d\Omega_o \tag{8.27}$$

$$\dot{\mathcal{W}}^{(I)} = \int_{\Omega_o} \tilde{\mathcal{T}} : \dot{\tilde{\mathcal{L}}}\, d\Omega_o = \int_\Omega \mathcal{T} : \widehat{\mathcal{D}}\, d\Omega \tag{8.28}$$

It is also customary to express them as

$$d\mathcal{W}^{(I)} = d\int_{\Omega_o} \varrho_o \tilde{w}^{(I)}\, d\Omega_o = \int_{\Omega_o} \varrho_o d\tilde{w}^{(I)}\, d\Omega_o \tag{8.29}$$

$$\frac{d\mathcal{W}^{(I)}}{dt} = \frac{d}{dt}\int_{\Omega_o} \varrho_o \tilde{w}^{(I)}\, d\Omega_o = \int_{\Omega_o} \varrho_o \frac{d\tilde{w}^{(I)}}{dt}\, d\Omega_o \tag{8.30}$$

or alternatively

$$\frac{d\mathcal{W}^{(I)}}{dt} = \frac{d}{dt}\int_\Omega \varrho w^{(I)}\, d\Omega = \int_\Omega \varrho \frac{dw^{(I)}}{dt}\, d\Omega \tag{8.31}$$

where

$$\varrho_o d\tilde{w}^{(I)} = \mathcal{T}^{(o)} \cdot\cdot \widetilde{\mathcal{F}} = \tilde{\mathcal{T}} : d\tilde{\mathcal{L}} \tag{8.32}$$

$$\varrho_o \frac{d\tilde{w}^{(I)}}{dt} = \tilde{\mathcal{T}} : \dot{\tilde{\mathcal{L}}} \tag{8.33}$$

$$\varrho \frac{dw^{(I)}}{dt} = \mathcal{T} : \widehat{\mathcal{D}} \tag{8.34}$$

and

- $\tilde{w}^{(t)}$ is known as the *stress energy* per unit volume;
- $d\tilde{w}^{(t)}/dt$ and $dw^{(t)}/dt$ are known as the *stress power* per unit volume.

Let us generally indicate anyone of the above scalar tensor products as

$$u = \mathbf{T} : d\mathbf{L} \tag{8.35}$$

where $\mathbf{T}$ represents a stress tensor and $\mathbf{L}$ the corresponding strain tensor in the above definitions. Interestingly, this scalar product may be decomposed in a volumetric and a deviatoric part as follows

$$u = pd\epsilon_v + \mathbf{S} : d\mathbf{E} \tag{8.36}$$

where:

- p and $\mathbf{S}$ are the mean pressure and the deviatoric stress tensor respectively defined as, Section 7.11,

$$\begin{aligned} p &= \frac{1}{3}\mathbf{I} : \mathbf{T} \\ \mathbf{S} &= \mathbf{T} - p\mathbf{I} \end{aligned}$$

- ϵ_v and $\mathbf{E}$ are the volumetric strain and the deviatoric strain tensor respectively defined as, Section 6.9,

$$\begin{aligned} \epsilon_v &= \mathbf{I} : \mathbf{L} \\ \mathbf{E} &= \mathbf{L} - \frac{1}{3}\epsilon_v \mathbf{I} \end{aligned}$$

In vector notation, this scalar quantity can be expressed as

$$u = \boldsymbol{\sigma}^T d\boldsymbol{\epsilon} = pd\epsilon_v + \mathbf{s}^T d\mathbf{e} \tag{8.37}$$

where, Sections 7.12 and 6.10,

$$\begin{aligned} \boldsymbol{\sigma} &= \{T_{11}, T_{22}, T_{33}, T_{12}, T_{13}, T_{21}, T_{23}, T_{31}, T_{32}\}^T \\ \boldsymbol{\epsilon} &= \{L_{11}, L_{22}, L_{33}, L_{12}, T_{13}, L_{21}, L_{23}, L_{31}, L_{32}\}^T \end{aligned}$$

and

$$\begin{aligned} p &= \frac{1}{3}\widehat{\mathbf{m}}^T \boldsymbol{\sigma} \\ \mathbf{s} &= \boldsymbol{\sigma} - p\widehat{\mathbf{m}} \end{aligned}$$

and

$$\begin{aligned}\epsilon_v &= \widehat{\mathbf{m}}^T \boldsymbol{\epsilon} \\ \mathbf{e} &= \boldsymbol{\epsilon} - \frac{\epsilon_v}{3}\widehat{\mathbf{m}}\end{aligned}$$

being

$$\widehat{\mathbf{m}} = \{1,1,1,0,0,0,0,0,0,0,0,0\}^T$$

We have stated that in general $d\mathcal{W}^{(I)}$ is not a perfect differential. However, if there exists an *internal energy function*

$$\tilde{w} = \varrho_o \tilde{u} \tag{8.38}$$

such that

$$\tilde{\mathcal{T}} = \frac{\partial \tilde{w}}{\partial \tilde{\mathcal{L}}} = \varrho_o \frac{\partial \tilde{u}}{\partial \tilde{\mathcal{L}}} \tag{8.39}$$

or alternatively

$$w = \varrho u \tag{8.40}$$

such that

$$\mathcal{T} = \frac{\partial w}{\partial \tilde{\mathcal{D}}} = \varrho \frac{\partial u}{\partial \tilde{\mathcal{D}}} \tag{8.41}$$

then $d\mathcal{W}^{(I)}$ is a *perfect differential*. In fact, for example, in the case stated by Eq. 8.38 we have

$$d\tilde{w}^{(I)} = \frac{1}{\varrho_o}\tilde{\mathcal{T}} : d\tilde{\mathcal{L}} = \frac{1}{\varrho_o}\frac{\partial \tilde{w}}{\partial \tilde{\mathcal{L}}} : d\tilde{\mathcal{L}} = \frac{\partial \tilde{u}}{\partial \tilde{\mathcal{L}}} : d\tilde{\mathcal{L}} = d\tilde{u}$$

and, consequently,

$$\begin{aligned}d\mathcal{W}^{(I)} &= \int_{\Omega_o} \varrho_o d\tilde{w}^{(I)}\, d\Omega_o = d\mathcal{U} \\ \frac{d\mathcal{W}^{(I)}}{dt} &= \frac{d\mathcal{U}}{dt}\end{aligned}$$

where

$$d\mathcal{U} = \int_{\Omega_o} \varrho_o d\tilde{u}\, d\Omega_o = d\int_{\Omega_o} \varrho_o \tilde{u}\, d\Omega_o$$

8.7 Energy Principles

The identity in Eq. 8.24 is also known as the *energy equation of a continuum*. If only mechanical forces are considered, this equation represents the mathematical statement for the following mechanical principles.

Theorem 8.7.1 *If only mechanical forces are considered, the power input of the external forces equals the rate of change of the kinetic energy plus the stress power, namely*

$$\frac{d\mathcal{W}^{(E)}}{dt} = \frac{d\mathcal{W}^{(I)}}{dt} + \frac{d\mathcal{E}^{(K)}}{dt} \tag{8.42}$$

where

$$\begin{aligned}
\frac{d\mathcal{W}^{(E)}}{dt} &= \int_\Omega \varrho \mathbf{b}^T \mathbf{v}\, d\Omega + \int_\Gamma \mathbf{t}^T \mathbf{v}\, d\Gamma \\
\frac{d\mathcal{W}^{(I)}}{dt} &= \int_\Omega \mathcal{T} : \widehat{\mathcal{D}}\, d\Omega = \int_{\Omega_o} \mathcal{T}^{(o)} \cdot \dot{\bar{\mathcal{F}}}\, d\Omega_o = \int_{\Omega_o} \tilde{\mathcal{T}} : \dot{\mathcal{L}}\, d\Omega_o \\
\frac{d\mathcal{E}^{(K)}}{dt} &= \int_\Omega \varrho \ddot{\mathbf{x}}^T \mathbf{v}\, d\Omega = \frac{d}{dt}\int_\Omega \varrho \frac{v^2}{2}\, d\Omega
\end{aligned}$$

Theorem 8.7.2 (Principle of Conservation of Energy.) *If only mechanical forces are considered and*

- *the external forces are conservative, i.e. there exists a potential energy $e^{(p)}$ so that the power input can be expressed as*

$$\frac{d\mathcal{W}^{(E)}}{dt} = \frac{d}{dt}\int_\Omega \varrho e^{(p)}\, d\Omega = \frac{d}{dt}\mathcal{E}^{(p)}$$

- *there exists an internal energy function $w = \varrho u$ so that the stress power can be expressed as*

$$\frac{d\mathcal{W}^{(I)}}{dt} = \frac{d}{dt}\int_\Omega \varrho u\, d\Omega = \frac{d}{dt}\mathcal{U}$$

then the total energy

$$\mathcal{E}^{(t)} = \int_\Omega \varrho e\, d\Omega$$

where

$$e = \frac{v^2}{2} + e^{(p)} + u$$

is constant.

Chapter 9

Thermodynamic Principles

9.1 Introduction

Nature has different forms of energies: the *mechanical energy* which we have presented in the previous Chapter, and the non-mechanical energies such as *thermal energy*, *chemical energy* or *electromagnetic energy*.

Thermodynamics is the science which comprises the study of transformation of energy and of the relationships between the properties whose values change as a consequence of these transformations. The study is approached from a macroscopical point of view, Section 9.2.

Although thermodynamic is a general science which may take into account all forms of energy existing in nature, we will assume that the continuum thermodynamic system is uncharged and chemically inert in any of its parts and not acted upon by electrical or magnetical field. Thus, only mechanical power and thermal energy, Section 9.3, are considered.

We have seen in the previous Chapter that a set of external forces supply a continuum with an amount of additional energy, power input. Part of this energy is spent as kinetic energy and the remaining part is used to deform the body. During the deformation process we may notice an increase of the body temperature which eventually is conducted away (dissipated) as heat. In many cases this heat production may be immediately visualized as the end result of friction developed between particles. Conversely, when thermal energy is supplied to a real material, part of it may be spent to deform the material structure.

The amount of these energy conversion processes depend on the nature of the material structure. However, the thermodynamic science has been able to establish the energy conversion rules which are common for any real material.

One of the most significant achievements in the evolution of physics is the recognition of the Principle of Conservation of Energy, also known as the *1st Principle of Thermodynamics*, Section 9.4. This principle states that any type of energy supplied to a thermodynamic system (here a continuum) may change its form but the total amount is preserved. However, it does not specify the extent and the reversibility of this conversion process.

This additional informations are supplied by the 2nd Principle of Thermodynamics which postulates the existence of an extensive property known as *entropy*, Section 9.5.

9.2 Thermodynamic Systems

Any portion, even the smallest, of our physical world may be considered as a *thermodynamic system*. It is postulated that any (thermodynamic) system possesses a certain level of*internal energy* U whose value can be modified by the contact with the surrounding systems.

Energy can be supplied or removed from the system under various forms. However, the only ones which are of interest for us are the *thermal* and *mechanical energies*, where the latter is defined as the rate of work done by the external forces on the system.

A mathematical description of a system requires the specification of its boundary conditions. In thermodynamics, the boundary conditions are defined through the constraints that they impose on the system. For our purposes it is sufficient to consider the following different kinds of thermodynamic systems:

- *Adiabatic* thermodynamic systems which are restrictive to the flow of heat preventing therefore thermal energy transfer.

- *Diathermal* thermodynamic systems which, instead, permit the flow of heat and thus thermal energy transfer.

- *Rigid* thermodynamic systems which are restrictive to any deformation process preventing, therefore, any mechanical work.

- A *not-rigid* thermodynamic system, instead, permits some or all deformation modes of the system allowing therefore mechanical work.

We remind that for a continuum we have to consider six possible contributions of mechanical work, each associated with one of the six independent strain components.

A system is then termed *closed* or *open* depending on whether the walls surrounding the system prevent or not any portion of matter to leave or enter into the system. A closed system is termed *isolated* or *not-isolated* depending on whether the walls surrounding the system prevent or not any exchange of any form of energies with the external system.

An isolated system tends to evolve spontaneously towards a terminal state called *thermodynamic equilibrium state*. In our case, where only mechanical and thermal energy are considered, a thermodynamic equilibrium state consists of:

- mechanical equilibrium in which all forces in the system are equilibrated;
- thermal equilibrium in which there is no flow of heat in the system.

Two or more systems constitute a *composite system*. Consistently with the above definitions, a continuum may be considered as a composite system where the most elementary subsystem is the material point of mass $dm = \varrho\, d\Omega$.

By definition, the *whole physical world is an isolated composite system*. Thus, the composite system made of a system and of all the surrounding systems is by definition isolated.

The *walls* of each individual subsystem represent the constraints of the composite system. The walls which result to be located inside or on the boundary of the composite system may be further distinguished as *internal* or *external constraints*, respectively.

If an isolated composite system is in equilibrium with respect to certain internal constraints and some of these are then removed, the system eventually comes into a new equilibrium state. The change from one to another equilibrium state is said a *thermodynamic process*.

A thermodynamic process is said to be *quasi-static* if at any instant the system is infinitely close to an equilibrated state. Consequently, the state assumed by the system at any stage of the process can be considered as an equilibrated state. Quasi-static processes may be obtained by removing very slowly the constraints. They are rather ideal processes and generally they can be properly simulated only in laboratory experiments. However, for any practical purpose, several real processes can be considered as quasi-static processes.

The equilibrium state of a system is characterized by the value assumed by the so-called *state variables* which are then classified as extensive and intensive according to the following definitions:

- an *extensive state variable* is that which may be used for characterizing a property of the whole system. An extensive variable is one that in an homogeneous system is proportional to the total mass; in general its total amount in the system is the sum of the amounts in all its parts. Extensive variables are, for example, those defined as

$$U = \int_{\Omega} \varrho u \, d\Omega$$

  where u is known as the density associated with the extensive variable U.

- an *intensive state variable* is one which may be used for characterizing a property of the system only at the point level. Its value does not depend on the mass. For example, the density of extensive variables, e.g. u, is an intensive variable.

Not all the state variables are necessarily independent and the functional relationships among them are called *equation of state*. Any state variable which may be expressed as a single-valued function of a set of other state variables is known as a *state function*.

A thermodynamic process is said *reversible* if whenever the independent variables return to their initial value, so do the dependent variables.

9.3 Thermal Energy

The rate of increase of total heat into a continuum is defined as

$$\frac{dQ}{dt} = \int_{\Omega} \varrho r \, d\Omega - \int_{\Gamma} \mathbf{c}^T \bar{\mathbf{n}} \, d\Gamma \tag{9.1}$$

where

- r is the internal heat supply per unit mass and unit time, for example the heat supply from a radiation field;
- $\mathbf{c}$ is the heat flux per unit area and unit time by conduction.

The negative sign is needed because $\int_\Gamma \mathbf{c}^T \hat{\mathbf{n}}\, d\Gamma$ is the outward heat flux. The special symbol $đ$ is again used to indicate that the quantity in Eq. 9.1 is not an exact differential.

The divergence theorem of Gauss allows to transform Eq. 9.1 as

$$\frac{đQ}{dt} = \int_\Omega \left(\varrho r - \frac{\partial c_i}{\partial x_i} \right) d\Omega = \int_\Omega (\varrho r - \operatorname{div} \mathbf{c})\, d\Omega \tag{9.2}$$

It is customary to express the rate of increase of total heat by the integral expression

$$\frac{đQ}{dt} = \frac{đ}{dt} \int_\Omega \varrho q \, d\Omega = \int_\Omega \varrho \frac{đq}{dt}\, d\Omega \tag{9.3}$$

where

$$\frac{đq}{dt} = r - \frac{\operatorname{div} \mathbf{c}}{\varrho} \tag{9.4}$$

is called the *local thermal energy*. According to Fourier's Law, the heat conduction is governed by

$$\mathbf{c} = -K \operatorname{grad} \theta \tag{9.5}$$

where K, known as the *thermal conductivity*, is a material constant.

9.4 The First Principle of Thermodynamics

In defining the power input and the rate of change of total heat Q, Section 8.5 and Section 9.3, it has been underlined that these two quantities are not exact differentials. In fact, generalizing from many experimental observations, it is found that when a system is subjected to a complete thermodynamic cycle returning to its initial state

$$\oint \frac{đ\mathcal{W}^{(E)}}{dt} dt \neq 0; \qquad\qquad \oint \frac{đQ}{dt} dt \neq 0$$

However, the same experimental evidence shows that

$$\oint \left(\frac{đ\mathcal{W}^{(E)}}{dt} + \frac{đQ}{dt} \right) dt = 0$$

These findings lead to the conclusion of the existence of a *total energy function* of the system E such that

$$dE = đ\mathcal{W}^{(E)} + đQ$$

is a perfect differential. Moreover, expressing the work done by external forces as in Eq. 8.23, we find that this total energy is equal to

$$dE = d\mathcal{E}^{(K)} + dU$$

where

$$dU = đ\mathcal{W}^{(I)} + đQ$$

is a perfect differential. Physically, the existence of a total energy implies the conservation of the energies supplied to the system and the interconvertibility of mechanical and thermal energy. These findings are explicitly stated by the following thermodynamic principle.

First Principle of Thermodynamics. *Thermodynamic systems have a definite and precise total energy E subjected to a definite conservation principle, so that*

$$dE = đ\mathcal{W}^{(E)} + đQ = d\mathcal{E}^{(K)} + dU \tag{9.6}$$

where

$$dU = đ\mathcal{W}^{(I)} + đQ \tag{9.7}$$

is known as internal energy and

$$\begin{aligned}
đ\mathcal{W}^{(E)} &= \int_{\Omega_o} \varrho_o \mathbf{b}^{(o)T} d\mathbf{u}\, d\Omega_o + \int_{\Gamma_o} \mathbf{t}^{(o)T} d\mathbf{u}\, d\Gamma_o \\
đ\mathcal{W}^{(I)} &= \int_{\Omega_o} \mathcal{T}^{(o)} \cdot d\widetilde{\mathcal{F}}\, d\Omega_o = \int_{\Omega_o} \tilde{\mathcal{T}} : d\tilde{\mathcal{L}}\, d\Omega_o \\
d\mathcal{E}^{(K)} &= d \int_{\Omega_o} \varrho_o \frac{v^2}{2}\, d\Omega_o \\
đQ &= \left[\int_{\Omega} \varrho r\, d\Omega - \int_{\Gamma} \mathbf{c}^T \hat{\mathbf{n}}\, d\Gamma \right] dt
\end{aligned}$$

Thus the power input plus the time rate of thermal energy supplied to (or removed from) the continuum equals the time rate of change of the kinetic energy plus the time rate of change of the internal energy. In a complete thermodynamic cycle returning to the initial state

$$\oint \frac{dE}{dt} dt = \oint \left(\frac{đ\mathcal{W}^{(E)}}{dt} + \frac{đQ}{dt} \right) dt = \oint \frac{d\mathcal{E}^{(K)}}{dt} dt + \oint \frac{dU}{dt} dt = 0 \tag{9.8}$$

We notice that since dU is a perfect differential

$$\oint \frac{dU}{dt} dt = 0 \tag{9.9}$$

we have that

$$\oint \frac{d\mathcal{W}^{(i)}}{dt}dt = -\oint \frac{dQ}{dt}dt \tag{9.10}$$

In Continuum Mechanics, it is customary to express the change of internal energy as

$$dU = d\int_{\Omega} \varrho u \, d\Omega = \int_{\Omega} \varrho du \, d\Omega \tag{9.11}$$

where u is known as the *specific internal energy* or as the *internal energy per unit mass*. According to Eqs 9.7, 9.4 and 8.32

$$\dot{u} = \frac{du}{dt} = \frac{dw^{(i)}}{dt} + \frac{dq}{dt} = \frac{1}{\varrho}\mathcal{T} : \widehat{\mathcal{D}} + r - \frac{\operatorname{div}\mathbf{c}}{\varrho} \tag{9.12}$$

9.5 The Second Principle of Thermodynamics

The 1st Principle of Thermodynamics establishes the interconvertibility of all forms of energy present in a thermodynamic system. However, it indicates neither the direction nor the extent of the conversion. This information is instead supplied by the 2nd Principle of Thermodynamics.

This principle is based on the concept of *entropy* which is thought as measure of the internal disorder of a thermodynamic system. The effectiveness of the use of this concept depends upon the fact that an isolated thermodynamic system always tends naturally to reach the maximum disorder.

The entropy concept expresses the fact that nature has its spontaneous tendencies. Any attempt to invert this natural behavior requires a certain amount of additional effort. For example, we know by experience that heat flows spontaneously only from a hotter to a colder part of the body. Eventually the system may reach a thermal equilibrium but the inverse heat flow will never occur spontaneously.

Another immediate example is given by the fact that mechanical work can be transformed through friction into heat. But this heat will never spontaneously change back into the mechanical energy.

The entropy and the 2nd Principle of Thermodynamics found their logical and rigorous explanation in the Quantum Statistical Theory. Herein, we adopt them as postulates, depending upon a posteriori rather then a priori justification.

- **I Postulate.** There exist particular states (called equilibrium states) which, macroscopically, are characterized completely by the specification of the internal energy U and a set of extensive parameters

$$\mathbf{N} = \{N_1, N_2 \ldots N_n\}^T$$

later to be specifically enumerated.

- **II Postulate.** There exists a function S, called *entropy*, of the extensive parameters

  $$S = S(U, \mathbf{N})$$

  defined for all equilibrium states. It is assumed that:

  - the entropy is an homogeneous first-order function of the extensive parameters, so that

    $$S(\lambda U, \lambda N_1, \lambda N_2, \ldots, \lambda N_n) = \lambda S(U, \mathbf{N})$$

  - the entropy is continuous and differentiable and is a monotonically increasing function of the internal energy, so that

    $$\left(\frac{\partial S}{\partial U}\right)_{\mathbf{N}} > 0$$

  - the value assumed by the extensive parameters in the absence of a constraint are those which maximize the entropy over the manifold constrained equilibrium states, that is

    $$\begin{aligned} dS &= 0 \\ d^2 S &< 0 \end{aligned}$$

- **III Postulate.** The entropy of a composite system is additive over the constituent subsystems, that is

  $$S = \sum_{i=1}^{m} S^{(i)}$$

  where

  $$S^{(i)} = S^{(i)}(U^{(i)}, \mathbf{N}^{(i)})$$

  is the entropy of the i-th subsystem.

- **IV Postulate: 2nd Principle of Thermodynamics.** The entropy of an isolated system can never decrease, so that

  $$\Delta S_{\text{isolated system}} \geq 0$$

- **V Postulate: 3rd Principle of Thermodynamics.** The entropy of any system vanishes in the state for which

$$\left(\frac{\partial S}{\partial U}\right)_{\mathrm{N}} = 0$$

that is the zero temperature.

The continuity, differentiability and monotone properties of S, indicate the possibility of solving uniquely the entropy function with respect of U. Thus, we can conversely state that the internal energy can be expressed as

$$U = U(S, \mathrm{N})$$

Moreover, always according to the hypotheses on S, it results that:

- U is a continuous and is monotonically increasing function of the entropy

$$\left(\frac{\partial U}{\partial S}\right)_{\mathrm{N}} > 0$$

- the value assumed by the extensive parameters in the absence of a constraint are those which minimize the internal energy over the manifold constrained equilibrium states, that is

$$\begin{aligned} dU &= 0 \\ d^2U &> 0 \end{aligned}$$

The above postulates can be reformulated in various alternative ways, as for example:

Entropy Maximum Principle. *The equilibrium value of any unconstrained internal parameter is such as to maximize the entropy S for a given value $\overline{U}$ of the total internal energy, that is*

$$\begin{aligned} dS(\overline{U}) &= 0 \\ d^2S(\overline{U}) &< 0 \end{aligned}$$

Energy Minimum Principle. *The equilibrium value of any unconstrained internal parameter is such as to minimize the internal energy U for a given value $\overline{S}$ of the total entropy, that is:*

$$\begin{aligned} dU(\overline{S}) &= 0 \\ d^2U(\overline{S}) &> 0 \end{aligned}$$

The 2nd Principle of Thermodynamics states that for an isolated system the entropy can never decrease. Thus, if an isolated system undergoes a process in which the entropy increases, the exact inverse transformation can not take place since it would imply a decrease of entropy. But, if the process does not increase the entropy, the inverse transformation may take place.

In an isolated system, therefore, thermodynamic processes producing an increase of entropy are irreversible, whereas if the entropy remains constant the process is reversible.

Real processes are always irreversible although in some cases, for practical purposes, they may be considered reversible. Irreversibility does not imply that we can not eventually restore a selected thermodynamic system to its original configuration. It only underlines that in such cases we have to input new energy into the system.

9.6 The Specific Entropy

The Continuum Mechanics Theory assumes that entropy is an extensive variable so that it can be expressed as

$$S = \int_{\Omega} \varrho s \, d\Omega \tag{9.13}$$

where s, known as the *specific entropy* per unit mass, is an intensive property of the continuum. Since a particle of unit mass is a thermodynamic system itself, we can postulate, according to the postulates reported in the previous Section, that

- The specific entropy is a function of the intensive parameters

  $$s = s(u, \boldsymbol{\nu}) \tag{9.14}$$

  where u is local internal energy u and

  $$\boldsymbol{\nu} = \{\nu_1, \nu_2, \ldots, \nu_n\}^T$$

  is a set of parameters later to be specifically enumerated.

- the entropy is continuous and differentiable and is a monotonically increasing function of the internal specific energy u, so that

$$\left(\frac{\partial s}{\partial u}\right)_{\nu} > 0 \tag{9.15}$$

- *2nd Principle of Thermodynamics*, the value assumed by the intensive parameters in the absence of a constraint are those which maximize the entropy over the manifold constrained equilibrium states, that is

$$ds = 0 \tag{9.16}$$
$$d^2 s < 0 \tag{9.17}$$

The continuity, differentiability and monotone properties of s, indicate the possibility of solving uniquely the entropy function with respect of u. Thus, we can conversely state that the internal specific energy can be expressed as, the *caloric equation of state*,

$$u = u(s, \nu) \tag{9.18}$$

Moreover, always according to the hypotheses on s, it results that:

- u is a continuous and is a monotonically increasing function of the entropy

$$\left(\frac{\partial u}{\partial s}\right)_{\nu} > 0 \tag{9.19}$$

- *2nd Principle of Thermodynamics*, the value assumed by the extensive parameters in the absence of a constraint are those which minimize the internal energy over the manifold constrained equilibrium states, that is

$$du = 0 \tag{9.20}$$
$$d^2 u > 0 \tag{9.21}$$

Some time Eq. 9.14 is alternatively written as

$$s = s(u, \nu, \mathbf{X})$$

where the presence of $\mathbf{X}$ in the equations permits that the functional dependence on u and ν can be different for different particles in inhomogeneous media. Accordingly, the inverse relationship in Eq. 9.18 takes on the form

$$u = u(s, \nu, \mathbf{X})$$

We suppose the dependence on $\mathbf{X}$ and on the other variables to be continuous, but for most purposes we do not explicitly include $\mathbf{X}$ in the equations.

9.7 Thermodynamic Pressure and Temperature

According to Eq. 9.18, for any change in the thermodynamic state of a given particle, the variation of the internal energy u is given by the *Gibbs relation*

$$du = \theta ds + \boldsymbol{\tau}^T d\boldsymbol{\nu} \tag{9.22}$$

where

- θ is an intensive parameter known as *thermodynamic temperature* and defined as
$$\theta = \left(\frac{\partial u}{\partial s}\right)_{\boldsymbol{\nu}} > 0 \tag{9.23}$$
The positiveness of θ is given by Eq. 9.19.

- $\boldsymbol{\tau}$ is a set of intensive parameters known as *thermodynamic pressure* and defined as
$$\boldsymbol{\tau} = \{\tau_1, \tau_2, \ldots, \tau_n\}^T$$
where
$$\tau_i = \left(\frac{\partial u}{\partial \nu_i}\right)_{s,\nu_j} \tag{9.24}$$
for all $j \neq i$.

We remark that according to Eq. 9.21,

$$d^2u = d\mathbf{z}^T\mathbf{H}d\mathbf{z} > 0 \tag{9.25}$$

for any $d\mathbf{z} \neq \mathbf{o}$, where

$$\begin{aligned} d\mathbf{z} &= \{ds, d\nu_1, d\nu_2, \ldots, d\nu_n\}^T \\ \mathbf{H} &= \begin{bmatrix} H_{11} & \mathbf{H}^{(12)} \\ \mathbf{H}^{(12)T} & \mathbf{H}^{(22)} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} H_{11} &= \frac{\partial \theta}{\partial s} = \frac{\partial^2 u}{\partial s^2} \\ \mathbf{H}^{(12)} &= \left[\frac{\partial \theta}{\partial \nu_j}\right] = \left[\frac{\partial^2 u}{\partial s \partial \nu_j}\right] \\ \mathbf{H}^{(22)} &= \left[\frac{\partial \tau_i}{\partial \nu_j}\right] = \left[\frac{\partial^2 u}{\partial \nu_i \partial \nu_j}\right] \end{aligned}$$

and $\mathbf{o}$ is a null vector. Hence, the Hessian matrix $\mathbf{H}$ of $\mathbf{u}$ is a symmetric and positive definite. Consequently, [38] Theorems 7.3.2 and 7.34, it is a nonsingular matrix and all its diagonal minors are symmetric and positive definite too. In particular,

$$\frac{\partial\theta}{\partial s} > 0 \tag{9.26}$$

and the matrix $\mathbf{H}^{(22)}$ is symmetric and positive definite.

9.8 Equations of State

Any relationship expressing intensive parameters in terms of independent parameters is called *equation of state*. Thus, Eqs. 9.14 and 9.18 are two equations of state; from these equations and the definition of the thermodynamic temperature and pressure in Eqs. 9.23 and 9.24, respectively, we can define the following set of equations of state:

- Since according to Eq. 9.18

$$u = u(s, \boldsymbol{\nu}) \tag{9.27}$$

  we can conclude that the thermodynamic temperature and pressure defined in Eqs. 9.23 and 9.24, respectively, can be expressed as

$$\theta = \theta(s, \boldsymbol{\nu}) \tag{9.28}$$
$$\boldsymbol{\tau} = \boldsymbol{\tau}(s, \boldsymbol{\nu}) \tag{9.29}$$

- According to Eq. 9.14

$$s = s(u, \boldsymbol{\nu}) \tag{9.30}$$

  which substituted in Eqs. 9.28 and 9.29 yields to

$$\theta = \theta(u, \boldsymbol{\nu}) \tag{9.31}$$
$$\boldsymbol{\tau} = \boldsymbol{\tau}(u, \boldsymbol{\nu}) \tag{9.32}$$

- The condition in Eq. 9.26 ensure the possibility of inverting Eq. 9.28 for s. This gives

$$s = s(\theta, \boldsymbol{\nu}) \tag{9.33}$$

  which substituted in Eqs. 9.27 and 9.29 yields to

$$u = u(\theta, \boldsymbol{\nu}) \tag{9.34}$$
$$\boldsymbol{\tau} = \boldsymbol{\tau}(\theta, \boldsymbol{\nu}) \tag{9.35}$$

- The positive definiteness of the diagonal submatrix $\mathbf{H}^{(22)}$ of $\mathbf{H}$ in Eq. 9.25 ensure the possibility of inverting Eq. 9.29 for $\boldsymbol{\nu}$. This gives

$$\boldsymbol{\nu} = \boldsymbol{\nu}(s, \boldsymbol{\tau}) \tag{9.36}$$

which substituted in Eqs. 9.27 and 9.28 yields to

$$u = u(s, \boldsymbol{\tau}) \tag{9.37}$$
$$\theta = \theta(s, \boldsymbol{\tau}) \tag{9.38}$$

- The condition in Eq. 9.26 ensure the possibility of inverting Eq. 9.38 for s. This gives

$$s = s(\theta, \boldsymbol{\tau}) \tag{9.39}$$

which substituted in Eqs. 9.37 and 9.36 yields to

$$u = u(\theta, \boldsymbol{\tau}) \tag{9.40}$$
$$\boldsymbol{\nu} = \boldsymbol{\nu}(\theta, \boldsymbol{\tau}) \tag{9.41}$$

It is customary to call

$$\begin{aligned} s &= s(u, \boldsymbol{\nu}) \\ u &= u(s, \boldsymbol{\nu}) \end{aligned}$$

as the *caloric equations of state*,

$$\begin{aligned} \boldsymbol{\tau} &= \boldsymbol{\tau}(\theta, \boldsymbol{\nu}) \\ \boldsymbol{\nu} &= \boldsymbol{\nu}(\theta, \boldsymbol{\tau}) \end{aligned}$$

as the *thermal equations of state*,

$$F(\theta, \boldsymbol{\nu}, \boldsymbol{\tau}) = 0$$

as the *kinetic equation of state*.

9.9 Thermodynamic Processes

Let us indicate the caloric equation of state in Eq. 9.18 as

$$u = u(\mathbf{z})$$

where

$$\mathbf{z} = \{z_0, z_1, \ldots, z_n\}^T = \{s, \nu_1, \ldots, \nu_n\}^T$$

Accordingly, we can indicate the definition of the thermodynamic temperature and pressure, Section 9.7, as

$$\mathbf{t} = \frac{\partial u}{\partial \mathbf{z}}$$

where

$$\mathbf{t} = \{t_0, t_1, \ldots, t_{n+1}\}^T = \{\theta, \tau_1, \ldots, \tau_n\}^T$$

Then, any variation of the parameters in $\mathbf{z}$ cause a variation of the parameters in $\mathbf{t}$ which can be calculated as

$$d\mathbf{t} = \mathbf{U}d\mathbf{z} \tag{9.42}$$

where

$$\mathbf{U} = \left[\frac{\partial t_i}{\partial z_j}\right] = \left[\frac{\partial^2 u}{\partial z_i \partial z_j}\right]$$

is the Hessian matrix of u which has been proved to be a symmetric positive definite matrix, Eq. 9.25. The relationship in Eq. 9.42 indicates a *thermodynamic process caused* by a variation $d\mathbf{z}$ leading thereby to an *effect* $d\mathbf{t}$. We notice that u is the potential function of this thermodynamic process.

However, not all thermodynamic processes are caused by a variation $d\mathbf{z}$. For example, the exact opposite may happen where the cause is a variation $d\mathbf{t}$ and the effect is a variation of $d\mathbf{z}$. Since $\mathbf{U}$ is a positive definite matrix and thus nonsingular, the relationship in Eq. 9.42 can be inverted and obtain

$$d\mathbf{z} = \mathbf{U}'d\mathbf{t} \tag{9.43}$$

where

$$\mathbf{U}' = \mathbf{U}^{-1}$$

As u is the potential function for the thermodynamic process in Eq. 9.43, it must exist a potential function, say

$$\varphi = \varphi(\mathbf{t})$$

such that

$$\begin{aligned} \mathbf{z} &= \frac{\partial \varphi}{\partial \mathbf{t}} \\ \mathbf{U}' &= \left[\frac{\partial^2 \varphi}{\partial t_i \partial t_j}\right] \end{aligned}$$

The identification of this potential function can be accomplished through the so-called Legendre Transformation Technique.

9.9.1 The Legendre Transformation Technique

Let

$$u = u(\mathbf{z}) \tag{9.44}$$

be a function of the independent variables $\mathbf{z}$. Accordingly, we can define

$$du = \mathbf{t}^T d\mathbf{z} \tag{9.45}$$

$$d^2u = d\mathbf{z}^T \mathbf{U} d\mathbf{z} \tag{9.46}$$

where

$$\mathbf{t} = \left\{ \frac{\partial u}{\partial z_i} \right\}^T \tag{9.47}$$

and

$$\mathbf{U} = \left[\frac{\partial t_i}{\partial z_j} \right] = \left[\frac{\partial^2 u}{\partial z_i \partial z_j} \right]$$

Finally, we can relate any variation of $d\mathbf{t}$ caused by $d\mathbf{z}$ through the linear transformation

$$d\mathbf{t} = \mathbf{U} d\mathbf{z} \tag{9.48}$$

We remark that under the above hypothesis, the matrix $\mathbf{U}$ is a symmetric matrix. If we assume that

$$d^2u > 0 \tag{9.49}$$

then u represents the potential function for the transformation in Eq. 9.48 and $\mathbf{U}$ results to be a positive definite matrix.

The inversion of the linear relationship in Eq. 9.48 yields to

$$-d\mathbf{z} = \mathbf{G} d\mathbf{t} \tag{9.50}$$

where

$$\mathbf{G} = -\mathbf{U}^{-1}$$

Moreover, if $\mathbf{U}$ is positive definite then $\mathbf{G}$ exists and, [38] Theorem 7.5.1,

$$d^2g = d\mathbf{t}^T \mathbf{G} d\mathbf{t} < 0$$

for all non trivial vectors $\mathbf{t}$.

According to the Legendre Transformation Technique, provided that it is possible to express

$$u = u(\mathbf{t})$$

the transformation matrix $\mathbf{G}$ can be calculated as

$$\mathbf{G} = \left[\frac{\partial z_i}{\partial t_j}\right] = \left[\frac{\partial^2 g}{\partial t_i \partial t_j}\right] \tag{9.51}$$

where the function g is defined as

$$g = g(\mathbf{t}) = u - \mathbf{t}^T\mathbf{z} \tag{9.52}$$

Moreover, it can be verified that

$$dg = -\mathbf{z}^T d\mathbf{t} \tag{9.53}$$

$$d^2 g = d\mathbf{t}^T \mathbf{G} d\mathbf{t} \tag{9.54}$$

where

$$-\mathbf{z} = \left\{\frac{\partial g}{\partial t_i}\right\}^T \tag{9.55}$$

The above relationships can be proved noticing that

$$\begin{aligned} \frac{\partial g}{\partial t_k} &= \frac{\partial}{\partial t_k}(u - t_i z_i) = \frac{\partial u}{\partial t_k} - \frac{\partial t_i}{\partial t_k} z_i - t_i \frac{\partial z_i}{\partial t_k} = \\ &= \frac{\partial u}{\partial t_k} - \delta_{ik} z_i - \frac{\partial u}{\partial z_i}\frac{\partial z_i}{\partial t_k} = \frac{\partial u}{\partial t_k} - z_k - \frac{\partial u}{\partial t_k} = -z_k \end{aligned}$$

which then verify Eq. 9.55. Consequently,

$$dg = \frac{\partial g}{\partial t_k} dt_k = -z_k dt_k = -\mathbf{z}^T d\mathbf{t}$$

This result can be also directly obtained from Eq. 9.52 as

$$\begin{aligned} dg &= d(u - \mathbf{t}^T\mathbf{z}) = du - d\mathbf{t}^T\mathbf{z} - \mathbf{t}^T d\mathbf{z} = \\ &= -d\mathbf{t}^T\mathbf{z} \end{aligned}$$

since from Eq. 9.45

$$du = \mathbf{t}^T d\mathbf{z}$$

9.9.2 The Generalized Legendre Transformation Technique

Let

$$u = u(\mathbf{z}^{(1)}, \mathbf{z}^{(2)}) \tag{9.56}$$

be a function of the independent variables $(\mathbf{z}^{(1)}, \mathbf{z}^{(2)})$. Accordingly, we can define

$$du = \mathbf{t}^{(1)T} d\mathbf{z}^{(1)} + \mathbf{t}^{(2)T} d\mathbf{z}^{(2)} \tag{9.57}$$

$$d^2u = d\mathbf{z}^{(1)T}\mathbf{U}^{(11)} d\mathbf{z}^{(1)} + 2d\mathbf{z}^{(1)T}\mathbf{U}^{(12)} d\mathbf{z}^{(2)} + d\mathbf{z}^{(2)T}\mathbf{U}^{(22)} d\mathbf{z}^{(2)} \tag{9.58}$$

where

$$\mathbf{t}^{(1)} = \left\{ \frac{\partial u}{\partial z_i^{(1)}} \right\}^T \tag{9.59}$$

$$\mathbf{t}^{(2)} = \left\{ \frac{\partial u}{\partial z_i^{(2)}} \right\}^T \tag{9.60}$$

and

$$\mathbf{U}^{(11)} = \left[\frac{\partial t_i^{(1)}}{\partial z_j^{(1)}} \right] = \left[\frac{\partial^2 u}{\partial z_i^{(1)} \partial z_j^{(1)}} \right]$$

$$\mathbf{U}^{(12)} = \left[\frac{\partial t_i^{(1)}}{\partial z_j^{(2)}} \right] = \left[\frac{\partial^2 u}{\partial z_i^{(1)} \partial z_j^{(2)}} \right]$$

$$\mathbf{U}^{(22)} = \left[\frac{\partial t_i^{(2)}}{\partial z_j^{(2)}} \right] = \left[\frac{\partial^2 u}{\partial z_i^{(2)} \partial z_j^{(2)}} \right]$$

Finally, we can relate any variation of $(d\mathbf{t}^{(1)}, d\mathbf{t}^{(2)})$ caused by $(d\mathbf{z}^{(1)}, d\mathbf{z}^{(2)})$ by the linear transformation

$$\left\{ \begin{array}{c} d\mathbf{t}^{(1)} \\ d\mathbf{t}^{(2)} \end{array} \right\} = \left[\begin{array}{cc} \mathbf{U}^{(11)} & \mathbf{U}^{(12)} \\ \mathbf{U}^{(12)T} & \mathbf{U}^{(22)} \end{array} \right] \left\{ \begin{array}{c} d\mathbf{z}^{(1)} \\ d\mathbf{z}^{(2)} \end{array} \right\} \tag{9.61}$$

We remark that under the above hypothesis, the matrix

$$\mathbf{U} = \left[\begin{array}{cc} \mathbf{U}^{(11)} & \mathbf{U}^{(12)} \\ \mathbf{U}^{(12)T} & \mathbf{U}^{(22)} \end{array} \right]$$

and the square submatrices $\mathbf{U}^{(ii)}$ are symmetric matrices. If we assume that

$$d^2u > 0 \tag{9.62}$$

the u represents the potential function for the linear transformation in Eq. 9.61 and $\mathbf{U}$ and $\mathbf{U}^{(ii)}$ result to be positive definite matrices, that is

$$\begin{aligned} d^2u &= d\mathbf{z}^T\mathbf{U}d\mathbf{z} > 0 \\ d^2\overline{u} &= d\mathbf{z}^{(1)T}\mathbf{U}^{(11)}d\mathbf{z}^{(1)} > 0 \\ d^2\tilde{u} &= d\mathbf{z}^{(2)T}\mathbf{U}^{(22)}d\mathbf{z}^{(2)} > 0 \end{aligned}$$

for all non trivial vectors $d\mathbf{z}, d\mathbf{z}^{(1)}$ and $d\mathbf{z}^{(2)}$.

It is possible to prove that, Theorem 7.6.3 in [38], the partial inversion of the linear relationship in Eq. 9.61 is given by

$$\left\{ \begin{array}{c} -d\mathbf{z}^{(1)} \\ d\mathbf{t}^{(2)} \end{array} \right\} = \left[\begin{array}{cc} \mathbf{G}^{(11)} & \mathbf{G}^{(12)} \\ \mathbf{G}^{(12)T} & \mathbf{G}^{(22)} \end{array} \right] \left\{ \begin{array}{c} d\mathbf{t}^{(1)} \\ d\mathbf{z}^{(2)} \end{array} \right\} \tag{9.63}$$

where

$$\begin{aligned} \mathbf{G}^{(11)} &= -\left[\mathbf{U}^{(11)}\right]^{-1} \\ \mathbf{G}^{(12)} &= -\mathbf{G}^{(11)}\mathbf{U}^{(12)} \\ \mathbf{G}^{(22)} &= \left[\mathbf{U}^{(22)} + \mathbf{U}^{(21)}\mathbf{G}^{(11)}\mathbf{U}^{(12)}\right] \end{aligned}$$

Moreover, if $\mathbf{U}$ is positive definite then $\mathbf{G}^{(ii)}$ exist and

$$\begin{aligned} d^2\overline{g} &= d\mathbf{t}^{(1)T}\mathbf{G}^{(11)}d\mathbf{t}^{(1)} < 0 \\ d^2\tilde{g} &= d\mathbf{z}^{(2)T}\mathbf{G}^{(22)}d\mathbf{z}^{(2)} > 0 \end{aligned}$$

for all non trivial vectors $d\mathbf{t}^{(1)}$ and $d\mathbf{z}^{(2)}$

According to the Generalized Legendre Transformation Technique, if it is possible to express

$$\mathbf{z}^{(1)} = \mathbf{z}^{(1)}(\mathbf{t}^{(1)}, \mathbf{z}^{(2)})$$

so that, Eq. 9.56,

$$u = u(\mathbf{t}^{(1)}, \mathbf{z}^{(2)})$$

then the matrices $\mathbf{G}^{(ij)}$ can be calculated as

$$\begin{aligned} \mathbf{G}^{(11)} &= \left[\frac{\partial z_i^{(1)}}{\partial t_j^{(1)}}\right] = \left[\frac{\partial^2 g}{\partial t_i^{(1)}\partial t_j^{(1)}}\right] \\ \mathbf{G}^{(12)} &= \left[\frac{\partial z_i^{(1)}}{\partial z_j^{(2)}}\right] = \left[\frac{\partial^2 g}{\partial t_i^{(1)}\partial z_j^{(2)}}\right] \\ \mathbf{G}^{(22)} &= \left[\frac{\partial t_i^{(2)}}{\partial z_j^{(2)}}\right] = \left[\frac{\partial^2 g}{\partial z_i^{(2)}\partial z_j^{(2)}}\right] \end{aligned}$$

where the function g is defined as

$$g = g(\mathbf{t}^{(1)}, \mathbf{z}^{(2)}) = u - \mathbf{t}^{(1)T}\mathbf{z}^{(1)} \tag{9.64}$$

Moreover, it can be verified that

$$\begin{aligned} dg &= -\mathbf{z}^{(1)T}d\mathbf{t}^{(1)} + \mathbf{t}^{(2)T}d\mathbf{z}^{(2)} & (9.65)\\ d^2g &= d\mathbf{t}^{(1)T}\mathbf{G}^{(11)}d\mathbf{t}^{(1)} + 2d\mathbf{t}^{(1)T}\mathbf{G}^{(12)}d\mathbf{z}^{(2)} + d\mathbf{z}^{(2)T}\mathbf{G}^{(22)}d\mathbf{z}^{(2)} & (9.66)\end{aligned}$$

where

$$\begin{aligned} -\mathbf{z}^{(1)} &= \left\{\frac{\partial g}{\partial t_i^{(1)}}\right\}^T & (9.67)\\ \mathbf{t}^{(2)} &= \left\{\frac{\partial g}{\partial z_i^{(2)}}\right\}^T & (9.68)\end{aligned}$$

The above relationships can be proved noticing that

$$\begin{aligned} \frac{\partial g}{\partial t_k^{(1)}} &= \frac{\partial}{\partial t_k^{(1)}}(u - t_i^{(1)}z_i^{(1)}) = \frac{\partial u}{\partial t_k^{(1)}} - \frac{\partial t_i^{(1)}}{\partial t_k^{(1)}}z_i^{(1)} - t_i^{(1)}\frac{\partial z_i^{(1)}}{\partial t_k^{(1)}} = \\ &= \frac{\partial u}{\partial t_k^{(1)}} - \delta_{ik}z_i^{(1)} - \frac{\partial u}{\partial z_i^{(1)}}\frac{\partial z_i^{(1)}}{\partial t_k^{(1)}} = \frac{\partial u}{\partial t_k^{(1)}} - z_k^{(1)} - \frac{\partial u}{\partial t_k^{(1)}} = -z_k^{(1)} \\ \frac{\partial g}{\partial z_k^{(2)}} &= \frac{\partial}{\partial z_k^{(2)}}(u - t_i^{(1)}z_i^{(1)}) = t_k^{(2)}\end{aligned}$$

which then verify Eqs. 9.67 and 9.68 Consequently,

$$\begin{aligned} dg &= \frac{\partial g}{\partial t_k^{(1)}}dt_k^{(1)} + \frac{\partial g}{\partial z_k^{(2)}}dz_k^{(2)} = -z_k^{(1)}dt_k^{(1)} + t_k^{(2)}dz_k^{(2)} = \\ &= -\mathbf{z}^{(1)T}d\mathbf{t}^{(1)} + \mathbf{t}^{(2)T}d\mathbf{z}^{(2)}\end{aligned}$$

This result can be also directly obtained from Eq. 9.64 as

$$\begin{aligned} dg &= d(u - \mathbf{t}^{(1)T}\mathbf{z}^{(1)}) = du - d\mathbf{t}^{(1)T}\mathbf{z}^{(1)} - \mathbf{t}^{(1)T}d\mathbf{z}^{(1)} = \\ &= -d\mathbf{t}^{(1)T}\mathbf{z}^{(1)} + \mathbf{t}^{(2)T}d\mathbf{z}^{(2)}\end{aligned}$$

since from Eq. 9.57

$$du - \mathbf{t}^{(1)T}d\mathbf{z}^{(1)} = \mathbf{t}^{(2)T}d\mathbf{z}^{(2)}$$

9.10 Potential Thermodynamic Functions

Thermodynamic processes of common interest are those where it is possible to control the four alternative pairs combinations of $ds, d\theta, d\boldsymbol{\nu}$ and $d\boldsymbol{\tau}$. The caloric equation of state $u = u(s, \boldsymbol{\nu})$ in Eq. 9.18 allows to control thermodynamic processes where $(ds, d\boldsymbol{\nu})$ are the causes and $(d\theta, d\boldsymbol{\tau})$ are the effects. The function controlling the other three alternative thermodynamic processes can be determined from $u = u(s, \boldsymbol{\nu})$ following the mathematical procedure reported in the Section 9.9. Hence:

- The *local internal function*

$$u = u(s, \boldsymbol{\nu}) \tag{9.69}$$

is the potential function for the thermodynamic process

$$\left\{ \begin{array}{c} d\theta \\ d\boldsymbol{\tau} \end{array} \right\} = \left[\begin{array}{cc} U_{11} & \mathbf{U}^{(12)} \\ \mathbf{U}^{(12)T} & \mathbf{U}^{(22)} \end{array} \right] \left\{ \begin{array}{c} ds \\ d\boldsymbol{\nu} \end{array} \right\} \tag{9.70}$$

where, Section 9.7,

$$\theta = \frac{\partial u}{\partial s} > 0 \tag{9.71}$$

$$\boldsymbol{\tau} = \left\{ \frac{\partial u}{\partial \nu_i} \right\}^T \tag{9.72}$$

and

$$\begin{aligned} U_{11} &= \frac{\partial \theta}{\partial s} = \frac{\partial^2 u}{\partial s^2} > 0 \\ \mathbf{U}^{(12)} &= \left[\frac{\partial \theta}{\partial \nu_j} \right] = \left[\frac{\partial^2 u}{\partial s \partial \nu_j} \right] \\ \mathbf{U}^{(22)} &= \left[\frac{\partial \tau_i}{\partial \nu_j} \right] = \left[\frac{\partial^2 u}{\partial \nu_i \partial \nu_j} \right] \end{aligned}$$

with

$$d\boldsymbol{\nu}^T \mathbf{U}^{(22)} d\boldsymbol{\nu} > 0$$

Finally,

$$du = \theta ds + \boldsymbol{\tau}^T d\boldsymbol{\nu} \tag{9.73}$$

$$d^2 u = U_{11} d^2 s + 2ds \mathbf{U}^{(12)} d\boldsymbol{\nu} + d\boldsymbol{\nu}^T \mathbf{U}^{(22)} d\boldsymbol{\nu} \tag{9.74}$$

According to the 2nd Principle of Thermodynamic, Section 9.6, equilibrium states are those which minimize the internal energy u. Notice that in a isentropic process, where $s = const$ that is $ds = 0$, u is the potential function for $\boldsymbol{\tau}$.

- The *Helmholtz free energy*

$$\psi = \psi(\theta, \boldsymbol{\nu}) = u - s\theta \tag{9.75}$$

where, according to Eqs. 9.33 and 9.34, we can express

$$\begin{aligned} s &= s(\theta, \boldsymbol{\nu}) \\ u &= u(\theta, \boldsymbol{\nu}) \end{aligned}$$

governs the thermodynamic process

$$\left\{ \begin{array}{c} -ds \\ d\boldsymbol{\tau} \end{array} \right\} = \left[\begin{array}{cc} \Psi_{11} & \boldsymbol{\Psi}^{(21)} \\ \boldsymbol{\Psi}^{(12)T} & \boldsymbol{\Psi}^{(22)} \end{array} \right] \left\{ \begin{array}{c} d\theta \\ d\boldsymbol{\nu} \end{array} \right\} \tag{9.76}$$

in the sense that

$$-s = \left(\frac{\partial \psi}{\partial \theta}\right)_{\boldsymbol{\nu}} \leq 0 \tag{9.77}$$

$$\boldsymbol{\tau} = \left\{\frac{\partial \psi}{\partial \nu_i}\right\}_{\theta}^{T} \tag{9.78}$$

and

$$\begin{aligned} \Psi_{11} &= \frac{\partial s}{\partial \theta} = \frac{\partial^2 \psi}{\partial \theta^2} < 0 \\ \boldsymbol{\Psi}^{(12)} &= \left[\frac{\partial s}{\partial \nu_j}\right] = \left[\frac{\partial^2 \psi}{\partial \theta \partial \nu_j}\right] \\ \boldsymbol{\Psi}^{(22)} &= \left[\frac{\partial \tau_i}{\partial \nu_j}\right] = \left[\frac{\partial^2 \psi}{\partial \nu_i \partial \nu_j}\right] \end{aligned}$$

with

$$d\boldsymbol{\nu}^T \boldsymbol{\Psi}^{(22)} d\boldsymbol{\nu} > 0$$

Finally,

$$d\psi = du - \theta ds - s d\theta = -s d\theta + \boldsymbol{\tau}^T d\boldsymbol{\nu} \tag{9.79}$$

$$d^2\psi = \Psi_{11} d^2\theta + d\theta \boldsymbol{\Psi}^{(12)} d\boldsymbol{\nu} + d\boldsymbol{\nu}^T \boldsymbol{\Psi}^{(22)} d\boldsymbol{\nu} \tag{9.80}$$

Notice that in an isothermal process, where $\theta = const$ that is $d\theta = 0$, ψ is the portion of internal energy for doing work. In this case, ψ is the potential function for $\boldsymbol{\tau}$ and equilibrium states are those which minimize ψ. The thermodynamic process in Eq. 9.76 can be considered to be obtained from that in Eq. 9.70 by inverting $d\theta$ with ds.

- The *entaply free energy*

$$h = h(s, \boldsymbol{\tau}) = u - \boldsymbol{\nu}^T \boldsymbol{\tau} \tag{9.81}$$

where, according to Eqs. 9.37 and 9.36, we can express

$$\begin{aligned} u &= u(s, \boldsymbol{\tau}) \\ \boldsymbol{\nu} &= \boldsymbol{\nu}(s, \boldsymbol{\tau}) \end{aligned}$$

governs the thermodynamic process

$$\left\{ \begin{array}{c} d\theta \\ -d\boldsymbol{\nu} \end{array} \right\} = \left[\begin{array}{cc} H_{11} & \mathbf{H}^{(12)} \\ \mathbf{H}^{(12)T} & \mathbf{H}^{(22)} \end{array} \right] \left\{ \begin{array}{c} ds \\ d\boldsymbol{\tau} \end{array} \right\} \tag{9.82}$$

in the sense that

$$\theta = \left(\frac{\partial h}{\partial s} \right)_{\boldsymbol{\tau}} \geq 0 \tag{9.83}$$

$$-\boldsymbol{\nu} = \left\{ \frac{\partial h}{\partial \tau_i} \right\}_s^T \tag{9.84}$$

and

$$\begin{aligned} H_{11} &= \frac{\partial \theta}{\partial s} = \frac{\partial^2 h}{\partial s^2} > 0 \\ \mathbf{H}^{(12)} &= \left[\frac{\partial \theta}{\partial \tau_j} \right] = \left[\frac{\partial^2 h}{\partial s \partial \tau_j} \right] \\ \mathbf{H}^{(22)} &= \left[\frac{\partial \nu_i}{\partial \tau_j} \right] = \left[\frac{\partial^2 h}{\partial \tau_i \partial \tau_j} \right] \end{aligned}$$

with

$$d\boldsymbol{\tau}^T \mathbf{H}^{(22)} d\boldsymbol{\tau} < 0$$

Finally,

$$dh = du - \boldsymbol{\tau}^T d\boldsymbol{\nu} - \boldsymbol{\nu}^T d\boldsymbol{\tau} = \theta ds - \boldsymbol{\nu}^T d\boldsymbol{\tau} \tag{9.85}$$

$$d^2 h = H_{11} d^2 s + 2ds \mathbf{H}^{(12)} d\boldsymbol{\tau} + d\boldsymbol{\tau}^T \mathbf{H}^{(22)} d\boldsymbol{\tau} \tag{9.86}$$

Notice that in a thermodynamic process where the pressure $\boldsymbol{\tau}$ is held constant, that is $d\boldsymbol{\tau} = \mathbf{o}$, h represents the portion of internal energy that can be released as heat and equilibrium states are those which minimize h. The thermodynamic process in Eq. 9.82 can be considered to be obtained from that in Eq. 9.70 by inverting $d\boldsymbol{\tau}$ with $d\boldsymbol{\nu}$.

- The *free entaply* or *Gibbs function*

$$g = g(\theta, \boldsymbol{\tau}) = u - s\theta - \boldsymbol{\tau}^T\boldsymbol{\nu} = h - s\theta \tag{9.87}$$

where, according to Eqs. 9.40, 9.39 and 9.41, we can express

$$\begin{aligned} u &= u(\theta, \boldsymbol{\tau}) \\ s &= s(\theta, \boldsymbol{\tau}) \\ \boldsymbol{\nu} &= \boldsymbol{\nu}(\theta, \boldsymbol{\tau}) \end{aligned}$$

governs the thermodynamic process

$$\left\{ \begin{array}{c} -ds \\ -d\boldsymbol{\nu} \end{array} \right\} = \left[\begin{array}{cc} G_{11} & \mathbf{G}^{(12)} \\ \mathbf{G}^{(12)T} & \mathbf{G}^{(22)} \end{array} \right] \left\{ \begin{array}{c} d\theta \\ d\boldsymbol{\tau} \end{array} \right\} \tag{9.88}$$

in the sense that

$$-s = \left(\frac{\partial g}{\partial \theta}\right)_{\boldsymbol{\tau}} \leq 0 \tag{9.89}$$

$$-\boldsymbol{\nu} = \left\{\frac{\partial g}{\partial \tau_i}\right\}_{\theta}^{T} \tag{9.90}$$

and

$$\begin{aligned} G_{11} &= \frac{\partial s}{\partial \theta} = \frac{\partial^2 g}{\partial \theta^2} < 0 \\ \mathbf{G}^{(12)} &= \left[\frac{\partial s}{\partial \tau_j}\right] = \left[\frac{\partial^2 g}{\partial \theta \partial \tau_j}\right] \\ \mathbf{G}^{(22)} &= \left[\frac{\partial \nu_i}{\partial \tau_j}\right] = \left[\frac{\partial^2 g}{\partial \tau_i \partial \tau_j}\right] \end{aligned}$$

with

$$d\boldsymbol{\tau}^T\mathbf{G}^{(22)}d\boldsymbol{\tau} < 0$$

Finally,

$$dg = -s d\theta - \boldsymbol{\nu}^T d\boldsymbol{\tau} \tag{9.91}$$

$$d^2 g = G_{11} d^2\theta + d\theta \mathbf{G}^{(12)} d\boldsymbol{\tau} + d\boldsymbol{\tau}^T \mathbf{G}^{(22)} d\boldsymbol{\tau} < 0 \tag{9.92}$$

Notice that the Gibbs function can be seen as the complementary internal energy and, according to the 2nd Principle of Thermodynamic, Section 9.6, equilibrium states are those which maximize the Gibbs function g. The thermodynamic process in Eq. 9.88 can be considered to be obtained by inverting that in Eq. 9.70.

9.11 The Clausius-Duhem Inequality

Let $\mathcal{M}$ be a continuum of density ϱ instantaneously occupying a volume Ω bounded by a surface Γ. According to the definition in Eq. 9.13, the rate of change of the entropy in $\mathcal{M}$ is given by

$$\frac{dS}{dt} = \frac{d}{dt}\int_\Omega \varrho s d\Omega = \int_\Omega \varrho \frac{ds}{dt} d\Omega \tag{9.93}$$

In Continuum Mechanics, the *rate of change of the entropy carried by heat transfer* is defined as

$$\frac{d\tilde{S}}{dt} = \int_\Omega \frac{\varrho r}{\theta} d\Omega - \int_\Gamma \frac{\mathbf{c}^T \hat{\mathbf{n}}}{\theta} d\Gamma \tag{9.94}$$

where r and $\mathbf{c}$ are defined in Section 9.3. The Gauss Divergence Theorem allows to transform Eq. 9.94 as

$$\frac{d\tilde{S}}{dt} = \int_\Omega \left[\frac{\varrho r}{\theta} - \operatorname{div}\frac{\mathbf{c}}{\theta}\right] d\Omega \tag{9.95}$$

The Clausius-Duhem inequality states that the rate of change of the entropy dS is greater than or equal to the rate of change of the entropy $d\tilde{S}$ carried by heat transfer, namely

$$\frac{dS}{dt} \geq \frac{d\tilde{S}}{dt} \tag{9.96}$$

where the inequality holds for irreversible processes. The relative local form is given by, Eqs. 9.93 and 9.95,

$$\frac{ds}{dt} \geq \frac{r}{\theta} - \frac{1}{\varrho}\operatorname{div}\frac{\mathbf{c}}{\theta} \tag{9.97}$$

The quantity

$$\gamma = \frac{ds}{dt} - \frac{r}{\theta} + \frac{1}{\varrho}\,\mathrm{div}\,\frac{\mathbf{c}}{\theta} \geq 0 \qquad (9.98)$$

is called *internal entropy production* per unit mass. Since

$$\mathrm{div}\,\frac{\mathbf{c}}{\theta} = \frac{\mathrm{div}\,\mathbf{c}}{\theta} - \frac{\mathbf{c}^T\,\mathrm{grad}\,\theta}{\theta^2}$$

The internal entropy production can be also expressed as

$$\gamma = \gamma^{(loc)} + \gamma^{(con)} \geq 0 \qquad (9.99)$$

where

$$\begin{aligned}\gamma^{(loc)} &= \frac{ds}{dt} - \frac{1}{\theta}\left[r - \frac{\mathrm{div}\,\mathbf{c}}{\varrho}\right] = \frac{ds}{dt} - \frac{1}{\theta}\frac{dq}{dt} \\ \gamma^{(con)} &= -\frac{\mathbf{c}^T\,\mathrm{grad}\,\theta}{\varrho\theta^2}\end{aligned}$$

and $\gamma^{(loc)}$ represents the *local entropy production* while $\gamma^{(con)}$ represents the *entropy production by heat conduction.*

Truesdell and Noll, [93], propose a stronger assumption than that in Eq. 9.99 requiring separately

$$\gamma^{(loc)} \geq 0 \quad \text{and} \quad \gamma^{(con)} \geq 0 \qquad (9.100)$$

and for reversible processes

$$\gamma^{(loc)} = 0 \quad \text{and} \quad \gamma^{(con)} = 0 \qquad (9.101)$$

Notice that the inequality $\gamma^{(con)} \geq 0$ simply implies that heat does not flow spontaneously towards a hotter part of the body. On the other hand, the equality $\gamma^{(loc)} = 0$ implies that in a reversible process

$$ds = \frac{dq}{\theta} \qquad (9.102)$$

9.12 Fully Recoverable Stress Power Processes

It is possible to prove that for two alternative thermodynamic processes the internal work $dw^{(I)}$ is fully recoverable and equal to

$$dw^{(I)} = \tau^T d\nu \qquad (9.103)$$

Moreover, there exists a potential function

$$w = w(\boldsymbol{\nu}) \tag{9.104}$$

for $\boldsymbol{\tau}$ such that

$$\begin{aligned} \boldsymbol{\tau} &= \frac{\partial w}{\partial \boldsymbol{\nu}} \\ dw &= \boldsymbol{\tau}^T d\boldsymbol{\nu} \\ d^2 w &= d\boldsymbol{\nu}^T \mathbf{W} d\boldsymbol{\nu} > 0 \end{aligned}$$

where

$$\mathbf{W} = \left[\frac{\partial \tau_i}{\partial \nu_j}\right] = \left[\frac{\partial^2 w}{\partial \nu_i \partial \nu_j}\right]$$

These two alternative thermodynamic processes are:

1. *Isentropic and adiabatic processes*, that is processes where

$$ds = 0 \quad \text{and} \quad đq = 0 \tag{9.105}$$

   In this case $w = u$, the internal energy function.

2. *Isothermal deformation with reversible heat transfer processes*, that is processes where

$$d\theta = 0 \quad \text{and} \quad đq = \theta ds \tag{9.106}$$

   In this case $w = \psi$, the Helmholtz free energy.

In fact, according to Eqs. 9.12 and 9.22,

$$\begin{aligned} du &= đw^{(I)} + đq \\ du &= \theta ds + \boldsymbol{\tau}^T d\boldsymbol{\nu} \end{aligned}$$

which, equalized for du, yield to

$$đw^{(I)} = \theta ds - đq + \boldsymbol{\tau}^T d\boldsymbol{\nu} \tag{9.107}$$

It is immediate to verify that both the hypotheses in Eqs. 9.105 and 9.106 reduce Eq. 9.107 into Eq. 9.103. We recall from Section 9.10 that in a isentropic process, 1st case, the internal function u is a potential function for $\boldsymbol{\tau}$. On the other hand, in a isothermal process, 2nd case, the Helmholtz free energy ψ is a potential function for $\boldsymbol{\tau}$.

9.13 Identification of the Thermodynamic Pressure

In any thermodynamic process where the internal work results to be equal to the thermodynamic work, that is

$$dw^{(I)} = \tau^T d\nu = \tau_{ij} d\nu_{ij} \tag{9.108}$$

we can identify the thermodynamic pressure as follows:

- Assuming

$$[\nu_{ij}] = [x_{i,j}] = \widetilde{\mathcal{F}} \tag{9.109}$$

the Lagrange Deformation Gradient tensor, we can identify

$$[\tau_{ij}] = \frac{1}{\varrho_o}[\sigma_{ji}^{(o)}] = \frac{1}{\varrho_o}\mathcal{T}^{(o)T} \tag{9.110}$$

the transpose of the 1st Piola-Kirchhoff stress tensor. In fact, according to Eqs. 8.32 and 9.108, we have

$$\begin{aligned} dw^{(I)} &= \frac{1}{\varrho_o}\mathcal{T}^{(o)} \cdot d\widetilde{\mathcal{F}} = \frac{1}{\varrho_o}\sigma_{ji}^{(o)} dx_{i,j} \\ dw^{(I)} &= \tau_{ij} d\nu_{ij} = \tau_{ij} dx_{i,j} \end{aligned}$$

which equalized yield to

$$\left[\frac{1}{\varrho_o}\sigma_{ji}^{(o)} - \tau_{ij}\right] dx_{i,j} = 0$$

Since dx_{ij} is arbitrary, we finally identify

$$\sigma_{ji}^{(o)} = \varrho_o \tau_{ij} \tag{9.111}$$

- Assuming

$$[\nu_{ij}] = [\widetilde{\mathcal{L}}_{ij}] = \widetilde{\mathcal{L}} \tag{9.112}$$

the Lagrange Strain tensor, we can identify

$$[\tau_{ij}] = \frac{1}{\varrho_o}[\widetilde{\sigma}_{ij}] = \frac{1}{\varrho_o}\widetilde{\mathcal{T}} \tag{9.113}$$

the 2nd Piola-Kirchhoff stress tensor. In fact, according to Eqs. 8.32 and 9.108, we have

$$\begin{aligned} dw^{(I)} &= \frac{1}{\varrho_o}\widetilde{\boldsymbol{T}} : d\widetilde{\boldsymbol{\mathcal{L}}} = \frac{1}{\varrho_o}\widetilde{\sigma}_{ij}d\widetilde{\mathcal{L}}_{ij} \\ dw^{(I)} &= \tau_{ij}d\nu_{ij} = \tau_{ij}d\widetilde{\mathcal{L}}_{ij} \end{aligned}$$

which equalized yield to

$$\left[\frac{1}{\varrho_o}\widetilde{\sigma}_{ij} - \tau_{ij}\right] d\widetilde{\mathcal{L}}_{ij} = 0$$

Since $d\widetilde{\mathcal{L}}_{ij}$ is arbitrary, we finally identify

$$\widetilde{\sigma}_{ij} = \varrho_o\tau_{ij} \tag{9.114}$$

- Assuming

$$[\dot{\nu}_{ij}] = [\widehat{\mathcal{D}}_{ij}] = \widehat{\boldsymbol{\mathcal{D}}} \tag{9.115}$$

the Euler Strain Rate tensor, we can identify

$$[\tau_{ij}] = \frac{1}{\varrho}[\sigma_{ij}] = \frac{1}{\varrho}\boldsymbol{T} \tag{9.116}$$

the Cauchy stress tensor. In fact, according to Eqs. 8.34 and 9.108, we have

$$\begin{aligned} \dot{w}^{(I)} &= \frac{1}{\varrho}\boldsymbol{T} : \widehat{\boldsymbol{\mathcal{D}}} = \frac{1}{\varrho}\sigma_{ij}\widehat{\mathcal{D}}_{ij} \\ \dot{w}^{(I)} &= \tau_{ij}\dot{\nu}_{ij} = \tau_{ij}\widehat{\mathcal{D}}_{ij} \end{aligned}$$

which equalized yield to

$$\left[\frac{1}{\varrho}\sigma_{ij} - \tau_{ij}\right] \widehat{\mathcal{D}}_{ij} = 0$$

Since $\widehat{\mathcal{D}}_{ij}$ is arbitrary, we finally identify

$$\sigma_{ij} = \varrho\tau_{ij} \tag{9.117}$$

Notice that in the case of small deformation, all the above findings reduce to

$$[\nu_{ij}] = [\mathcal{L}_{ij}] = \boldsymbol{\mathcal{L}} \tag{9.118}$$

$$[\tau_{ij}] = \frac{1}{\varrho}[\sigma_{ij}] = \frac{1}{\varrho}\boldsymbol{T} \tag{9.119}$$

and

$$\sigma_{ij} = \varrho \tau_{ij} \tag{9.120}$$

where $\mathcal{L}$ is the Linear Lagrange Strain tensor.

9.14 Non Fully Recoverable Stress Power Processes

In all real thermodynamic processes the stress power is not fully recoverable. Although in many practical cases the non-recoverable part may be neglected, there is an increasing demand from the industries for a mathematically correct description of the complete real thermodynamic process. Since this need has become of relevance only recently, non-reversible thermodynamic processes are still a subject of intensive research studies.

Let us generally indicate the internal work as

$$đw^{(I)} = \frac{1}{\varrho}\boldsymbol{\sigma}^T d\boldsymbol{\epsilon} \tag{9.121}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are the appropriate conjugated stress and strain tensors, respectively. We have seen in Section 9.12 that for two particular thermodynamic processes we can identify

$$\boldsymbol{\sigma} = \varrho \boldsymbol{\tau} \tag{9.122}$$

$$\boldsymbol{\epsilon} = \boldsymbol{\nu} \tag{9.123}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are the thermodynamic pressure and strain. Moreover, there exists a potential function

$$\phi = \varrho w \tag{9.124}$$

for $\boldsymbol{\sigma}$ such that

$$\boldsymbol{\sigma} = \frac{\partial \phi}{\partial \boldsymbol{\epsilon}} = \varrho \frac{\partial w}{\partial \boldsymbol{\epsilon}} \tag{9.125}$$

$$đw^{(I)} = \frac{1}{\varrho}\boldsymbol{\sigma}^T d\boldsymbol{\epsilon} = \frac{1}{\varrho}\left(\frac{\partial \phi}{\partial \boldsymbol{\epsilon}}\right)^T d\boldsymbol{\epsilon} = dw \tag{9.126}$$

$$d^2 w^{(I)} = \frac{1}{\varrho} d\boldsymbol{\sigma}^T d\boldsymbol{\epsilon} = d\boldsymbol{\epsilon}^T \mathbf{W} d\boldsymbol{\epsilon} > 0 \tag{9.127}$$

where

$$\mathbf{W} = \frac{1}{\varrho}\left[\frac{\partial \sigma}{\partial \epsilon_i}\right] = \left[\frac{\partial^2 w}{\partial \epsilon_i \partial \epsilon_j}\right]$$

In a non-reversible thermodynamic process, we obviously cannot expect to identify a potential function. However, it is of interest to identify the sign for $đw^{(I)}$ so that we can properly set up the appropriate constitutive equation.

According to the Clausius-Duhem inequality, the internal entropy production respects the condition, Eq. 9.99,

$$\gamma = \gamma^{(loc)} + \gamma^{(con)} \geq 0 \tag{9.128}$$

where

$$\begin{aligned} \gamma^{(loc)} &= \frac{ds}{dt} - \frac{1}{\theta}\frac{đq}{dt} = \frac{1}{\theta}\left(\frac{đw^{(I)}}{dt} - \tau^T \frac{d\nu}{dt}\right) \\ \gamma^{(con)} &= -\frac{\mathbf{c}^T \operatorname{grad} \theta}{\rho\theta^2} \end{aligned}$$

The above alternative form of $\gamma^{(loc)}$ is obtained taking into account that, from Eq. 9.107, the change of entropy can be expressed

$$ds = \frac{1}{\theta}(đw^{(I)} - \tau^T d\nu) + \frac{đq}{\theta}$$

If we accept the strong hypothesis in Eqs. 9.100 and 9.101, we may conclude that in general

$$đw^{(I)} - \tau^T d\nu \geq 0 \tag{9.129}$$

and, for reversible thermodynamic processes,

$$đw^{(I)} = \tau^T d\nu \tag{9.130}$$

When the internal work is not fully recoverable, two alternative common assumptions are made:

- The stress tensor can be split into

$$\sigma = \sigma^{(R)} + \sigma^{(D)} \tag{9.131}$$

  where $\sigma^{(R)}$ and $\sigma^{(N)}$ are the recoverable and the dissipative (non-recoverable) part, respectively.

- or, the strain tensor can be split into

$$\epsilon = \epsilon^{(R)} + \epsilon^{(D)} \tag{9.132}$$

  where $\epsilon^{(R)}$ and $\epsilon^{(D)}$ are the recoverable and the dissipative (non-recoverable) part, respectively.

In both cases we obtain that the internal work can be expressed as

$$đw^{(I)} = đw^{(IR)} + đw^{(ID)} \tag{9.133}$$

where $đw^{(IR)}$ and $đw^{(ID)}$ are the recoverable and the dissipative (non-recoverable) part, respectively. Moreover, if we assume that the recoverable part is given by

$$đw^{(IR)} = \boldsymbol{\tau}^T d\boldsymbol{\nu} \tag{9.134}$$

we arrive to the conclusion that the dissipative is subjected to the condition that

$$đw^{(ID)} \geq 0 \tag{9.135}$$

This phenomenological condition should then be respected by any constitutive equation relating stress and strain.

9.15 Fluids

Experimental evidence shows that for the case of fluids the only substate variable in $\boldsymbol{\nu}$ is the specific volume v, defined as the inverse of the mass density function ϱ, that is

$$v = \frac{1}{\varrho} \tag{9.136}$$

Accordingly, the caloric equation in Eq. 9.18 can be indicated as

$$u = u(s, v) \tag{9.137}$$

and its differential form as

$$du = \theta ds - \pi dv \tag{9.138}$$

where

$$\begin{aligned} \theta &= \left(\frac{\partial u}{\partial s}\right)_v > 0 \\ -\pi &= \left(\frac{\partial u}{\partial v}\right)_s \end{aligned}$$

and π is known as the *thermodynamic pressure*. Interestingly, according to the Eulerian form of the continuity equation, Eq. 7.11,

$$\frac{dv}{dt} = \frac{d}{dt}\left(\frac{1}{\varrho}\right) = -\frac{1}{\varrho^2}\frac{d\varrho}{dt} = \frac{\operatorname{div} \mathbf{v}}{\varrho} = \frac{1}{\varrho}\mathbf{I} : \widehat{\mathcal{D}} \tag{9.139}$$

where $\widehat{\mathcal{D}}$ is the Euler Strain Rate tensor.

Obviously, all the mathematical developments presented in the previous Sections hold for the case of fluids as well; the only difference consists in replacing ν by v or, equivalently, by $1/\varrho$. In particular, according to Eqs. 9.33 and 9.35, we have that

$$s = s(\theta, v) \tag{9.140}$$
$$\pi = \pi(\theta, v) \tag{9.141}$$

As an example of explicit forms of these two equations of state, we have the following experimental laws:

- The *Law for Perfect Gas* which states that

$$\pi v = R\theta$$

  where R is the gas constant for the particular gas.

- The *Boltzman Formula for Gas* which states that, the total entropy S in a sample of gas of volume Ω containing N molecules can be calculated as

$$S = kN\left(\ln\Omega + \frac{3}{2}\ln\theta\right) + c$$

  where k and c are the Boltzmann constants.

Usually, in the solution of fluid problems, we refer to the kinetic and Caloric equations of state in Eqs. 9.34 and 9.35 written in the form

$$\pi = \pi(\varrho, \theta) \tag{9.142}$$
$$u = u(\varrho, \theta) \tag{9.143}$$

It is immediate to verify that, for the case of fluids, the local form of the internal entropy production respects the condition, the Clausius-Duhem inequality, Eq. 9.99,

$$\gamma = \gamma^{(loc)} + \gamma^{(con)} \geq 0 \tag{9.144}$$

where

$$\gamma^{(loc)} = \frac{ds}{dt} - \frac{1}{\theta}\frac{dq}{dt} = \frac{1}{\varrho\theta}(\mathcal{T} + \pi\mathbf{I}) : \widehat{\mathcal{D}}$$
$$\gamma^{(con)} = -\frac{\mathbf{c}^T \operatorname{grad}\theta}{\varrho\theta^2}$$

The above alternative form of $\gamma^{(loc)}$ is obtained specializing the alternative form in Eq. 9.128 as

$$\gamma^{(loc)} = \frac{1}{\theta}(\frac{đw^{(I)}}{dt} - \tau^{T}\frac{d\nu}{dt}) = \frac{1}{\theta}(\frac{đw^{(I)}}{dt} + \pi\frac{dv}{dt})$$

and taking into account that, Eqs. 8.34 and 9.139,

$$\begin{aligned} \frac{đw^{(I)}}{dt} &= \frac{1}{\varrho}\mathcal{T} : \widehat{\mathcal{D}} \\ \frac{dv}{dt} &= \frac{1}{\varrho}\mathbf{I} : \widehat{\mathcal{D}} \end{aligned}$$

If we accept the strong hypothesis in Eqs. 9.100 and 9.101, we may conclude that in general

$$(\mathcal{T} + \pi\mathbf{I}) : \widehat{\mathcal{D}} \geq 0 \tag{9.145}$$

and, for reversible thermodynamic processes,

$$\mathcal{T} = -\pi\mathbf{I} \tag{9.146}$$

that is

$$\mathcal{T}_{kk} = -\pi \tag{9.147}$$

$$\mathcal{T}_{ij} = 0 \tag{9.148}$$

for $i \neq j$.

Chapter 10

Constitutive Equations Theory

10.1 Introduction

The set of functional relationships relating stress and deformation at a material point are called *constitutive equations*. As the name suggests, a constitutive equation describes the (macroscopic) behavior resulting from the internal constitution of the material. Symbolically, a constitutive equation may be indicated as

$$Stress\ Tensor = f(Deformation\ Tensor) \tag{10.1}$$

where f is a tensor valued function, known as *response function*.

In this Chapter, we report the mathematical developments of the *Purely Mechanical Theory*, Sections 10.2 and 10.3, to single out the appropriate pairs of stress and deformation tensors and the requirements that the response function must satisfy.

Attention is focused upon the simplest, nevertheless most used, constitutive relationships for *elastic solid materials*, Sections 10.4-10.9, and for *Newtonian fluids*, Section 10.10. Possible forms of constitutive relationships for non-elastic solid materials are presented in the next Chapters.

10.2 Purely Mechanical Theory Postulates

The Purely Mechanical Theory ignores thermal effects and, for any constitutive theory attempting to model the behavior of a continuum, the following postulates are assumed to be valid, [71, 93, 94]:

1. *Principle of Determinism for Stress.* The stress in a body is determined by the history of the motion of that body.

2. *Principle of Local Action.* In determining the stress at given particle P, the motion outside an arbitrary neighborhood of P may be disregarded.

3. *The Principle of Material Frame-Indifference.* Constitutive equations must be invariant under a change of frame of reference.

4. *Principle of Physical Admissibility.* All constitutive equations must be consistent with the basic physical laws of conservation of energy and the Clausius-Duhem inequality.

The first two principles seem rather obvious for a purely mechanical theory. The Principle of Determinist of Stress establishes that the past behavior may certainly influence the current behavior of the material; however, it excludes any possible dependence of the material behavior on future events. The Principle of Local Action excludes any dependence of the stress-strain relationship on any type of action-at-a-distance. The Principle of Frame-Indifference guarantees that two observers, even if in relative motion with respect to each other, observe the same stress in a given body.

Let $F = Ox_1x_2x_3$ and $F' = O'\xi_1\xi_2\xi_3$ be two CaORS in relative motion and with the same orientation at some initial time $t = 0$. In general, functions and fields whose values are scalars or tensors are called *frame-indifferent* or *objective* if both the dependent and the independent vector and tensor variables transform according to the following relationships:

1. Events, Section 4.3:
$$\boldsymbol{\xi} = \mathbf{c} + \mathbf{R}\mathbf{x}$$
where

   - $\mathbf{x} = \mathbf{x}(t)$ is the location of a material point P with respect to F;
   - $\boldsymbol{\xi} = \boldsymbol{\xi}(t)$ is the location of the same material point P but with respect to F';
   - $\mathbf{c} = \mathbf{c}(t)$ is the relative distance between the origins O' and O.
   - $\mathbf{R} = \mathbf{R}(t)$ is the orthogonal tensor which rotates F' to the orientation of F. By hypothesis, at $t = 0$ the two CaORS had the same orientation so that $\mathbf{R}(t = 0) \equiv \mathbf{I}$.

2. Vectors, Section 6.8:

$$\mathbf{v}' = \mathbf{R}\mathbf{v}$$

   where

   - $\mathbf{v}$ is the vector as identified from F;
   - $\mathbf{v}'$ is the same vector as identified from F';

3. Linear vector transformation tensors, Table 6.1:

$$\mathbf{T}' = \mathbf{R}\mathbf{T}\mathbf{R}^T$$

   where

   - $\mathbf{T}$ is the linear transformation tensor as defined in F;
   - $\mathbf{T}'$ is the same linear transformation tensor but as defined in F';

4. Deformation Gradient, Table 6.1:

$$\overline{\overline{\mathcal{F}}}' = \mathbf{R}\overline{\overline{\mathcal{F}}}$$

With regard to the Principle of Physical Admissibility, the strong condition on the Clausius-Duhem inequality, Eqs. 9.100 and 9.101, leads to the conclusion that, Eq. 9.129,

$$đw^{(i)} - \boldsymbol{\tau}^T d\boldsymbol{\nu} \geq 0 \tag{10.2}$$

and, for reversible thermodynamic processes, Eq. 9.130,

$$đw^{(i)} = \boldsymbol{\tau}^T d\boldsymbol{\nu} \tag{10.3}$$

where $\boldsymbol{\tau}$ and $\boldsymbol{\nu}$ are the thermodynamic pressure and strain. Let us generally indicate the internal work as

$$đw^{(i)} = \frac{1}{\varrho}\boldsymbol{\sigma}^T d\boldsymbol{\epsilon} \tag{10.4}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are the appropriate conjugated stress and strain tensors, respectively. We have seen in Section 9.13 that when the internal work results to be equal to the thermodynamic work, we can identify

$$\boldsymbol{\sigma} = \varrho\boldsymbol{\tau} \tag{10.5}$$

$$\boldsymbol{\epsilon} = \boldsymbol{\nu} \tag{10.6}$$

Moreover, we have seen in Section 9.12 that for two particular thermodynamic processes we can identify a potential function

$$\phi = \varrho w \tag{10.7}$$

for $\boldsymbol{\sigma}$ such that

$$\begin{aligned}
\boldsymbol{\sigma} &= \frac{\partial \phi}{\partial \boldsymbol{\epsilon}} = \varrho \frac{\partial w}{\partial \boldsymbol{\epsilon}} & (10.8)\\
dw &= \frac{1}{\varrho}\left(\frac{\partial \phi}{\partial \boldsymbol{\epsilon}}\right)^T d\boldsymbol{\epsilon} = \frac{1}{\varrho}\boldsymbol{\sigma}^T d\boldsymbol{\epsilon} = đw^{(I)} & (10.9)\\
d^2 w^{(I)} &= d\boldsymbol{\sigma}^T d\boldsymbol{\epsilon} = d\boldsymbol{\epsilon}^T \boldsymbol{\Phi} d\boldsymbol{\epsilon} > 0 & (10.10)
\end{aligned}$$

where

$$\boldsymbol{\Phi} = \left[\frac{\partial \sigma_i}{\partial \epsilon_j}\right] = \left[\frac{\partial^2 \phi}{\partial \epsilon_i \partial \epsilon_j}\right]$$

In a non-reversible thermodynamic process, the conditions to be imposed on a constitutive equation in order to respect the Clausius-Duhem inequality in Eq. 10.2 are not so obvious. In the hypothesis that the internal work can be expressed as

$$đw^{(I)} = đw^{(IR)} + đw^{(ID)} \tag{10.11}$$

where $đw^{(IR)}$ and $đw^{(ID)}$ are the recoverable and the dissipative (non-recoverable) part, respectively, and assuming that the recoverable part is given by

$$đw^{(IR)} = \boldsymbol{\tau}^T d\boldsymbol{\nu} \tag{10.12}$$

we arrive to the conclusion that the non-recoverable part is subjected to the condition that

$$đw^{(ID)} \geq 0 \tag{10.13}$$

For these reasons alternative criteria for physical admissibility have been proposed. Among them, the best known is the so-called Drucker's Stability Postulate, Section 10.3.

10.3 Drucker's Stability Criterion

According to Drucker, [31], a *stable material* is one respecting the following two conditions:

1. *Stability in small.* The work done by an added stress set $d\boldsymbol{\sigma}$ on the corresponding change of strain $d\epsilon$ is positive, that is

$$d\sigma^T d\epsilon > 0 \tag{10.14}$$

2. *Stability in cycle.* The net work over a complete cycle of application and removal done by an added stress set $d\boldsymbol{\sigma}$ on the corresponding change of strain $d\epsilon$ is non negative, that is

$$\oint d\sigma^T d\epsilon \geq 0 \tag{10.15}$$

We will see in the next Sections that, in general, a constitutive equation may be expressed by an incremental relationship of the type

$$d\sigma = \mathbf{C} d\epsilon \tag{10.16}$$

where $\mathbf{C}$ is known as the *tangential constitutive matrix.* Accordingly,

$$d\sigma^T d\epsilon = d\epsilon^T \mathbf{C}^T d\epsilon$$

We notice that the condition in Eq. 10.14 can be satisfied if $\mathbf{C}$ is a symmetric and positive definite matrix.

Moreover, it is easy to verify that if there exists a potential function

$$\phi = \phi(\epsilon) \tag{10.17}$$

for $\boldsymbol{\sigma}$ such that

$$\begin{aligned}
\boldsymbol{\sigma} &= \frac{\partial \phi}{\partial \epsilon} \\
d\phi &= \boldsymbol{\sigma}^T d\epsilon \\
d^2\phi &= d\sigma^T d\epsilon = d\epsilon^T \mathbf{C} d\epsilon > 0
\end{aligned}$$

where

$$\mathbf{C} = \left[\frac{\partial \sigma_i}{\partial \epsilon_j}\right] = \left[\frac{\partial^2 \phi}{\partial \epsilon_i \partial \epsilon_j}\right]$$

then the equality in Eq. 10.15 holds.

Thus, Drucker's criterion includes, as a particular case, the mathematical requirements of a fully recoverable thermodynamic process. The extension consists in requiring that useful net energy cannot be extracted from the material and the system of forces acting upon it, in a cycle of application and removal of the added set of forces and displacements. Furthermore, energy must be put in if only irrecoverable deformation is to take place.

Clearly, the conditions posed by Drucker's criterion are more restrictive than those strictly required by the Clausius-Duhem inequality. However, it turns out that Drucker's stability criterion provides the sufficient conditions for establishing the uniqueness of the solution for an important class of solid continuum problems.

10.4 Finite Elastic Constitutive Equations

A material is defined *elastic* when the continuum recovers its initial configuration upon removal of the applied load. In this Section we will prove that, according to the postulate of the Purely Mechanical Theory, Section 10.2, a possible form of the constitutive functional relationship for elastic material is of the type

$$\tilde{\mathcal{T}} = \tilde{\mathrm{g}}(\tilde{\mathcal{L}}) \qquad (10.18)$$

where $\tilde{\mathcal{T}}$ is the 2nd Piola-Kirchhoff stress tensor and $\tilde{\mathrm{g}}$, known as the *elastic response function*, is in general a non linear tensor value function of the Lagrange Strain tensor $\tilde{\mathcal{L}}$, Sections 7.8 and 6.3.1. For small rotations but finite deformations this constitutive relationship may be reduced to

$$\mathcal{T} = \tilde{\mathrm{g}}(\tilde{\mathcal{L}}) \qquad (10.19)$$

where $\mathcal{T}$ is the Cauchy stress tensor, Section 7.7. Finally, when also the deformations are sufficiently small

$$\mathcal{T} = \tilde{\mathrm{g}}(\mathcal{L}) \qquad (10.20)$$

where $\mathcal{L}$ is the Linear Lagrange Strain tensor, Section 6.6.1. The vectorial form of this elastic constitutive functional relationship can be indicated as

$$\tilde{\sigma} = \tilde{\mathrm{g}}(\tilde{\epsilon}) \qquad (10.21)$$

where $\tilde{\sigma}$ is the 2nd Piola-Kirchhoff stress vector and $\tilde{\epsilon}$ is the Lagrange Strain vector, Sections 7.12 and 6.10.

In fact, the application of the Principles of Determinism for Stress and of Local Action leads to the conclusion that for such materials the stress depends only on the current deformation state. Accordingly, we can state that the general form of the constitutive equation for an elastic material is of the type

$$\mathcal{T} = \mathbf{h}(\widetilde{\mathcal{F}}) \tag{10.22}$$

where the response function $\mathbf{h}$ is, in general, a nonlinear tensor value function of the Lagrange Deformation Gradient $\widetilde{\mathcal{F}}$.

The Principle of Frame-Indifference requires the response function to be invariant under a change of reference system, that is

$$\mathcal{T}' = \mathbf{R}\mathbf{h}(\widetilde{\mathcal{F}})\mathbf{R}^T = \mathbf{h}(\mathbf{R}\widetilde{\mathcal{F}}) \tag{10.23}$$

for all rotation (orthogonal) matrices $\mathbf{R}$. In fact, under a change of reference system rotated by $\mathbf{R}$, the tensors $\mathbf{T}$ and $\widetilde{\mathbf{F}}$ transform according to the rule, Section 10.2,

$$\begin{aligned} \mathcal{T}' &= \mathbf{R}\mathcal{T}\mathbf{R}^T \\ \widetilde{\mathcal{F}}' &= \mathbf{R}\widetilde{\mathcal{F}} \end{aligned}$$

We recall that the polar decomposition of $\widetilde{\mathcal{F}}$ yields to, Eq. 6.26,

$$\widetilde{\mathcal{F}} = \widetilde{\mathcal{R}}\widetilde{\mathcal{M}}$$

where $\widetilde{\mathcal{R}}$ is the (orthogonal) Lagrange Rotation tensor and

$$\widetilde{\mathcal{M}} = \widetilde{\mathcal{G}}^{\frac{1}{2}}$$

is the Right Lagrange Strecth tensor. Hence, if we make the special choice $\mathbf{R} \equiv \widetilde{\mathcal{R}}^T$, the constitutive functional relationship in Eq. 10.23 takes on the form

$$\mathcal{T}' = \mathbf{h}(\widetilde{\mathcal{M}}) = \widetilde{\mathcal{R}}^T\mathbf{h}(\widetilde{\mathcal{F}})\widetilde{\mathcal{R}}$$

and, conversely,

$$\mathcal{T} = \widetilde{\mathcal{R}}\mathcal{T}'\widetilde{\mathcal{R}}^T = \widetilde{\mathcal{R}}\mathbf{h}(\widetilde{\mathcal{M}})\widetilde{\mathcal{R}}^T = \mathbf{h}(\widetilde{\mathcal{F}})$$

Thus, the constitutive relationship in Eq. 10.22 may take on the following alternative reduced forms

$$\mathcal{T} = \widetilde{\mathcal{R}}\mathbf{h}(\widetilde{\mathcal{M}})\widetilde{\mathcal{R}}^T \tag{10.24}$$

$$\mathcal{T} = \widetilde{\mathcal{R}}\mathbf{f}(\widetilde{\mathcal{G}})\widetilde{\mathcal{R}}^T \tag{10.25}$$

$$\mathcal{T} = \widetilde{\mathcal{R}}\mathbf{g}(\widetilde{\mathcal{L}})\widetilde{\mathcal{R}}^T \tag{10.26}$$

where

$$\begin{aligned} \mathrm{f}(\tilde{\mathcal{G}}) &= \mathrm{h}(\tilde{\mathcal{G}}^{\frac{1}{2}}) = \mathrm{h}(\widetilde{\mathcal{M}}) \\ \mathrm{g}(\tilde{\mathcal{L}}) &= \mathrm{f}(1+2\tilde{\mathcal{L}}) = \mathrm{h}(\tilde{\mathcal{G}}^{\frac{1}{2}}) = \mathrm{h}(\widetilde{\mathcal{M}}) \end{aligned}$$

We recall that the Cauchy tensor $\mathcal{T}$ is related to the 2nd Piola-Kirchhoff $\tilde{\mathcal{T}}$ as, Eq. 7.38,

$$\mathcal{T} = \frac{\varrho}{\varrho_o}\widetilde{\mathcal{F}}\tilde{\mathcal{T}}\widetilde{\mathcal{F}}^T = \frac{\varrho}{\varrho_o}\widetilde{\mathcal{R}}\widetilde{\mathcal{M}}\tilde{\mathcal{T}}\widetilde{\mathcal{M}}^T\widetilde{\mathcal{R}}^T \tag{10.27}$$

Thus, substituting Eq. 10.27 into Eq. 10.24, we may eventually find the following alternative forms of constitutive relationships

$$\tilde{\mathcal{T}} = \tilde{\mathrm{h}}(\widetilde{\mathcal{M}}) \tag{10.28}$$

$$\tilde{\mathcal{T}} = \tilde{\mathrm{f}}(\tilde{\mathcal{G}}) \tag{10.29}$$

$$\tilde{\mathcal{T}} = \tilde{\mathrm{g}}(\tilde{\mathcal{L}}) \tag{10.30}$$

where

$$\tilde{\mathrm{h}}(\widetilde{\mathcal{M}}) = \frac{\varrho_o}{\varrho}\widetilde{\mathcal{M}}^{-1}\mathrm{h}(\widetilde{\mathcal{M}})\widetilde{\mathcal{M}}^{-T}$$

and

$$\begin{aligned} \tilde{\mathrm{f}}(\tilde{\mathcal{G}}) &= \tilde{\mathrm{h}}(\tilde{\mathcal{G}}^{\frac{1}{2}}) = \tilde{\mathrm{h}}(\widetilde{\mathcal{M}}) \\ \tilde{\mathrm{g}}(\tilde{\mathcal{L}}) &= \tilde{\mathrm{f}}(1+2\tilde{\mathcal{L}}) = \tilde{\mathrm{h}}(\tilde{\mathcal{G}}^{\frac{1}{2}}) = \tilde{\mathrm{h}}(\widetilde{\mathcal{M}}) \end{aligned}$$

10.5 Hyperelastic Materials

A material is called *hyperelastic*, or *Green elastic*, if the elastic response function in Eq. 10.18 is further restricted by the existence of a potential function

$$\phi = \varrho w(\bar{\epsilon}) \tag{10.31}$$

known as the *strain energy function* or as the *elastic potential function*, such that

$$\begin{aligned} \bar{\sigma} &= \frac{\partial \phi}{\partial \bar{\epsilon}} \\ d\phi &= \bar{\epsilon}^T d\bar{\sigma} \\ d^2\phi &= d\bar{\sigma}^T d\bar{\epsilon} = d\bar{\epsilon}^T \mathrm{C} d\bar{\epsilon} > 0 \end{aligned}$$

$$\mathbf{C} = \left[\frac{\partial^2 \phi}{\partial \tilde{\epsilon}_i \partial \tilde{\epsilon}_j}\right]$$

We notice that:

- $\mathbf{C}$ is a symmetric positive definite matrix;
- hyperelastic constitutive equations respect Drucker's stability criteria, Section 10.3;
- the internal work of hyperelastic elastic material results to be equal to

  $$dw^{(I)} = \frac{1}{\varrho}\tilde{\sigma}^T d\tilde{\epsilon} = dw$$

  and

  $$d^2 w = \frac{1}{\varrho} d\tilde{\sigma}^T d\tilde{\epsilon} = \frac{1}{\varrho} d\tilde{\epsilon}^T \mathbf{C} d\tilde{\epsilon} > 0$$

  as in the case of the fully recoverable thermodynamic process presented in Section 9.12.

According to the above definition, we have that the constitutive equation for an hyperelastic material is given by the following incremental form

$$d\tilde{\sigma} = \mathbf{C} d\tilde{\epsilon} \tag{10.32}$$

where $\mathbf{C}$, the *tangential stiffness matrix*, is a symmetric and positive definite matrix. Since $\mathbf{C}$ is symmetric positive definite, we can invert Eq. 10.32 and obtain

$$d\tilde{\epsilon} = \mathbf{D} d\tilde{\sigma} \tag{10.33}$$

where

$$\mathbf{D} = \mathbf{C}^{-1}$$

the *tangential compliance matrix*, is a symmetric and positive definite matrix, too. Following the Legendre Transformation technique in Section 9.9.1, we can prove that the inverse transformation in Eq. 10.33 has potential function defined as

$$\psi = \rho\omega(\tilde{\sigma}) = \tilde{\sigma}^T \tilde{\epsilon} - \phi \tag{10.34}$$

also known as the *stress energy function* or as the *complementary elastic potential function*, such that

$$\begin{aligned}
\tilde{\epsilon} &= \frac{\partial \phi}{\partial \tilde{\sigma}} \\
d\psi &= \tilde{\epsilon}^T d\tilde{\sigma} \\
d^2\psi &= d\tilde{\epsilon}^T d\tilde{\sigma} = d\tilde{\sigma}^T \mathbf{D} d\tilde{\sigma} > 0
\end{aligned}$$

and

$$\mathbf{D} = \left[\frac{\partial^2 \psi}{\partial \tilde{\sigma}_i \partial \tilde{\sigma}_j}\right]$$

It is important to remark that the positive definiteness property of $\mathbf{C}$ leads to the conclusion that in the 9D strain space, the surface corresponding to

$$\phi(\tilde{\epsilon}) = const \tag{10.35}$$

is convex. The same applies for

$$\psi(\sigma) = const \tag{10.36}$$

in the 9D stress space $\tilde{\sigma}$. In fact, the Taylor expansion of Eq. 10.33, neglecting the higher order terms, in given by

$$\begin{aligned}\psi(\tilde{\sigma}^{(B)}) - \psi(\tilde{\sigma}^{(A)}) &= \left(\frac{\partial \psi}{\partial \tilde{\sigma}}\right)_A \Delta\tilde{\sigma} + \frac{1}{2}\Delta\tilde{\sigma}^T \mathbf{D} \Delta\tilde{\sigma} = \\ &= \tilde{\epsilon}^{(A)}\Delta\tilde{\sigma} + \frac{1}{2}\Delta\tilde{\sigma}^T \mathbf{D}\Delta\tilde{\sigma}\end{aligned}$$

where

$$\Delta\tilde{\sigma} = \left(\tilde{\sigma}^{(B)} - \tilde{\sigma}^{(A)}\right)$$

Assume that $\tilde{\sigma}^{(A)}$ and $\tilde{\sigma}^{(B)}$ are two next stress points on $\psi = const$. Then, since the term on the left-hand side is equal to zero, while the second term on the right-hand side is positive definite, we can write

$$\tilde{\epsilon}^{(A)}\Delta\tilde{\sigma} < 0$$

which is condition of strict convexity on ψ. Similarly we can prove the convexity of $\phi(\tilde{\epsilon}) = const$.

We recall that in a 3D space, a surface $f(x_1, x_2, x_3) = const$ is said to be convex, if and only if any line segment joining two points on the surface lies on or inside the surface. It follows that each tangent plane to a convex surface of constant $f(x_1, x_2, x_3)$ is a supporting plane that does not intersect the surface.

10.6 Isotropic Materials

A material is called *isotropic* when its mechanical response is identical in all directions. This implies that the functional relationship in Eq. 10.18 must respect the condition

$$\tilde{\mathcal{T}}' = \mathbf{Q}\tilde{\mathbf{g}}(\tilde{\mathcal{L}})\mathbf{Q}^T = \tilde{\mathbf{g}}(\mathbf{Q}\tilde{\mathcal{L}}\mathbf{Q}^T) \tag{10.37}$$

for all instantaneous (orthogonal) rotation matrix $\mathbf{Q}$, where $\mathbf{Q}$ represents the instantaneous rotation of the current reference frame. In fact, in this case, both tensors $\mathcal{T}$ and $\tilde{\mathcal{L}}$ transform according to the rule

$$\begin{aligned} \tilde{\mathcal{T}}' &= \mathbf{Q}\tilde{\mathcal{T}}\mathbf{Q}^T \\ \tilde{\mathcal{L}}' &= \mathbf{Q}\tilde{\mathcal{L}}\mathbf{Q}^T \end{aligned}$$

Notice that, if in particular we select $\mathbf{Q}$ as the orthogonal matrix $\mathbf{Q}'$ which rotates $\tilde{\mathcal{L}}$ into its principal reference system, then we have

$$\mathbf{Q}\tilde{\mathbf{g}}(\tilde{\mathcal{L}})\mathbf{Q}^T = \tilde{\mathbf{g}}(\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2, \tilde{\mathcal{L}}_3) = \mathbf{g}(I_1^{(\tilde{\mathcal{L}})}, I_2^{(\tilde{\mathcal{L}})}, I_3^{(\tilde{\mathcal{L}})})$$

where $\tilde{\mathcal{L}}_i$ and $I_i^{(\tilde{\mathcal{L}})}$ are the principal strain value and the strain invariants of $\tilde{\mathcal{L}}$, respectively.

It can be proved, [83], that the response functions $\tilde{\mathbf{g}}$ respecting the condition in Eq. 10.37 have a representation of the form

$$\tilde{\mathcal{T}} = \tilde{\mathbf{g}}(\tilde{\mathcal{L}}) = \tilde{g}_1\mathbf{I} + \tilde{g}_2\tilde{\mathcal{L}} + \tilde{g}_3\tilde{\mathcal{L}}\tilde{\mathcal{L}} \tag{10.38}$$

that is, in vectorial form,

$$\tilde{\boldsymbol{\sigma}} = \tilde{\mathbf{g}}(\tilde{\boldsymbol{\epsilon}}) = \tilde{g}_1\mathbf{I} + \tilde{g}_2\tilde{\boldsymbol{\epsilon}} + \tilde{g}_3\tilde{\boldsymbol{\epsilon}}^T\tilde{\boldsymbol{\epsilon}} \tag{10.39}$$

where $\tilde{g}_i$, for $i = 1, 2, 3$, are scalar invariant functions of the Lagrangian strain invariants $(I_1^{(\tilde{\mathcal{L}})}, I_2^{(\tilde{\mathcal{L}})}, I_3^{(\tilde{\mathcal{L}})})$ Accordingly, we can state that for an hyperelastic material the strain energy function must have a representation of the type

$$\phi = \phi(I_1^{(\tilde{\mathcal{L}})}, I_2^{(\tilde{\mathcal{L}})}, I_3^{(\tilde{\mathcal{L}})}) \tag{10.40}$$

so that, according to Eq. 10.38 and Appendix A, the stresses result to be equal to

$$\begin{aligned} \tilde{\sigma}_{ij} &= \frac{\partial\phi}{\partial\tilde{\epsilon}_{ij}} = \frac{\partial\phi}{\partial I_1^{(\tilde{\mathcal{L}})}}\frac{\partial I_1^{(\tilde{\mathcal{L}})}}{\partial\tilde{\mathcal{L}}_{ij}} + \frac{\partial\phi}{\partial I_2^{(\tilde{\mathcal{L}})}}\frac{\partial I_2^{(\tilde{\mathcal{L}})}}{\partial\tilde{\mathcal{L}}_{ij}} + \frac{\partial\phi}{\partial I_3^{(\tilde{\mathcal{L}})}}\frac{\partial I_3^{(\tilde{\mathcal{L}})}}{\partial\tilde{\mathcal{L}}_{ij}} = \\ &= \tilde{g}_1\delta_{ij} + \tilde{g}_2\tilde{\mathcal{L}}_{ij} + \tilde{g}_3\tilde{\mathcal{L}}_{ik}\tilde{\mathcal{L}}_{jk} \end{aligned}$$

where

$$\tilde{g}_i = \tilde{g}_i(I_1^{(\tilde{\mathcal{L}})}, I_2^{(\tilde{\mathcal{L}})}, I_3^{(\tilde{\mathcal{L}})}) = \frac{\partial\phi}{\partial I_i^{(\tilde{\mathcal{L}})}}$$

We notice that in an isotropic material the principal values of $\tilde{\mathcal{T}}$ can be calculated as

$$\tilde{\sigma}_i = \tilde{g}_1 + \tilde{g}_2\tilde{\mathcal{L}}_i + \tilde{g}_3\tilde{\mathcal{L}}_i^2 \tag{10.41}$$

from which we can conclude that the principal directions of $\tilde{\mathcal{L}}$ coincide with those of $\tilde{\mathcal{T}}$. In fact, the strain tensor $\tilde{\mathcal{L}}$ in its principal direction has a representation of the type

$$\tilde{\mathcal{L}} = [\delta_{ij}\tilde{\mathcal{L}}_i]$$

which substituted in Eq. 10.38 yields to

$$\begin{aligned} \tilde{\sigma}_{ij} &= \tilde{g}_1\delta_{ij} + \tilde{g}_2\delta_{ij}\tilde{\mathcal{L}}_i + \tilde{g}_3\delta_{ik}\tilde{\mathcal{L}}_i\delta_{jk}\tilde{\mathcal{L}}_j = \\ &= \delta_{ij}\left(\tilde{g}_1 + \tilde{g}_2\tilde{\mathcal{L}}_i + \tilde{g}_3\tilde{\mathcal{L}}_i^2\right) \end{aligned}$$

By induction, we can conclude that for an hyperelastic material, the complementary energy must have a representation of the type

$$\psi = \psi(I_1^{(\tilde{\mathcal{T}})}, I_2^{(\tilde{\mathcal{T}})}, I_3^{(\tilde{\mathcal{T}})}) \tag{10.42}$$

where $I_i^{(\tilde{\mathcal{T}})}$, for $i = 1, 2, 3$, are the stress tensor invariants. Consequently, the strains result to be equal to

$$\tilde{\mathcal{L}} = \tilde{c}_1\mathbf{I} + \tilde{c}_2\tilde{\mathcal{T}} + \tilde{c}_3\tilde{\mathcal{T}}\tilde{\mathcal{T}} \tag{10.43}$$

that is, in vectorial form,

$$\tilde{\epsilon} = \tilde{c}_1\tilde{\mathbf{m}} + \tilde{c}_2\tilde{\sigma} + \tilde{c}_3\tilde{\sigma}^T\tilde{\sigma} \tag{10.44}$$

where

$$\tilde{c}_i = \tilde{c}_i(I_1^{(\tilde{\mathcal{T}})}, I_2^{(\tilde{\mathcal{T}})}, I_3^{(\tilde{\mathcal{T}})}) = \frac{\partial\psi}{\partial I_i^{(\tilde{\mathcal{T}})}}$$

In fact,

$$\begin{aligned} \tilde{\mathcal{L}}_{ij} &= \frac{\partial\psi}{\partial\tilde{\sigma}_{ij}} = \frac{\partial\psi}{\partial I_1^{(\tilde{\mathcal{T}})}}\frac{\partial I_1^{(\tilde{\mathcal{T}})}}{\partial\tilde{\sigma}_{ij}} + \frac{\partial\psi}{\partial I_2^{(\tilde{\mathcal{T}})}}\frac{\partial I_2^{(\tilde{\mathcal{T}})}}{\partial\tilde{\sigma}_{ij}} + \frac{\partial\psi}{\partial I_3^{(\tilde{\mathcal{T}})}}\frac{\partial I_3^{(\tilde{\mathcal{T}})}}{\partial\tilde{\sigma}_{ij}} = \\ &= \tilde{c}_1\delta_{ij} + \tilde{c}_2\tilde{\sigma}_{ij} + \tilde{c}_3\tilde{\sigma}_{ik}\tilde{\sigma}_{jk} \end{aligned}$$

10.7 Energy Function for Isotropic Materials

We recall from Section 6.9 that $(\tilde{\epsilon}_v, \tilde{\epsilon}_s, \theta_{\tilde{\epsilon}})$ are a set of alternative strain invariants. Thus, the elastic energy function in Eq. 10.40 can alternatively be expressed as

$$\phi = \phi(\tilde{\epsilon}_v, \tilde{\epsilon}_s, \theta_{\tilde{\epsilon}}) \tag{10.45}$$

It can be shown that, Appendix A,

$$\tilde{\sigma} = \frac{\partial \phi}{\partial \tilde{\epsilon}} = \tilde{f}_1 \widetilde{\mathrm{m}} + \tilde{f}_2 \tilde{\mathrm{e}} + \tilde{f}_3 \tilde{\mathrm{d}} \tag{10.46}$$

where

$$\begin{aligned} \tilde{f}_1 &= \frac{\partial \phi}{\partial \tilde{\epsilon}_v} \\ \tilde{f}_2 &= \frac{2}{3\tilde{\epsilon}_s}\left(\frac{\partial \phi}{\partial \epsilon_s} - \frac{\tan 3\theta_{\tilde{\epsilon}}}{\tilde{\epsilon}_s}\frac{\partial \phi}{\partial \theta_{\tilde{\epsilon}}}\right) \\ \tilde{f}_3 &= -\frac{4}{3\tilde{\epsilon}_s^3 \cos 3\theta_{\tilde{\epsilon}}}\frac{\partial \phi}{\partial \theta_{\tilde{\epsilon}}} \end{aligned}$$

and the elements of $\tilde{\mathrm{d}}$ are given by

$$\tilde{d}_{ij} = \overline{d}_{ij} - \frac{\delta_{ij}}{3}\overline{d}_{kk}$$

where

$$[\overline{d}_{ij}] = \begin{bmatrix} (\tilde{e}_{22}\tilde{e}_{33} - \tilde{e}_{23}^2) & (\tilde{e}_{13}\tilde{e}_{23} - \tilde{e}_{33}\tilde{e}_{12}) & (\tilde{e}_{12}\tilde{e}_{23} - \tilde{e}_{22}\tilde{e}_{13}) \\ & (\tilde{e}_{11}\tilde{e}_{33} - \tilde{e}_{13}^2) & (\tilde{e}_{12}\tilde{e}_{13} - \tilde{e}_{11}\tilde{e}_{23}) \\ \text{Symmetric} & & (\tilde{e}_{11}\tilde{e}_{22} - \tilde{e}_{12}^2) \end{bmatrix}$$

Moreover,

$$\tilde{\mathrm{s}} = \left\{\frac{\partial \phi}{\partial \tilde{e}_{kl}}\frac{\partial \tilde{e}_{kl}}{\partial \tilde{\epsilon}_{ij}}\right\}^T = \tilde{f}_2 \tilde{\mathrm{e}} + \tilde{f}_3 \tilde{\mathrm{d}} \tag{10.47}$$

$$\tilde{p} = 3\frac{\partial \phi}{\partial \tilde{\epsilon}_v} = \tilde{f}_1 \tag{10.48}$$

$$\tilde{q} = \left[\left(\frac{\partial \phi}{\partial \tilde{\epsilon}_s}\right)^2 + \frac{1}{\tilde{\epsilon}_s^2}\left(\frac{\partial \phi}{\partial \theta_{\tilde{\epsilon}}}\right)^2\right]^{1/2} \tag{10.49}$$

where, by definition, Sections 6.10 and 7.12,

$$\begin{aligned}\tilde{\mathbf{e}} &= \tilde{\boldsymbol{\epsilon}} - \widetilde{\mathbf{m}}\frac{\tilde{\epsilon}_v}{3} \\ \tilde{\mathbf{s}} &= \tilde{\boldsymbol{\sigma}} - \widetilde{\mathbf{m}}\tilde{p}\end{aligned}$$

We recall from Section 6.9 that $(\tilde{p}, \tilde{q}, \theta_{\tilde{\sigma}})$ are a set of alternative stress invariants. Thus, the complementary elastic energy function in Eq. 10.42 can alternatively be expressed as

$$\psi = \psi(\tilde{p}, \tilde{q}, \theta_{\tilde{\sigma}}) \tag{10.50}$$

It can be shown that, Appendix A,

$$\tilde{\boldsymbol{\epsilon}} = \frac{\partial \psi}{\partial \tilde{\boldsymbol{\sigma}}} = \tilde{h}_1 \widetilde{\mathbf{m}} + \tilde{h}_2 \tilde{\mathbf{s}} + \tilde{h}_3 \tilde{\mathbf{a}} \tag{10.51}$$

where

$$\begin{aligned}\tilde{h}_1 &= \frac{1}{3}\frac{\partial \psi}{\partial \tilde{p}} \\ \tilde{h}_2 &= \frac{3}{2\tilde{q}}\left(\frac{\partial \psi}{\partial \tilde{q}} - \frac{\tan 3\theta_{\tilde{\sigma}}}{\tilde{q}}\frac{\partial \psi}{\partial \theta_{\tilde{\sigma}}}\right) \\ \tilde{h}_3 &= -\frac{9}{2\tilde{q}^3 \cos 3\theta_{\tilde{\sigma}}}\frac{\partial \psi}{\partial \theta_{\tilde{\sigma}}}\end{aligned}$$

and the elements of $\tilde{\mathbf{a}}$ are given by

$$\tilde{a}_{ij} = \bar{a}_{ij} - \frac{\delta_{ij}}{3}\bar{a}_{kk}$$

where

$$[\bar{a}_{ij}] = \begin{bmatrix} (\tilde{s}_{22}\tilde{s}_{33} - \tilde{s}_{23}^2) & (\tilde{s}_{13}\tilde{s}_{23} - \tilde{s}_{33}\tilde{s}_{12}) & (\tilde{s}_{12}\tilde{s}_{23} - \tilde{s}_{22}\tilde{s}_{13}) \\ & (\tilde{s}_{11}\tilde{s}_{33} - \tilde{s}_{13}^2) & (\tilde{s}_{12}\tilde{s}_{13} - \tilde{s}_{11}\tilde{s}_{23}) \\ \text{Symmetric} & & (\tilde{s}_{11}\tilde{s}_{22} - \tilde{s}_{12}^2) \end{bmatrix}$$

Moreover,

$$\tilde{\mathbf{e}} = \left\{\frac{\partial \psi}{\partial \tilde{s}_{kl}}\frac{\partial \tilde{s}_{kl}}{\partial \tilde{\sigma}_{ij}}\right\}^T = \tilde{h}_2 \tilde{\mathbf{s}} + \tilde{h}_3 \tilde{\mathbf{a}} \tag{10.52}$$

$$\tilde{\epsilon}_v = \frac{\partial \psi}{\partial \tilde{p}} = 3\tilde{h}_1 \tag{10.53}$$

$$\tilde{\epsilon}_s = = \sqrt{\frac{3}{2}}\left[\left(\frac{\partial \psi}{\partial \tilde{q}}\right)^2 + \frac{1}{\tilde{q}^2}\left(\frac{\partial \psi}{\partial \theta_{\tilde{\sigma}}}\right)^2\right]^{1/2} \tag{10.54}$$

10.8 Linear Elastic Materials

Linear elasticity implies that stress and strain are linearly related. Accordingly, for a linear hyperelastic material, the constitutive equation in Eq. 10.32 reduces to

$$\tilde{\sigma} = \mathbf{C}\tilde{\epsilon} \tag{10.55}$$

where $\mathbf{C}$, the *stiffness matrix*, is a constant symmetric and positive definite matrix. Since $\mathbf{C}$ is symmetric positive definite, we can invert Eq. 10.55 and obtain

$$d\tilde{\epsilon} = \mathbf{D}d\tilde{\sigma} \tag{10.56}$$

where

$$\mathbf{D} = \mathbf{C}^{-1}$$

the *compliance matrix*, is a constant symmetric and positive definite matrix, too.

Thus, in an linear hyperelastic elastic material it must exist a quadratic potential function

$$\phi = \phi(\tilde{\epsilon}) = \frac{1}{2}\tilde{\epsilon}^T\mathbf{C}\tilde{\epsilon} \tag{10.57}$$

where the *stiffness matrix* $\mathbf{C}$ is a symmetric positive definite matrix such that

$$\begin{aligned}
\tilde{\sigma} &= \frac{\partial\phi}{\partial\tilde{\epsilon}} = \mathbf{C}\tilde{\epsilon} \\
d\phi &= \tilde{\sigma}^T d\tilde{\epsilon} = \tilde{\epsilon}^T\mathbf{C}d\tilde{\epsilon} \\
d^2\phi &= d\tilde{\sigma}^T d\tilde{\epsilon} = d\tilde{\epsilon}^T\mathbf{C}d\tilde{\epsilon} > 0
\end{aligned}$$

Conversely, we can state that the complementary elastic energy

$$\psi = \psi(\tilde{\sigma}) = \tilde{\sigma}^T\tilde{\epsilon} - \phi \tag{10.58}$$

is quadratic function of the type

$$\psi = \psi(\tilde{\sigma}) = \frac{1}{2}\tilde{\sigma}^T\mathbf{D}\tilde{\sigma} \tag{10.59}$$

where the *compliance matrix* $\mathbf{D}$ is a symmetric positive definite matrix such that

$$\begin{aligned}
\tilde{\epsilon} &= \frac{\partial\psi}{\partial\tilde{\sigma}} = \mathbf{D}\tilde{\sigma} \\
d\psi &= \tilde{\epsilon}^T d\tilde{\sigma} = \tilde{\sigma}^T\mathbf{D}d\tilde{\sigma} \\
d^2\psi &= d\tilde{\epsilon}^T d\tilde{\epsilon} = d\tilde{\sigma}^T\mathbf{D}d\tilde{\sigma} > 0
\end{aligned}$$

and

$$\mathbf{D} = (\mathbf{C})^{-1} \tag{10.60}$$

10.9 Linear Elasticity Isotropic Materials

For an isotropic linear hyperelastic material the constitutive relationship in Eq. 10.46 must be reduced of the quadratic terms and, therefore, take on a linear form of the type

$$\tilde{\boldsymbol{\sigma}} = \tilde{f}_1 \widehat{\mathbf{m}} + \tilde{f}_2 \tilde{\mathbf{e}} \tag{10.61}$$

This can be obtained if, and only if, the strain energy function is independent of $\theta_{\tilde{\epsilon}}$ invariant so that

$$\begin{aligned}
\tilde{f}_1 &= \frac{\partial \phi}{\partial \tilde{\epsilon}_v} \\
\tilde{f}_2 &= \frac{2}{3\tilde{\epsilon}_s} \frac{\partial \phi}{\partial \tilde{\epsilon}_s} \\
\tilde{f}_3 &= 0
\end{aligned}$$

and $\tilde{f}_1$ is a linear function of $(\tilde{\epsilon}_v, \tilde{\epsilon}_s)$ while $\tilde{f}_2$ is a constant.

Accordingly, we can state that the elastic strain energy function for an hyperelastic linear isotropic material must have a representation of the type

$$\phi = \phi(\tilde{\epsilon}_v, \tilde{\epsilon}_s) \tag{10.62}$$

where ϕ is a quadratic homogeneous function, for example

$$\phi(\tilde{\epsilon}_v, \tilde{\epsilon}_s) = \frac{1}{2}(B\tilde{\epsilon}_v^2 + 3G\tilde{\epsilon}_s^2) \tag{10.63}$$

such that

$$\tilde{\boldsymbol{\sigma}} = \frac{\partial \phi}{\partial \tilde{\boldsymbol{\epsilon}}} = B\widehat{\mathbf{m}}\tilde{\epsilon}_v + 2G\tilde{\mathbf{e}} \tag{10.64}$$

and

$$\begin{aligned}
d\phi &= \tilde{\boldsymbol{\sigma}}^T d\tilde{\boldsymbol{\epsilon}} = B d\tilde{\epsilon}_v + 3G d\epsilon_s \\
d^2\phi &= d\tilde{\boldsymbol{\sigma}}^T d\tilde{\boldsymbol{\epsilon}} = B d\tilde{\epsilon}_v^2 + 3G d\epsilon_s^2 > 0
\end{aligned}$$

Since $\tilde{\epsilon}_v^2$ and $d\epsilon_s^2$ are two independent positive variables, the inequality $d^2\phi > 0$ is always satisfied if and only if

$$B > 0; \qquad G > 0 \tag{10.65}$$

where B is known as the *bulk elastic modulus* and G is known as the *shear elastic modulus*.

The constitutive equation for linear elastic material is commonly given in the form, *Hooke's Law*,

$$\tilde{\sigma}_{ij} = \left[\lambda\delta_{kl}\delta_{ij} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right]\tilde{\epsilon}_{kl} \tag{10.66}$$

or, equivalently,

$$\tilde{\sigma}_{ij} = \lambda\tilde{\epsilon}_v\delta_{ij} + 2\mu\tilde{\epsilon}_{ij} \tag{10.67}$$

where

$$\begin{aligned} \lambda &= B - \frac{2G}{3} \\ \mu &= G \end{aligned}$$

are known as the *Lame's elastic constants*. This form can be derived from Eq. 10.64 as follows

$$\begin{aligned} \tilde{\sigma}_{ij} &= B\tilde{\epsilon}_v\delta_{ij} + 2G\tilde{e}_{ij} = B\tilde{\epsilon}_v\delta_{ij} + 2G\left(\tilde{\epsilon}_{ij} - \delta_{ij}\frac{\tilde{\epsilon}_v}{3}\right) = \\ &= \left(B - \frac{2G}{3}\right)\tilde{\epsilon}_v\delta_{ij} + 2G\tilde{\epsilon}_{ij} = \lambda\tilde{\epsilon}_v\delta_{ij} + 2G\tilde{\epsilon}_{ij} = \\ &= \lambda\delta_{kl}\tilde{\epsilon}_{kl}\delta_{ij} + G\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\tilde{\epsilon}_{kl} \end{aligned}$$

It is possible to verify that the vectorial form of Eq. 10.65 is given by

$$\tilde{\sigma} = \mathbf{C}\tilde{\epsilon} \tag{10.68}$$

where the stiffness matrix $\mathbf{C}$ is a positive definite matrix of only two independent coefficients defined as

$$\mathbf{C} = \frac{\mathrm{E}}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & \nu & 0 & . & 0 \\ \nu & (1-\nu) & \nu & 0 & . & 0 \\ \nu & \nu & (1-\nu) & 0 & . & 0 \\ 0 & 0 & 0 & (1-2\nu) & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & (1-2\nu) \end{bmatrix}_{(9\times9)}$$

where

$$\begin{aligned} E &= \frac{\mu\left(2\mu + 3\lambda\right)}{\mu + \lambda} \\ \nu &= \frac{\lambda}{2\left(\lambda + \mu\right)} \end{aligned}$$

Table 10.1: Relationship among elastic moduli

	E	ν	B	$G=\mu$	λ
E,ν	E	ν	$\frac{E}{3(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	$\frac{E\nu}{(1+\nu)(1-2\nu)}$
E,B	E	$\frac{3K-E}{6K}$	B	$\frac{3BE}{9B-E}$	$\frac{B(9B-3E)}{9B-E}$
E,G	E	$\frac{E-2G}{2G}$	$\frac{EG}{9G-3E}$	G	$\frac{G(E-2G)}{3G-E}$
ν,B	$3K(1-2\nu)$	ν	B	$\frac{3K(1-2\nu)}{2(1+\nu)}$	$\frac{3K\nu}{1+\nu}$
ν,G	$2G(1+\nu)$	ν	$\frac{2G(1+\nu)}{3(1-2\nu)}$	G	$\frac{2G\nu}{1-2\nu}$
B,G	$\frac{9GK}{3K+G}$	$\frac{3K-2G}{2(3K+G)}$	B	G	$K-\frac{2G}{3}$
B,λ	$\frac{9B(B-\lambda)}{3B-\lambda}$	$\frac{\lambda}{3K-\lambda}$	B	$\frac{3(K-\lambda)}{2}$	λ
G,λ	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{\lambda}{2(\lambda+G)}$	$\lambda+\frac{2G}{3}$	G	λ

are respectively known as the *Youngs modulus* and the *Poisson ratio*. The relationships among the elastic moduli are shown in Table 10.1, [18]. The respect of the conditions in Eq. 10.65 yields to the conclusion that the possible range constants E and ν is given by

$$E > 0; \qquad -1 \le \nu \le \frac{1}{2} \tag{10.69}$$

However, the measured range of variation of ν for real materials is

$$0 \le \nu \le 0.49 \tag{10.70}$$

In a small deformation process the constitutive equation for an hyperelastic material in Eq. 10.68 reduces to

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\epsilon} \tag{10.71}$$

The values of the parameter E and ν can be experimentally measured in a small deformation 1D axial load test in which $\sigma_{11} \neq 0$ while all the other stress components are equal to zero. In this case, in fact, we have that

$$E = \frac{\sigma_{11}}{\epsilon_{11}}; \qquad \nu = -\frac{\epsilon_{22}}{\epsilon_{11}} = -\frac{\epsilon_{33}}{\epsilon_{11}} \tag{10.72}$$

The inversion of Eq. 10.68 yields to

$$\tilde{\boldsymbol{\epsilon}} = \mathbf{D}\tilde{\boldsymbol{\sigma}} \tag{10.73}$$

where the compliance matrix $\mathbf{D} = \mathbf{C}^{-1}$ is positive definite matrix of only two independent coefficients defined as

$$\mathbf{D} = \frac{1}{\mathrm{E}} \begin{bmatrix} 1 & -\nu & -\nu & 0 & \vdots & 0 \\ -\nu & 1 & -\nu & 0 & \vdots & 0 \\ -\nu & -\nu & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & (1+\nu) & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \vdots & (1+\nu) \end{bmatrix}_{(9\times 9)}$$

This inverse relationship can be obtained from a complementary elastic energy function

$$\psi = \psi(\tilde{p}, \tilde{q}) = \phi - \tilde{\boldsymbol{\sigma}}^T \tilde{\boldsymbol{\epsilon}} \tag{10.74}$$

where ψ is a quadratic homogeneous function of the type

$$\psi(\tilde{p}, \tilde{q}) = \frac{\tilde{p}^2}{2B} + \frac{\tilde{q}^2}{6G} \tag{10.75}$$

such that

$$\tilde{\epsilon} = \frac{\partial \psi}{\partial \tilde{\sigma}} = \frac{\tilde{p}}{3B}\widehat{\mathrm{m}} + \frac{1}{2G}\tilde{\mathrm{s}} \tag{10.76}$$

and

$$\begin{aligned} d\psi &= \tilde{\sigma}^T d\tilde{\epsilon} = \frac{d\tilde{p}}{B}\widehat{\mathrm{m}} + \frac{1}{3G}d\tilde{q} \\ d^2\psi &= d\tilde{\sigma}^T d\tilde{\epsilon} = \frac{d\tilde{p}^2}{B} + \frac{d\tilde{q}^2}{3G} > 0 \end{aligned}$$

Finally, it is of interest to notice that, Eqs. 10.47-10.49,

$$\tilde{\mathrm{s}} = 2G\tilde{\mathrm{e}} \tag{10.77}$$

$$\tilde{p} = B\tilde{\epsilon}_v \tag{10.78}$$

$$\tilde{q} = 3G\tilde{\epsilon}_s \tag{10.79}$$

and the inverse relationships are given by, Eqs. 10.52-10.54,

$$\tilde{\mathrm{e}} = \frac{\tilde{\mathrm{s}}}{2G} \tag{10.80}$$

$$\tilde{\epsilon}_v = \frac{\tilde{p}}{B} \tag{10.81}$$

$$\tilde{\epsilon}_s = \frac{\tilde{q}}{3G} \tag{10.82}$$

10.10 Newtonian Fluids

In Continuum Mechanics, the term *fluids* usually denotes a material that, unlike solid materials, has the following two main characteristics:

- Fluids cannot support a shear stress; in a fluid, the shear deformation will continue as long as any shear stress is applied.
- Fluids carry no memory of their initial state; in a fluid, the stress at a point depends exclusively on the instantaneous rate of deformation.

The classical Fluid Mechanics Theory assumes that the Cauchy stress tensor $\boldsymbol{T}$ can be expressed as

$$\tilde{\boldsymbol{T}} = \tilde{\boldsymbol{T}}^{(R)} + \tilde{\boldsymbol{T}}^{(D)} \tag{10.83}$$

where

- $\tilde{\mathcal{T}}^{(R)}$ is a recoverable part assumed to be equal to, Eq. 9.146,

$$\tilde{\mathcal{T}}^{(R)} = -\pi \mathbf{I} \tag{10.84}$$

where π is the thermodynamic pressure obeying the thermal equation of state, Eq. 9.35,

$$\pi = \pi(\theta, v)$$

where $v = 1/\varrho$ is the specific volume.

- $\tilde{\mathcal{T}}^{(D)}$ is a dissipative (non-recoverable) part assumed to be equal to

$$\tilde{\mathcal{T}}^{(D)} = \mathbf{g}(\widehat{\mathcal{D}}) \tag{10.85}$$

where $\mathbf{g}$ is a tensor value function of the Euler Strain Rate tensor $\widehat{\mathcal{D}}$.

In general, therefore, the constitutive equation for fluids is assumed to be given by

$$\mathcal{T} = -\pi \mathbf{I} + \mathbf{g}(\widehat{\mathcal{D}}) \tag{10.86}$$

and, depending on the functional form of $\mathbf{g}$, fluids are further classified as follows:

- If $\mathbf{g}$ is a non-linear tensor function, the fluid is called *Stokesian*.
- If $\mathbf{g}$ is a linear tensor function, the fluid is called *Newtonian*.

According to Principle of Frame Indifference, $\mathbf{g}$ must respect the condition

$$\mathcal{T}'^{(D)} = \mathbf{R}\mathbf{g}(\widehat{\mathcal{D}})\mathbf{R}^T = \mathbf{g}(\mathbf{R}\widehat{\mathcal{D}}\mathbf{R}^T) \tag{10.87}$$

for all rotation (orthogonal) matrices $\mathbf{R}$. In fact, under a change of reference system rotate by $\mathbf{R}$, both the tensors $\mathcal{T}$ and $\widehat{\mathcal{D}}$ transform according to the rule

$$\begin{aligned} \mathcal{T}'^{(D)} &= \mathbf{R}\mathcal{T}\mathbf{R}^T \\ \widehat{\mathcal{D}}' &= \mathbf{R}\widehat{\mathcal{D}}\mathbf{R}^T \end{aligned}$$

Notice that the condition in Eq. 10.87 is analogous to that in Eq. 10.37 and thus it implies also a condition of isotropicity for fluids.

In a Newtonian fluid, similarly to the case of linear isotropic elastic solids, Eqs. 10.66 and 10.67, the constitutive equation for the dissipative part is set to

$$\mathcal{T}_{ij} + \pi\delta_{ij} = \left[\lambda\delta_{kl}\delta_{ij} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)\right]\widehat{\mathcal{D}}_{kl} \tag{10.88}$$

or, equivalently,

$$\sigma_{ij} = -\pi\delta_{ij} + \lambda d_{kk}\delta_{ij} + \mu d_{ij} \tag{10.89}$$

where

$$\begin{aligned} \sigma_{ij} &= \mathcal{T}_{ij} \\ d_{ij} &= \widehat{\mathcal{D}}_{ij} \end{aligned}$$

and λ and μ are two independent constant. The respect of the inequality in Eq. 9.145, leads to the conclusion that

$$\begin{aligned} k &= \lambda + \frac{2}{3}\mu \geq 0 \\ \mu &\geq 0 \end{aligned}$$

In vectorial form the constitutive equation for a Newtonian fluid can be indicated as

$$\boldsymbol{\sigma} = -\pi\widehat{\mathbf{m}} + \mathbf{C}\dot{\boldsymbol{\epsilon}} \tag{10.90}$$

where $\dot{\boldsymbol{\epsilon}}$ collects the components of $\widehat{\boldsymbol{\mathcal{D}}}$ and $\mathbf{C}$ is a symmetric positive definite (or semidefinite) matrix of the same form of that in Eq. 10.68.

An equivalent form of the Newtonian constitutive relationship in Eq. 10.89 is given by, *Navier-Poisson Law of a Newtonian Fluid*,

$$\begin{cases} \tau_{ij} &= 2\mu e_{ij} \\ p &= \pi - \kappa d_{kk} \end{cases} \tag{10.91}$$

where κ, known as the *bulk viscosity*, is defined as

$$\kappa = \lambda + \frac{2}{3}\mu$$

τ_{ij} and e_{ij} are the deviatoric stress and strain rate tensors respectively defined as

$$\begin{aligned} \tau_{ij} &= \sigma_{ij} + \delta_{ij}p \\ e_{ij} &= d_{ij} - \delta_{ij}\frac{d_{kk}}{3} \end{aligned}$$

Finally, p is the mean compressive pressure defined as

$$p = -\frac{\sigma_{kk}}{3}$$

In fact, according to the above positions, the constitutive relationship in Eq. 10.89 yields to

$$\tau_{ij} - p\delta_{ij} = -\pi\delta_{ij} + \lambda d_{kk}\delta_{ij} + 2\mu\left(e_{ij} + \delta_{ij}\frac{d_{kk}}{3}\right)$$

that is

$$\tau_{ij} = [(p-\pi) + \kappa d_{kk}]\delta_{ij} + 2\mu e_{ij}$$

from which, setting $i = j$ and summing, we find

$$(p-\pi) + \kappa d_{kk} = 0$$

since $\tau_{kk} = 0$ and $e_{kk} = 0$.

We notice that according to Eq. 10.91, the stress power can be expressed as

$$\dot{w} = \dot{w}^{(n)} + \dot{w}^{(D)} \tag{10.92}$$

where

$$\begin{aligned} \dot{w} &= \sigma_{ij}d_{ij} = -pd_{kk} + \tau_{ij}e_{ij} = \\ &= -\pi d_{kk} + \kappa d_{kk}^2 + 2\mu e_{ij}e_{ij} \\ \dot{w}^{(n)} &= \sigma_{ij}^{(n)}d_{ij} = -\pi d_{kk} \\ \dot{w}^{(D)} &= \sigma_{ij}^{(D)}d_{ij} = \kappa d_{kk}^2 + 2\mu e_{ij}e_{ij} \end{aligned}$$

Finally, if the fluid respects one of the following conditions:

- perfect (or inviscid) fluid conditions

$$\lambda = \mu = 0 \tag{10.93}$$

  so that $\kappa = 0$.

- Stokes condition

$$\lambda = -\frac{2}{3}\mu \tag{10.94}$$

  so that $\kappa = 0$.

- incompressibility condition

$$d_{kk} = 0 \tag{10.95}$$

then

$$\pi = p \tag{10.96}$$

Chapter 11

Hypoelastic Constitutive Equations

11.1 Introduction

The hyperelastic constitutive theory presented in the previous Chapter is based on the main hypothesis that all deformations are recoverable upon removal of the applied forces. It is common knowledge, instead, that a large class of real solid materials do not exhibit such a characteristic when subjected to cycles of loading and unloading.

A typical example of the response of a non-hyperlastic material subjected to cycles of triaxial loading condition, Section 11.2, is shown in Fig. 11.1. In this diagram we can distinguish the following important features:

- stress vs. strain relationships are not linear and are dependent on the current state of stress;
- not all the deformations are recoverable upon removal of the applied load;
- there exists an ultimate value for stress.

Irreversible deformations resulting from mechanisms of *slip*, or from dislocations at the atomic level, and thereby leading to permanent dimensional changes, are known as *plastic deformations*. With reference to Fig. 11.1, the strains $\epsilon_a^{(p)}$ and $\epsilon_a^{(e)}$ represent the plastic (irreversible) and the elastic (reversible) components of the total strain ϵ_a, respectively.

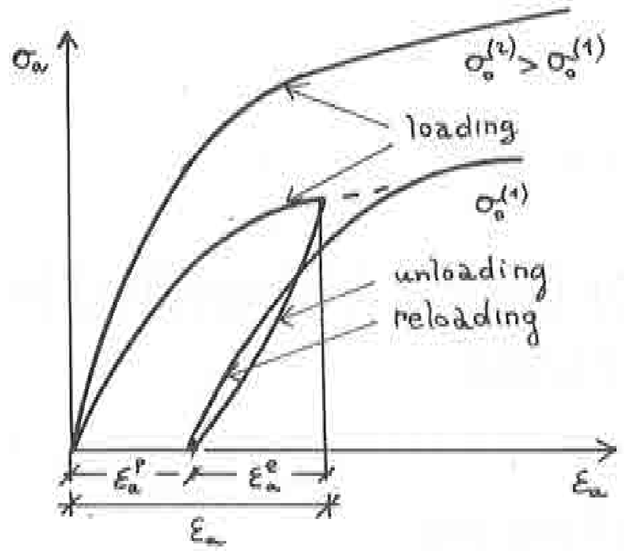


Figure 11.1: Typical triaxial response of a real solid material

Experimental evidence shows the existence of an ultimate stress intensity value for any real material, known as *state of failure*. The envelope of all possible states of stress producing failure describe a surface $Q(\sigma)$ in the stress space known as *failure surface*, Section 11.3.

The three distinct material responses for loading, unloading and reloading triaxial stress condition are as well observable for any other different stress path. The envelope of all the transition states of stress from one type of response to another defines in the stress space σ a surface $G(\sigma, k)$ known as *yield surface*, Section 11.5.

A possible way to model such a type of material response, in the hypothesis of small deformations, is provided by the so-called *hypoelastic constitutive model*, Section 11.6, according to which a possible constitutive equation is of the form

$$d\sigma = \mathbf{C} d\epsilon \tag{11.1}$$

where σ is the Cauchy stress vector, ϵ the Linear Lagrange strain vector and $\mathbf{C}$ is the tangent constitutive matrix assumed to depend on the current stress σ, that is

$$\mathbf{C} = \mathbf{C}(\sigma) \tag{11.2}$$

Various hypoelastic constitutive equations have been developed and an exhaustive review can be found in [18, 28]. In practice, the most used ones are the so-called *Variable Moduli Models*, Section 11.7, according to which the material response is assumed to be incrementally isotopic. Under this hypothesis, the tangent **C** matrix contains only two independent material parameters

$$\mathbf{C} = \mathbf{C}(E, \nu) \tag{11.3}$$

or, alternatively,

$$\mathbf{C} = \mathbf{C}(B, G) \tag{11.4}$$

The law of variation of E and ν, or of B and G, with $\boldsymbol{\sigma}$, may be determined on the basis of the material response under triaxial loading conditions. An example of this type of model is fully described in Section 11.9.

11.2 Triaxial Apparatus

The properties of a solid material can be determined experimentally in various ways. Usually laboratory tests are performed on a small cylindrical sample of material. The simplest test consists in loading axially the sample; more information about the material properties can be obtained by applying a confining pressure all around the sample.

The possibility of controlling at the same time axial and confining pressure is provided by the so called *simple triaxial test apparatus*, Fig. 11.2. Today several types of triaxial apparatuses are available on the market. In the most sophisticated one, the applied axial and confining pressures are controlled through computer facilities. Also, these apparatuses provide the complete monitoring of the axial and radial strains as well as of the volume change.

As a general rule, a triaxial apparatus can test a material sample for any desired loading history of the following stress components, Fig. 11.2:

- axial stress $\sigma_a = \sigma_z$;
- confining pressure $\sigma_o = \sigma_x = \sigma_y$;
- mean pressure $p = (\sigma_a + 2\sigma_o)/3$;
- deviatoric stress $q = |\sigma_a - \sigma_o|$.

The material response may be then controlled by monitoring the following strain components:

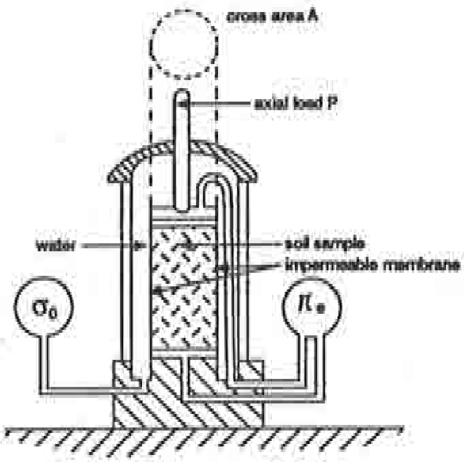


Figure 11.2: Simple triaxial apparatus

- Axial strain $\epsilon_a = \epsilon_z$
- Hoop strain $\epsilon_o = \epsilon_x = \epsilon_y$
- Volumetric strain $\epsilon_v = \epsilon_a + 2\epsilon_o$
- Shear strain $\epsilon_s = \frac{2}{3}|\epsilon_a - \epsilon_o|$

Among the different stress paths which is possible to follow through the triaxial apparatus, we may distinguish the following three main paths, Fig. 11.3:

- Isotropic Compression, IC path in Fig. 11.3a, of stress components

$$\begin{aligned} d\sigma_a &= d\sigma_o \\ dp &= d\sigma_a \\ dq &= 0 \\ \frac{dq}{dp} &= 0 \end{aligned}$$

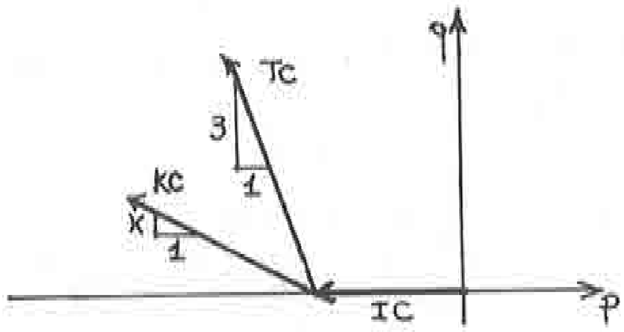


Figure 11.3: Stress paths in a simple triaxial test

- K-Compression, KC path in Fig. 11.3b, of stress components

$$
\begin{aligned}
\frac{d\sigma_a}{d\sigma_o} &= \frac{3+2K}{3-K} \\
dp &= \frac{d\sigma_o}{3}\left[\frac{d\sigma_a}{d\sigma_o}+2\right] \\
dq &= \left|d\sigma_o\left[\frac{d\sigma_a}{d\sigma_o}-1\right]\right| \\
\frac{dq}{dp} &= aK
\end{aligned}
$$

where

$$a = \frac{|d\sigma_o|}{d\sigma_o}\frac{|3K-1|}{3K-1}$$

- Triaxial Compression, TC path in Fig. 11.3b, of stress components

$$
\begin{aligned}
d\sigma_a &\neq 0 \\
d\sigma_o &= 0 \\
dp &= \frac{d\sigma_a}{3} \\
dq &= |d\sigma_a| \\
\frac{dq}{dp} &= 3\frac{|d\sigma_a|}{d\sigma_a}
\end{aligned}
$$

11.3 Failure Criteria

Coulomb (1776) was apparently the first to suggest a criterion for the definition of failure. Let τ and σ_n be the shear and normal stresses acting on a plane of normal $\tilde{\mathbf{n}}$. According to the *Coulomb failure criterion*, failure takes place along this plane if

$$\tau = c - \sigma_n \tan\varphi \tag{11.5}$$

where c and φ, known as the *cohesion* and the *friction angle*, respectively, are material constants. For pure cohesive materials, where $\varphi = 0$, Eq. 11.5 simplifies into

$$\tau = c \tag{11.6}$$

which is known as the *Tresca failure criterion*, (Tresca, 1868).

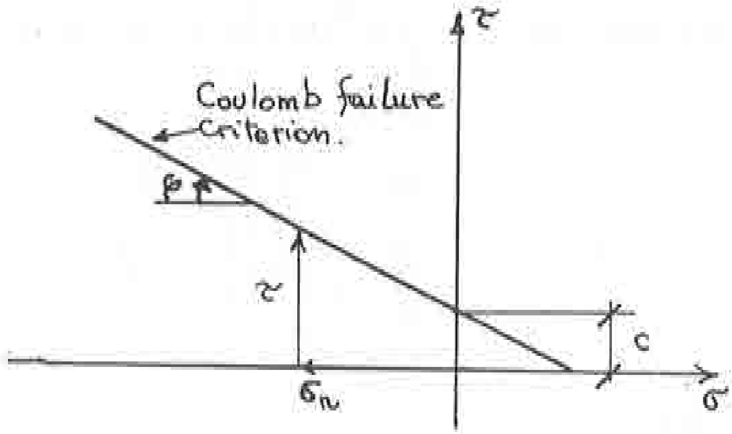


Figure 11.4: Mohr-Coulomb failure criterion in the σ vs. τ Mohr-plane

Mohr (1882) interpreted the Coulomb criterion in terms of principal stresses and the relative geometrical representation of the complete *Mohr-Coulomb failure surface* in the principal stress space is shown in Fig. 11.5a. This failure surface is a pyramid, symmetric about the space diagonal and the normal section at any point is an irregular hexagon, Fig. 11.5b. For pure cohesive material, the failure surface changes into a prism whose normal section at any point is a regular hexagon, Fig. 11.5b.

A convenient representation of the Mohr-Coulomb failure surface in the 3D principal stress space was given by Nayak and Zienkiewicz, [70], in term

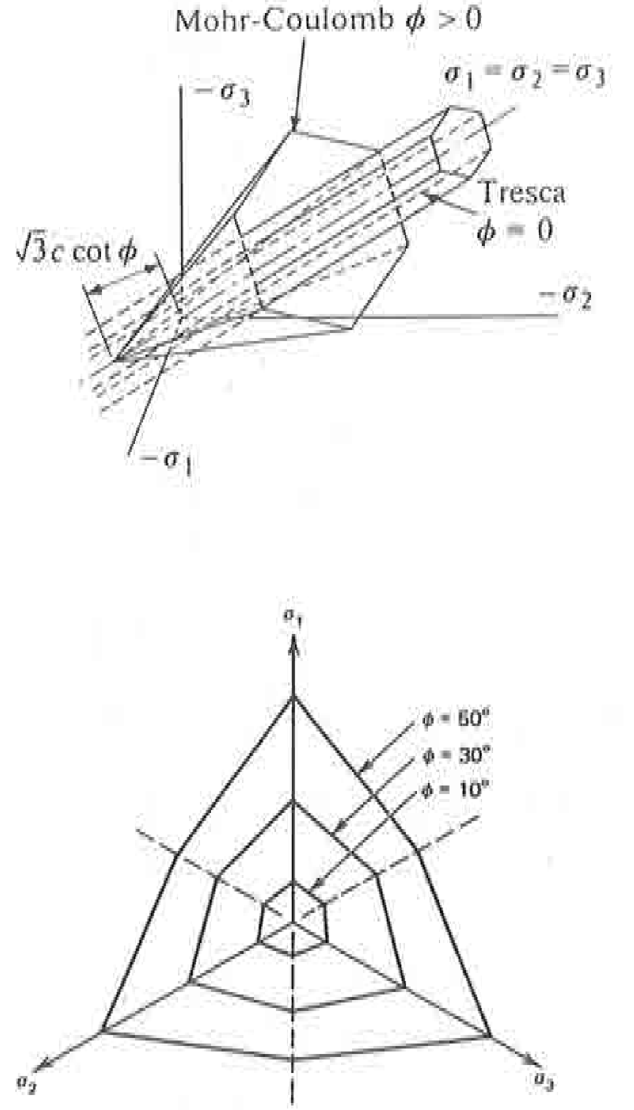


Figure 11.4: Mohr-Coulomb and Tresca failure surfaces

of stress invariant in the following form

$$I_1 \frac{\sin\varphi}{3} + J_2^{\frac{1}{2}}\left(\cos\theta - \frac{\sin\theta\sin\varphi}{\sqrt{3}}\right) = c\cos\varphi \tag{11.7}$$

The Tresca yield criterion can be then easily obtained by setting $\varphi = 0$; this gives

$$J_2^{\frac{1}{2}}\cos\theta = c \tag{11.8}$$

The above expression in Eq. 11.7 may be derived noticing in Fig. 11.4 that

$$r = (d - b)\sin\varphi$$

where

$$\begin{aligned} r &= \frac{\sigma_1 - \sigma_3}{2} \\ b &= \frac{\sigma_1 + \sigma_3}{2} \\ d &= \frac{c}{\tan\varphi} \end{aligned}$$

Thus,

$$\frac{\sigma_1 - \sigma_3}{2} = \left(\frac{c}{\tan\varphi} - \frac{\sigma_1 + \sigma_3}{2}\right)\sin\varphi \tag{11.9}$$

According to the relationships in Eq. 3.28, the principal stresses σ_1 and σ_3 can be expressed as

$$\left\{\begin{array}{c}\sigma_1\\ \\ \sigma_3\end{array}\right\} = \frac{I_1}{3}\left\{\begin{array}{c}1\\ \\ 1\end{array}\right\} + 2\left(\frac{J_2}{3}\right)^{\frac{1}{2}}\left\{\begin{array}{c}\sin\left(\theta + \frac{2}{3}\pi\right)\\ \\ \sin\left(\theta - \frac{2}{3}\pi\right)\end{array}\right\}$$

from which

$$\begin{aligned} \frac{\sigma_1 - \sigma_3}{2} &= \left(\frac{J_2}{3}\right)^{\frac{1}{2}}\left\{\sin\left(\theta + \frac{2}{3}\pi\right) - \sin\left(\theta - \frac{2}{3}\pi\right)\right\} = \\ &= \left(\frac{J_2}{3}\right)^{\frac{1}{2}}\left\{2\cos\frac{1}{2}\left[\left(\theta + \frac{2}{3}\pi\right) + \left(\theta - \frac{2}{3}\pi\right)\right]\right. \\ &\qquad \left.\sin\frac{1}{2}\left[\left(\theta + \frac{2}{3}\pi\right) - \left(\theta - \frac{2}{3}\pi\right)\right]\right\} = \\ &= \left(\frac{J_2}{3}\right)^{\frac{1}{2}}\left\{2\cos\theta\sin\frac{2}{3}\pi\right\} = J_2^{1/2}\cos\theta \end{aligned} \tag{11.10}$$

$$\begin{aligned}\frac{\sigma_1+\sigma_3}{2} &= \frac{I_1}{3}+\left(\frac{J_2}{3}\right)^{\frac{1}{2}}\left\{\sin\left(\theta+\frac{2}{3}\pi\right)+\sin\left(\theta-\frac{2}{3}\pi\right)\right\}=\\ &= \frac{I_1}{3}+\left(\frac{J_2}{3}\right)^{\frac{1}{2}}\left\{2\sin\frac{1}{2}\left[\left(\theta+\frac{2}{3}\pi\right)+\left(\theta-\frac{2}{3}\pi\right)\right]\right.\\ &\quad \left.\cos\frac{1}{2}\left[\left(\theta+\frac{2}{3}\pi\right)-\left(\theta-\frac{2}{3}\pi\right)\right]\right\}=\\ &= \frac{I_1}{3}+\left(\frac{J_2}{3}\right)^{\frac{1}{2}}\left\{2\sin\theta\cos\frac{2}{3}\pi\right\}=\frac{I_1}{3}-\left(\frac{J_2}{3}\right)^{\frac{1}{2}}\sin\theta \quad (11.11)\end{aligned}$$

Substituting Eqs. 11.10 and 11.11 into Eq. 11.9 we obtain the relationship in Eq. 11.7

Von Mises (1913), tried to reduce the mathematical difficulties associated with the determination of the angle θ in Eq. 11.8, by replacing the hexagonal prism by an inscribed circular cylinder, Fig. 11.6. Thus, the Von Mises

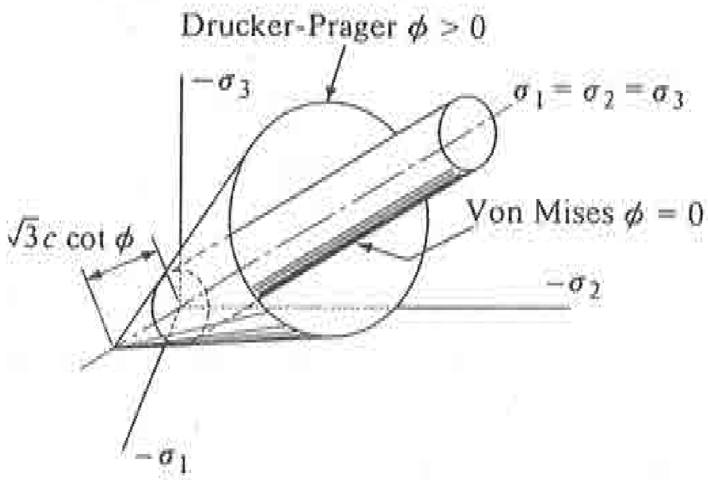


Figure 11.6: Drucker-Prager and Von-Mises failure surfaces

failure criterion can be obtained by setting $\theta = 30°$ in Eq. 11.8, that is

$$J_2^{\frac{1}{2}} = \frac{2}{\sqrt{3}}c \qquad (11.12)$$

Drucker and Prager (1952), [32], then proposed the following generalized form of Eq. 11.12

$$\alpha I_1 + J_2^{\frac{1}{2}} = k \tag{11.13}$$

where α and k are to be considered as two independent material parameters. In this case the yield surface in the principal stress space is a right circular cone whose axis of revolution is the diagonal space, Fig. 11.6.

It is easy to verify that all four Mohr-Coulomb based failure criteria in Eqs. 11.7, 11.8, 11.12 and 11.13 may be generally expressed as

$$Q = q + Mp - N = 0 \tag{11.14}$$

where p is the mean pressure (negative for compression), q the deviatoric stress and:

- In the Mohr-Coulomb failure surface we have that

$$\begin{aligned} N &= N(\theta) = \frac{3c\cos\varphi}{\sqrt{3}\cos\theta - \sin\theta\sin\varphi} \\ M &= M(\theta) = \frac{3\sin\varphi}{\sqrt{3}\cos\theta - \sin\theta\sin\varphi} \end{aligned}$$

- In the Tresca failure surface, being $\varphi = 0$, we have that

$$\begin{aligned} N &= N(\theta) = \frac{3c}{\sqrt{3}\cos\theta} \\ M &= 0 \end{aligned}$$

- In the Drucker-Prager failure surface N and M are independent of θ.
- In the Von Mises failure surface $M = 0$ and N is independent of θ.

In particular:

- The Drucker-Prager cone that crosses the outer apexes of the Mohr-Coulomb hexagon at any section can be obtained by setting in the Mohr-Coulomb relationships $\theta = 30^o$. This yields to

$$\begin{aligned} N &= \frac{6c\cos\varphi}{3 - \sin\varphi} \\ M &= \frac{6\sin\varphi}{3 - \sin\varphi} \end{aligned}$$

- The Drucker-Prager cone that crosses the inner apexes of the Mohr-Coulomb hexagon at any section can be obtained by setting in the Mohr-Coulomb relationships $\theta = -30^o$. This yields to

$$\begin{aligned} N &= \frac{6c\cos\varphi}{3+\sin\varphi} \\ M &= \frac{6\sin\varphi}{3+\sin\varphi} \end{aligned}$$

- The Drucker-Prager cone that is tangent to the Mohr-Coulomb hexagon edges at any section can be obtained by setting in the Mohr-Coulomb relationships

$$\theta = -\arctan\left(\frac{\sin\varphi}{\sqrt{3}}\right)$$

This yields to

$$\begin{aligned} N &= \frac{3c\cos\phi}{(3+\sin^2\varphi)^{1/2}} \\ N &= \frac{3\sin\phi}{(3+\sin^2\varphi)^{1/2}} \end{aligned}$$

Finally, for the Von-Mises failure surface the N and M expressions can be obtained setting $\varphi = 0$ in the above listed Drucker-Prager expressions.

More recently other types of failure criteria have been proposed. An excellent review of some of them can be found in [18].

11.4 A Modified Mohr-Coulomb Surface

We have seen in the previous Section that the Mohr-Coulomb criterion in Eq. 11.14 is a generalization of the simple Coulomb frictional law postulated in Eq. 11.5. The relative surface in the $3D$ principal stress space presents six sharp corners, Fig. 11.5. But experimental results performed on real material with a true $3D$ stress states seem to indicate smoothed corner zones.

A possible modification of the Mohr-Coulomb equation accounting for smoothed corner zones can be of the type, [26],

$$q = -M(\theta)\,(p - p_M) \qquad (11.15)$$

where

$$M(\theta) = \begin{cases} M_1(\theta) & \text{for } -\frac{\pi}{6} \leq \theta \leq \theta_1 \\ M_o(\theta) & \text{for } \theta_1 \leq \theta \leq \theta_2 \\ M_2(\theta) & \text{for } \theta_2 \leq \theta \leq \frac{\pi}{6} \end{cases}$$

with

$$\begin{aligned} M_o(\theta) &= \frac{3\sin\phi}{\sqrt{3}\cos\theta - \sin\theta\sin\phi} \\ M_i(\theta) &= a_i\left(g_i(\theta) + \sqrt{g_i^2(\theta) - b_i}\right) \end{aligned}$$

and

$$\begin{aligned} a_i &= \frac{a_\phi b_\phi}{b_\phi - d_i} \\ b_i &= d_i\left[2c_ie_i - d_i\right] + 2e_i - 1 \\ c_i &= \begin{cases} 0 & \text{for } i = 1 \\ \sqrt{3} & \text{for } i = 2 \end{cases} \\ d_i &= \text{cotan}\left(\frac{\pi}{3} - \theta_i\right) \\ e_i &= \frac{1 + b_\phi d_i}{1 + b_\phi c_i} \\ g_i(\theta) &= e_i\left[\sin\left(\frac{\pi}{3} - \theta\right) + c_i\cos\left(\frac{\pi}{3} - \theta\right)\right] \end{aligned}$$

for $i = 1, 2$, in which

$$\begin{aligned} a_\phi &= \frac{6\sin\phi}{3 + \sin\phi} \\ b_\phi &= \frac{1}{\sqrt{3}}\frac{\sin\phi + 3}{\sin\phi - 1} \end{aligned}$$

Finally, $(p_M, \theta_1, \theta_2, \phi)$ are material constants whose physical meaning can be understood from the graphical representation of Eq. 11.15 in Fig. 11.7. In particular:

- p_M represents the distance OV shown in Fig. 11.7b;

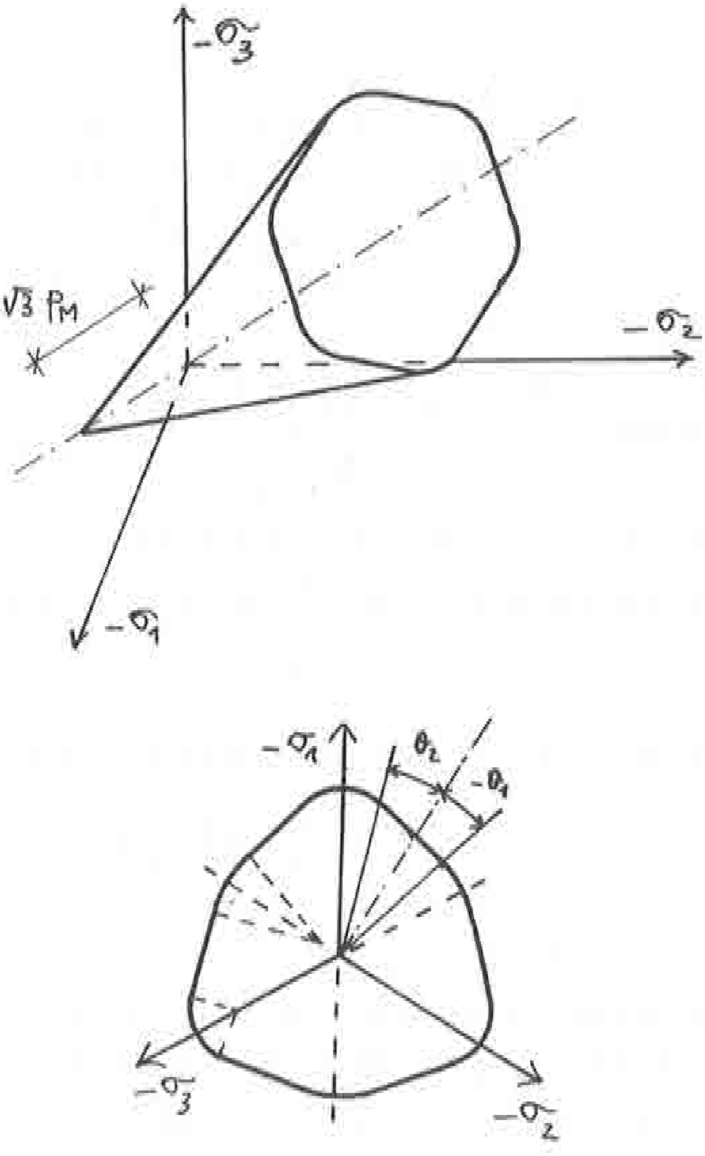


Figure 11.7: A modified Mohr-Coulomb surface

- θ_1 and θ_2 are the values of the angles where the smoothed corner zones begin; their values result to be bounded as

$$-\frac{\pi}{6} \leq \theta_1 \leq \theta_2 \leq \frac{\pi}{6}$$

- ϕ is the (friction) angle which controls the value of the inclination M, Fig. 11.7b; its value can be determined by solving the implicit equation

$$M_2\left(\theta = \frac{\pi}{6}\right) = -\frac{q^{(f)}}{\left(p^{(f)} - p_M\right)}$$

where $\left(p^{(f)}, q^{(f)}\right)$ is a pair of experimental stress invariant values measured on the inclined line in Fig. 11.7b. The value of ϕ results to be bounded as

$$0 < \phi < \frac{\pi}{2}$$

As a particular case of this modified Mohr-Coulomb criterion, we have:

- The standard Mohr-Coulomb criterion which can be obtained setting

$$\theta_1 = -\theta_2 = -\frac{\pi}{6}$$

- The Drucker-Prager criterion which can be obtained setting

$$\theta_1 = \theta_2 = -\arctan\left(\frac{\sin\phi}{\sqrt{3}}\right)$$

Interestingly, by setting

$$\theta_1 = -\theta_2 = -25^o$$

the corresponding M values calculated according to Eq. 11.15 and that calculated according to Eq. 11.14 specialized for the standard Mohr-Coulomb criterion, differ at most of 8 %, for $\phi < 60^o$.

For our purposes, it is useful to report the following derivatives

$$\begin{aligned}
\frac{dM_o(\theta)}{d\theta} &= M_o(\theta)^2 \frac{\sqrt{3}\sin\theta + \cos\theta\sin\phi}{3\sin\phi} \\
\frac{d\widetilde{M_o}(\theta)}{d\theta} &= \frac{dM_o(\theta)}{d\theta}\frac{1}{\cos 3\theta} = M_o(\theta)^2 \frac{\sqrt{3}\sin\theta + \cos\theta\sin\phi}{3\cos 3\theta\sin\phi}
\end{aligned}$$

$$\frac{dM_i(\theta)}{d\theta} = \frac{M_i(\theta)g_i'(\theta)}{\sqrt{g_i^2(\theta) - b_i}}$$

$$\frac{d\widetilde{M_i}(\theta)}{d\theta} = \frac{dM_i(\theta)}{d\theta}\frac{1}{\cos 3\theta} = \frac{M_i(\theta)e_i}{\sqrt{g_i^2(\theta) - b_i}}\frac{1 + 3(-1)^i}{4\cos\theta\left(\cos\theta + (-1)^i\sqrt{3}\sin\theta\right)}$$

where

$$g_i'(\theta) = e_i\left[-\cos\left(\frac{\pi}{3} - \theta\right) + e_i\sin\left(\frac{\pi}{3} - \theta\right)\right]$$

for $i = 1, 2$.

11.5 Yield Surface

Experimental evidence has shown that the three distinct material behaviors observed for loading, unloading and reloading triaxial conditions are as well observable for any other type of stress paths.

Clearly, in a general stress path involving a contemporaneous change of all six independent stress components, the terms loading, unloading and reloading no longer have such a clear-cut meaning as in the triaxial case. In fact, it is possible that the material is loaded in one direction while unloaded in the others.

Supported by experimental evidence, we assume the existence in stress space $\boldsymbol{\sigma}$ of a so-called *yield surface* $G(\boldsymbol{\sigma}, k)$ where k is some hardening parameter controlling the evolution of G. For loading conditions, the yield surface evolves following the current stress point $\boldsymbol{\sigma}$, so that:

$$G(\boldsymbol{\sigma}, k) = 0 \tag{11.16}$$

Consequently any other possible stress state may lie only within the space region inscribed by the yield surface, that is

$$G(\boldsymbol{\sigma}, k) < 0 \tag{11.17}$$

If $d\boldsymbol{\sigma}$ represents the infinitesimal stress increment from the current $\boldsymbol{\sigma}$, we define loading/unloading/reloading conditions as follows, Fig. 11.8:

$$G(\boldsymbol{\sigma}, k) = 0 \quad \text{and} \quad \begin{cases} G(\boldsymbol{\sigma} + d\boldsymbol{\sigma}, k) \geq 0; & \text{loading.} \\ G(\boldsymbol{\sigma} + d\boldsymbol{\sigma}, k) < 0; & \text{unloading.} \end{cases}$$

$$G(\boldsymbol{\sigma}, k) < 0 \quad \text{and} \quad \begin{cases} G(\boldsymbol{\sigma} + d\boldsymbol{\sigma}, k) \geq G(\boldsymbol{\sigma}, k); & \text{reloading.} \\ G(\boldsymbol{\sigma} + d\boldsymbol{\sigma}, k) < G(\boldsymbol{\sigma}, k); & \text{unloading.} \end{cases}$$

The specific form of G is material dependent.

11.6 Hypoelastic Theory

A general class of incremental constitutive equations has been proposed in the form of

$$\dot{\mathcal{T}} = \mathrm{g}(\mathcal{T}, \dot{\mathcal{L}}) \tag{11.18}$$

where $\mathcal{T}$ and $\mathcal{L}$ denote the Cauchy stress and the linear Lagrange strain tensors, respectively. The overdot denotes the rate of change with respect to time.

This incremental constitutive equation is strictly applicable under the hypothesis of small deformations in which also rigid rotations are negligible. In fact, Eq. 11.18 does not satisfy the principle of Frame-indifference for any arbitrary motion. Part of the difficulty is that $\dot{\mathcal{T}}$ does not transform according to the rule in Section 10.2.

For (initially) isotropic materials, the response value function must respect the condition, Eq. 10.37,

$$\dot{\mathcal{T}}' = \mathbf{R}\mathbf{g}(\mathcal{T}, \dot{\mathcal{L}})\mathbf{R}^T = \mathbf{g}(\mathcal{T}', \dot{\mathcal{L}}') \tag{11.19}$$

where

$$\begin{aligned} \mathcal{T}' &= \mathbf{R}\mathcal{T}\mathbf{R}^T \\ \dot{\mathcal{L}}' &= \mathbf{R}\dot{\mathcal{L}}\mathbf{R}^T \end{aligned}$$

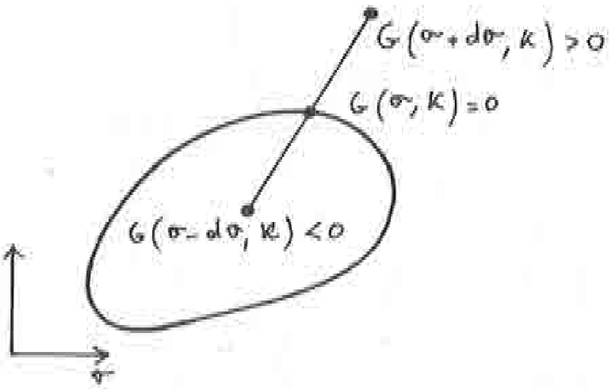
for all the rotational (orthogonal) matrices $\mathbf{R}$. It can be proved, [83], that response functions $\mathbf{g}$ respecting the condition in Eq. 11.19 have a representation of the type

$$\begin{aligned} \dot{\mathcal{T}} &= \mathbf{g}(\mathcal{T}, \dot{\mathcal{L}}) = \\ &= \alpha_0\mathbf{I} + \alpha_1\dot{\mathcal{L}} + \alpha_2\dot{\mathcal{L}}\dot{\mathcal{L}} + \alpha_3\mathcal{T} + \alpha_4\mathcal{T}\mathcal{T} + \\ &\quad +\alpha_5(\dot{\mathcal{L}}\mathcal{T} + \mathcal{T}\dot{\mathcal{L}}) + \alpha_6(\dot{\mathcal{L}}\dot{\mathcal{L}}\mathcal{T} + \mathcal{T}\dot{\mathcal{L}}\dot{\mathcal{L}}) + \alpha_7(\dot{\mathcal{L}}\mathcal{T}\mathcal{T} + \mathcal{T}\mathcal{T}\dot{\mathcal{L}}) + \\ &\quad +\alpha_8(\dot{\mathcal{L}}\dot{\mathcal{L}}\mathcal{T}\mathcal{T} + \mathcal{T}\mathcal{T}\dot{\mathcal{L}}\dot{\mathcal{L}}) \end{aligned} \tag{11.20}$$

where α_i, for $i = 0, 1, \ldots, 8$, are scalar invariant functions of the six $\dot{\mathcal{L}}$ and $\mathcal{T}$ invariants and of the following four joint invariants

$$Q_1 = \dot{\mathcal{L}} : \mathcal{T}; \qquad Q_2 = (\dot{\mathcal{L}}\mathcal{T}) : \mathcal{T}$$

$$Q_3 = (\dot{\mathcal{L}}\dot{\mathcal{L}}) : \mathcal{T}; \quad Q_4 = (\dot{\mathcal{L}}\dot{\mathcal{L}}) : (\mathcal{T}\mathcal{T})$$



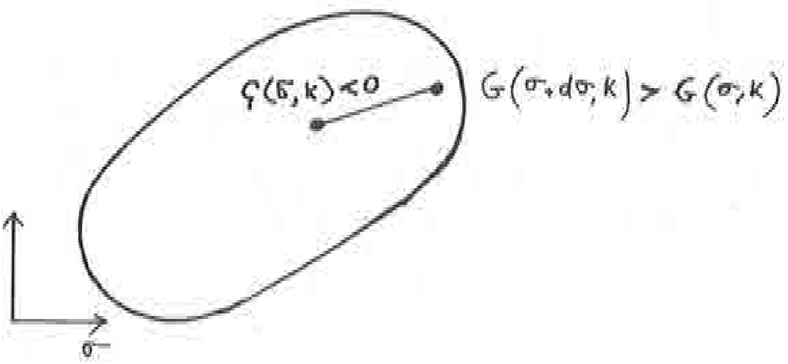


Figure 11.8: Loading/unloading/reloading criterion

For *time-independent* materials, the constitutive relationship in Eq. 11.18 must be homogeneous in time, that is time must appear to the same order in all terms so that it may be dropped from the equation. In this case, [18], the constitutive equation in Eq. 11.20 reduces to

$$\begin{aligned}\dot{\mathcal{T}} &= \mathrm{g}(\mathcal{T}, \dot{\mathcal{L}}) = \\ &= \alpha_0 \mathrm{I} + \alpha_1 \dot{\mathcal{L}} + \alpha_3 \mathcal{T} + \alpha_4 \mathcal{T}\mathcal{T} + \\ &\quad + \alpha_5(\dot{\mathcal{L}}\mathcal{T} + \mathcal{T}\dot{\mathcal{L}}) + \alpha_7(\dot{\mathcal{L}}\mathcal{T}\mathcal{T} + \mathcal{T}\mathcal{T}\dot{\mathcal{L}}) \end{aligned} \tag{11.21}$$

where α_i, for $i = 0, 1, 3, 4, 5, 7$, are scalar invariant functions of the three $\mathcal{T}$ invariants only. In particular:

- α_1, α_5 and α_7 must be independent of $\dot{\mathcal{L}}$ and so functions of the three $\mathcal{T}$ invariants only.
- α_0, α_3 and α_4 must be of degree one in $\dot{\mathcal{L}}$ so that

$$\begin{aligned}\alpha_0 &= \beta_o \dot{\mathcal{L}}_{kk} + \beta_1 Q_1 + \beta_2 Q_2 \\ \alpha_3 &= \beta_3 \dot{\mathcal{L}}_{kk} + \beta_4 Q_1 + \beta_5 Q_2 \\ \alpha_4 &= \beta_6 \dot{\mathcal{L}}_{kk} + \beta_7 Q_1 + \beta_8 Q_2\end{aligned}$$

  where β_i, or $i = 1, 2, \ldots, 8$ are response coefficients functions of the three $\mathcal{T}$ invariants.

In fact, the requirement of being homogeneous in time can be satisfied only by eliminating all terms containing second and higher powers of $\dot{\mathcal{L}}$.

We notice that the right-hand-side of Eq. 11.21 is a linear function of $d\mathcal{L}$. This suggests that the constitutive relationship in Eq. 11.21 may be conveniently written in the *incrementally linear form*

$$d\sigma_{ij} = C_{ijkl} d\epsilon_{kl} \tag{11.22}$$

where

$$C_{ijkl} = C_{ijkl}(\sigma_{mn})$$

and, because of the symmetry of σ_{ij} and $d\epsilon_{kl}$,

$$C_{ijkl} = C_{jikl} = C_{jilk} = C_{ijlk}$$

while, because of the (initial) isotropic condition,

$$C_{pqrs}(\sigma'_{ab}) = R_{pi} R_{qj} R_{rk} R_{sl} C_{ijkl}(\sigma_{mn})$$

where

$$\sigma'_{ab} = R_{am} R_{bn} \sigma_{mn}$$

The most general form of C_{ijkl} satisfying the above conditions may be written as

$$\begin{aligned} C_{ijkl} \; = \; & A_1 \delta_{ij} \delta_{kl} + A_2 (\delta_{ik} \delta_{jl} + \delta_{jk} \delta_{il}) + A_3 \sigma_{ij} \delta_{kl} + A_4 \delta_{ij} \sigma_{kl} + \\ & + A_5 (\delta_{ik} \sigma_{jl} + \delta_{il} \sigma_{jk} + \delta_{jk} \sigma_{il} + \delta_{jl} \sigma_{ik}) + \\ & + A_6 \delta_{ij} \sigma_{km} \sigma_{ml} + A_7 \delta_{kl} \sigma_{im} \sigma_{mj} + \\ & + A_8 (\delta_{ik} \sigma_{jm} \sigma_{ml} + \delta_{il} \sigma_{jm} \sigma_{mk} + \delta_{jk} \sigma_{im} \sigma_{ml} + \delta_{jl} \sigma_{im} \sigma_{mk}) + \\ & + A_9 \sigma_{ij} \sigma_{kl} + A_{10} \sigma_{ij} \sigma_{km} \sigma_{ml} + A_{11} \sigma_{im} \sigma_{mj} \sigma_{kl} + \\ & + A_{12} \sigma_{im} \sigma_{mj} \sigma_{kn} \sigma_{nl} \end{aligned} \qquad (11.23)$$

in which the 12 material coefficients A_i depend only on the invariants of the stress tensor σ_{ij}.

Depending on the degree of dependence of C_{ijkl} on the components of the stress tensor, (initially) isotropic hypoelastic models of various order can be formulated. The simplest one is the Hooke's law for linear elastic isotropic material in Eq. 10.66 which can be obtained setting A_1 and A_2 to be constants while $A_i = 0$ for $i = 3, 4, \ldots, 12$.

It is important to notice that the constitutive relationship in Eq. 11.22 presents the following characteristics:

- If a material element is subjected to an infinitesimal strain increment and then return to its original state of strain, the stress components resume their initial values to within higher-order quantities. Thus infinitesimal (incremental) deformations in an hypoelastic material under initial stress are reversible.

- The behavior of a hypoelastic model is in general stress (or strain) path dependent.

- In general, the matrix C_{ijkl} is not symmetric. Consequently, neither the Drucker's stability conditions in Section 10.3 nor the invertibility of the constitutive relationship can be ensured.

An important characteristic exhibited by the hypoelastic behavior described above is the *stress-* or *strain-induced anisotropy*, [18]. In fact, the tangential stiffness matrix in Eq. 11.23 does not generally have the same isotropic form of the stiffness matrix for isotropic materials in Eq. 10.66. Thus, depending on the stress path, an initial isotropy of the material may be destroyed, resulting in a generally anisotropic incremental stiffness. As a result of the induced anisotropy, there is a *coupling (interaction)* between the volumetric response and the deviatoric action. The stress-induced anisotropy and the coupling effects are important features in modeling the behavior of real materials such as concrete and soils for which inelastic *dilatations* or *compactions* are important effects.

11.7 Variable Moduli Models

The hypoelastic Variable Moduli Models assume the material to be isotropic and that the constitutive relationship can be expressed as, *Generalized Hooke's Law*,

$$d\boldsymbol{\sigma} = \mathbf{C}d\boldsymbol{\epsilon} \tag{11.24}$$

where the tangent stiffness matrix

$$\mathbf{C} = \mathbf{C}(\boldsymbol{\sigma})$$

is a symmetric matrix of the type in Eq. 10.68. In this case, the two independent material parameters E and ν are assumed to be function of the stress invariants, that is

$$\begin{aligned} E &= E(p, q, \theta) \\ \nu &= E(p, q, \theta) \end{aligned}$$

This constitutive relationship can be obtained from the general hypoelastic constitutive equation in Eq. 11.22 setting

$$\begin{aligned} A_1 &= A_1(p, q, \theta) \\ A_2 &= A_2(p, q, \theta) \\ A_3 &= A_4 = \ldots = A_{12} = 0 \end{aligned}$$

from which we can prove that

$$\begin{aligned} E &= \frac{A_2(2A_2 + 3A_1)}{A_1 + A_2} \\ \nu &= \frac{A_1}{2(A_1 + A_2)} \end{aligned}$$

It can be proved that Eq. 11.24 may be alternatively expressed as, Section 10.9,

$$d\boldsymbol{\sigma} = B\widehat{\mathbf{m}}d\epsilon_v + 2Gd\mathbf{e} \tag{11.25}$$

where the (B, G) are two independent scalar functions of (p, q) related to (E, ν) as reported in Table 10.1. Finally,

$$\begin{aligned} d\mathbf{s} &= 2Gd\mathbf{e} \\ dp &= Bd\epsilon_v \\ dq &= 3Gd\epsilon_s \end{aligned}$$

Interesting to notice that in triaxial stress conditions

$$\begin{aligned} d\epsilon_a &= \frac{1}{E}[d\sigma_a - 2\nu d\sigma_o] \\ d\epsilon_o &= \frac{1}{E}[d\sigma_o - \nu(d\sigma_a + d\sigma_o)] \end{aligned}$$

Then, the laws of variation of the parameters (E, ν) or alternatively of (B, G) can be established from triaxial test as follows, Section 11.2:

- TC test with $d\sigma_o = 0$. From the experimental diagram σ_a vs. ϵ_a we can compute
$$E = \frac{d\sigma_a}{d\epsilon_a}$$
while from the experimental diagram ϵ_o vs. ϵ_a we can compute
$$\nu = -\frac{d\epsilon_o}{d\epsilon_a}$$
- IC or KC or TC tests. From the experimental diagram p vs. ϵ_v we can compute
$$B = \frac{dp}{d\epsilon_v} = \frac{1}{3}\left(\frac{d\sigma_a + 2d\sigma_o}{d\epsilon_a + 2d\epsilon_o}\right)$$
while from the experimental diagram q vs. ϵ_s we can compute
$$G = \frac{1}{3}\frac{dq}{d\epsilon_s} = \frac{1}{2}\frac{|d\sigma_a - d\sigma_o|}{|d\epsilon_a - d\epsilon_o|}$$

11.8 Hyperbolic Models

The name *hyperbolic models* usually denotes a class of incrementally isotropic hypoelastic models, i.e. variable moduli models, in which stress-strain relationships are approximated by hyperbolic equations. This Section presents some of the best known hyperbolic relationships applied for the best fitting of experimental σ_a vs. ϵ_a curves obtained from uniaxial tests, that is TC tests with $\sigma_a = \sigma$ and $\sigma_o = 0$.

The simplest hyperbolic stress/strain relationship has been proposed by Kondner, [55], in the form of

$$\sigma = \frac{\epsilon}{a + b\epsilon} \tag{11.26}$$

This equation represents a curve, Fig. 11.9a, with an initial slope $E_o = 1/a$ and an asymptotic value $\sigma_s = 1/b$. In fact, the inversion of Eq. 11.26 gives

$$\epsilon = \frac{a\sigma}{1 - b\sigma} \tag{11.27}$$

while the differentiation of Eq. 11.26 yields to

$$E = \frac{d\sigma}{d\epsilon} = \frac{a}{(a + b\epsilon)^2} \tag{11.28}$$

which, substituting the expression of ϵ in Eq. 11.27, takes on the form

$$E = \frac{1}{a}(1 - b\sigma)^2 = E_o\left(1 - \frac{\sigma}{\overline{\sigma}}\right)^2 \tag{11.29}$$

Thus, from Eq. 11.26, for $\epsilon \to \infty$

$$\sigma = \frac{1}{b} = \overline{\sigma}$$

while, from Eq. 11.29,

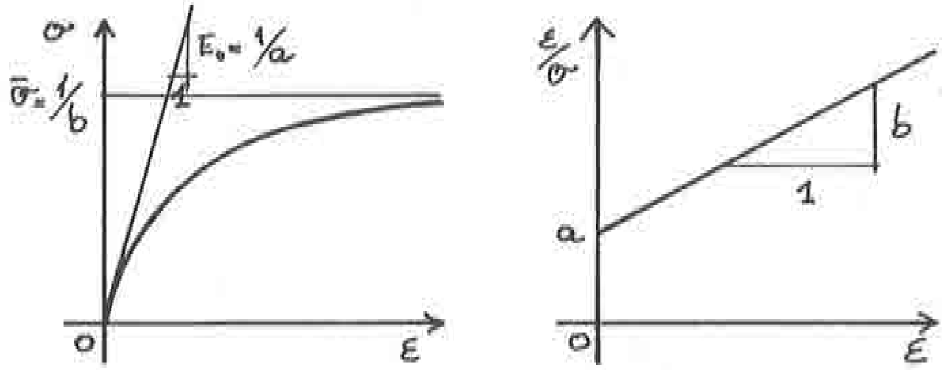
$$E = \begin{cases} \frac{1}{a} = E_o; & \text{for } \epsilon = 0. \\ 0; & \text{for } \epsilon \to \infty. \end{cases}$$

A modified stress/strain relationship has been proposed by Hansen, [49], in the form of

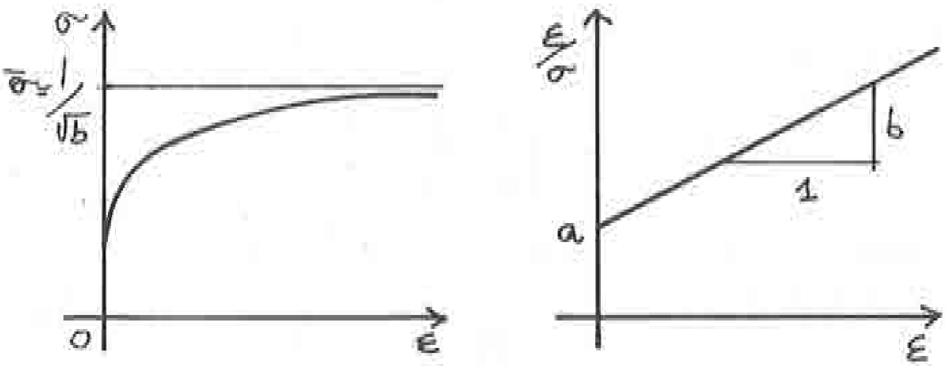
$$\sigma = \left(\frac{\epsilon}{a + b\epsilon}\right)^{\frac{1}{2}} \tag{11.30}$$

This relationship represents a curve, Fig. 11.9b, with an initial vertical slope and an asymptotic value at $\overline{\sigma} = 1/\sqrt{b}$. In fact, the inversion of Eq. 11.30 gives

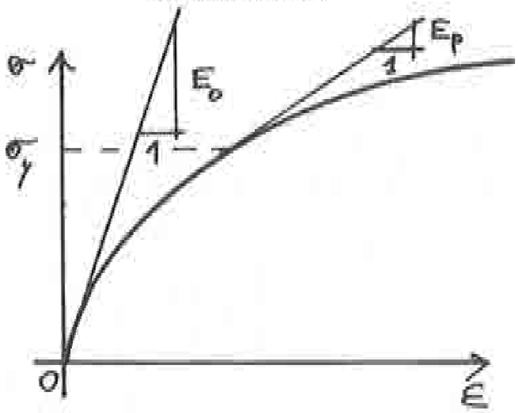
$$\epsilon = \frac{a\sigma^2}{1 - b\sigma^2} \tag{11.31}$$



Kondner's model



Hansen's model



Ramberg-Osgood model

Figure 11.9: Hyperbolic stress-strain curve

while the differentiation of Eq. 11.30 yields to

$$E = \frac{d\sigma}{d\epsilon} = \frac{a}{2\left[\epsilon\,(a + b\epsilon)^3\right]^{1/2}} \tag{11.32}$$

which, substituting the expression of ϵ in Eq. 11.31, takes on the form

$$E = \frac{1}{2a\sigma}\left(1 - b\sigma^2\right)^2 = \frac{1}{2a\sigma}\left(1 - \frac{\sigma^2}{\overline{\sigma}^2}\right)^2 \tag{11.33}$$

Thus, from Eq. 11.30, for $\epsilon \to \infty$

$$\sigma = \frac{1}{\sqrt{b}} = \overline{\sigma}$$

while, from Eq. 11.33,

$$E = \begin{cases} \infty; & \text{for } \epsilon = 0 \ . \\ 0; & \text{for } \epsilon \to \infty \ . \end{cases}$$

Ramberg and Osgood, [76], proposed a sort of generalized hyperbolic model in the form of

$$\sigma = \frac{E_r \epsilon}{\left[1 + \left(\dfrac{E_r \epsilon}{\sigma_y}\right)^m\right]^{1/m}} + E_p \epsilon \tag{11.34}$$

where, Fig. 11.9c,

$$E_r = E_o - E_p$$

and E_o is the initial slope, σ_y a yield stress, E_p the slope at σ_y, while m defines the order of the curve. The differentiation of Eq. 11.34 gives

$$E_t = \frac{d\sigma}{d\epsilon} = \frac{E_r}{\left[1 + \left(\dfrac{E_r \epsilon}{\sigma_y}\right)^m\right]^{\frac{m+1}{m}}} + E_p \tag{11.35}$$

Notice that for $\sigma_y \equiv \overline{\sigma}$, $E_p = 0$ and $m = 1$, Eq. 11.34 reduces to Kondner's hyperbolic relationship in Eq. 11.26.

For an optimal approximation of an experimental curve, the following best fitting procedure has been proposed, [27]:

- measure at $\epsilon = 0$ the value of the slope E_o;
- choose a convenient value of E_p;
- select two points (ϵ_1, σ_1) and (ϵ_2, σ_2) on the experimental curve and then calculate
$$E_1 = \frac{\sigma_1}{\epsilon_1}; \quad E_2 = \frac{\sigma_2}{\epsilon_2}; \quad R = \frac{\epsilon_2}{\epsilon_1}$$
- calculate the value of m by solving the equation
$$A^m - \frac{B^m}{R^m} + \left(\frac{1}{R^m} - 1\right) = 0$$
where
$$A = \frac{E_r}{E_1 - E_p}; \quad B = \frac{E_r}{E_2 - E_p}$$
- evaluate the yield stress as
$$\sigma_y = \frac{E_r}{(A^m - 1)^{1/m}}\epsilon_1$$

11.9 A Complete Variable Moduli Model

Variable Moduli Models have been widely used in geotechnical engineering for describing the behavior of soils. Among the various proposals, the most popular ones are perhaps those which assume that, under triaxial loading conditions, the σ_a vs. ϵ_a and the ϵ_v vs. ϵ_a curves can be represented by hyperbolic curves.

Based on many available experimental and theoretical suggestions, we describe in this Section a possible complete hypoelastic model for isotropic soils. The main hypothesis of the model are:

- the material response in a triaxial test is of the type shown in Fig. 11.10
- failure conditions are represented by a Mohr-Coulomb surface, so that at failure, Eq. 11.9,
$$(\tilde{\sigma}_1 - \tilde{\sigma}_3) + (\tilde{\sigma}_1 + \tilde{\sigma}_3)\sin\varphi = 2c\cos\varphi \tag{11.36}$$
or, equivalently,
$$\frac{(\tilde{\sigma}_1 - \tilde{\sigma}_3)(1 - \sin\varphi)}{2(c\cos\varphi - \tilde{\sigma}_1\sin\varphi)} = 1 \tag{11.37}$$
where

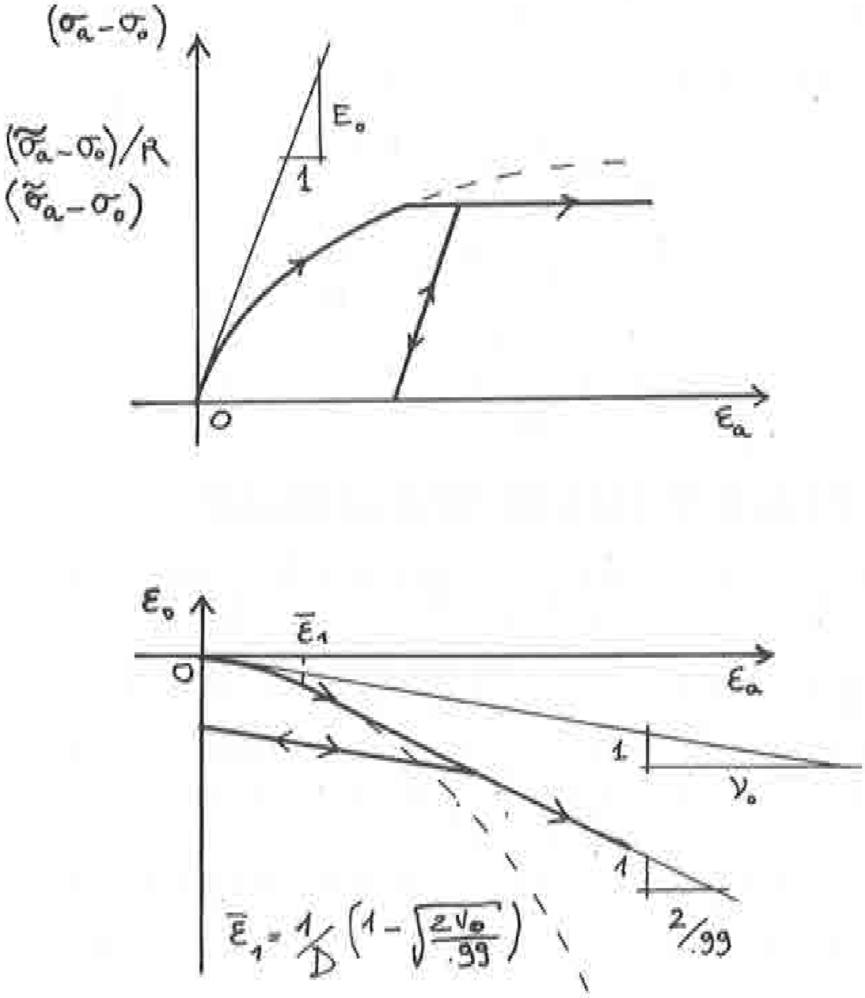


Figure 11.10: Material response under triaxial conditions

- $\tilde{\sigma}_1$ and $\tilde{\sigma}_3$ are the maximum and the minimum principal stress values at failure, respectively.
- c and φ are material coefficients.

- The material response is controlled by a *stress level parameter* defined as
$$s = \frac{(\sigma_1 - \sigma_3)(1 - \sin\varphi)}{2(c\cos\varphi - \sigma_1 \sin\varphi)} \tag{11.38}$$
where σ_1 and σ_3 are the maximum and the minimum current principal stress values, respectively. Comparing Eq. 11.37 with Eq. 11.38, we immediately realize that this parameter is a sort of a measure of the current distance of the current stress σ from the failure surface and its range of variation is bounded by
$$s = \begin{cases} 0; & \text{for } \sigma_1 = \sigma_3. \\ 1; & \text{for } \sigma_1 = \tilde{\sigma}_1 \text{ and } \sigma_3 = \tilde{\sigma}_3. \end{cases}$$

- the loading criterion for a given stress load $d\sigma$ is given by
$$\begin{aligned} s_n &= k \begin{cases} s_{n+1} \geq s_n; & \text{loading.} \\ s_{n+1} < s_n; & \text{unloading.} \end{cases} \\ s_n &< k \begin{cases} s_{n+1} \geq s_n; & \text{reloading.} \\ s_{n+1} < s_n; & \text{unloading.} \end{cases} \end{aligned}$$
where
$$\begin{aligned} s_n &= s_n(\sigma) \\ s_{n+1} &= s_n(\sigma + d\sigma) \end{aligned}$$
where k is the maximum stress level parameter ever measured.

- Under triaxial loading conditions, the σ_a vs. ϵ_a and the ϵ_o vs. ϵ_a curves can be represented by hyperbolic curves.

Under the above hypothesis, the constitutive relationship is given by
$$d\sigma = \mathbf{C}d\epsilon \tag{11.39}$$
where $\mathbf{C}$ is a symmetric matrix of the type in Eq. 10.68 function of the two independent material parameters E and ν. For loading conditions
$$E = \begin{cases} E_o[1 - Rs]^2; & \text{for } 0 \leq s < 1. \\ \\ 0.01p_a; & \text{for } s = 1. \end{cases} \tag{11.40}$$

$$\nu = \begin{cases} \dfrac{\nu_o}{(1-A)^2}; & \text{for } |1-A| > \sqrt{\nu_o/0.495} \\ 0.495; & \text{for } |1-A| \leq \sqrt{\nu_o/0.495} \end{cases} \qquad (11.41)$$

where

$$E_o = K p_a \left(\frac{|\sigma_1|}{p_a}\right)^n > 0$$

$$\nu_o = G - m \log\left(\frac{|\sigma_1|}{p_a}\right) < \frac{1}{2}$$

$$A = \frac{D}{E}(\sigma_1 - \sigma_3)(1 - Rs)$$

and p_a is the atmospheric pressure. The coefficients to be determined experimentally from triaxial tests are

- K, initial slope in the $(\sigma_a - \sigma_o)/p_a$ vs. ϵ_a diagram for $\sigma_o = p_a$, Fig. 11.10a;
- R, factor controlling the asymptote in the $(\sigma_a - \sigma_o)/p_a$ vs. ϵ_a diagram for $\sigma_o = p_a$, Fig. 11.10a;
- n, exponent controlling the rate of change of E_o with σ_o;
- G, initial slope in the ϵ_o vs. ϵ_a diagram for $\sigma_o = p_a$, Fig. 11.10b;
- D, factor controlling the asymptote in the ϵ_o vs. ϵ_a diagram for $\sigma_o = p_a$, Fig. 11.10b;
- m, exponent controlling the rate of change of ν_o with σ_o.

On the other hand, unloading and reloading conditions

$$E = K_u p_a \left(\frac{|\sigma_1|}{p_a}\right)^n > 0 \qquad (11.42)$$

$$\nu = \nu_o \qquad (11.43)$$

where K_u is a material constant to be determined experimentally.

The original theoretical and experimental sources from which this model has been developed are reported in the following Sections. In geotechnical engineering, it is customary to assume compression stress and strain components as positive. In order to be consistent with the works which will be cited, we report the mathematical developments adopting the same sign convention, that is:

- Maximum compressive stress $\sigma_a = -\sigma_3$ and strain $\epsilon_a = -\epsilon_3$
- Minimum compressive stress $\sigma_o = -\sigma_1$ and strain $\epsilon_o = -\epsilon_1$

11.9.1 Law of variation of the Young modulus

According to the Kondner formula in Eq. 11.26, uniaxial experimental data may be approximated by an hyperbolic curve of the type

$$\sigma = \frac{\epsilon}{a + b\epsilon} \tag{11.44}$$

and the relative tangent results to be equal to

$$E = \frac{1}{a}(1 - b\sigma)^2 = E_o\left(1 - \frac{\sigma}{\overline{\sigma}}\right)^2 \tag{11.45}$$

where $E_o = 1/a$ and $\overline{\sigma} = 1/b$ are the initial slope and the asymptote of the curve, respectively. The extension of this formula to the case of triaxial test can be obtained by setting

$$\sigma = \sigma_a - \sigma_o \tag{11.46}$$

$$\epsilon = \epsilon_a \tag{11.47}$$

and

$$\overline{\sigma} = \overline{\sigma}_a - \sigma_o = \frac{\tilde{\sigma}_a - \sigma_o}{R} \tag{11.48}$$

where $\tilde{\sigma}_a$ is the measured ultimate axial stress and R is an opportune factor. Substituting these positions in Eq. 11.45 we obtain

$$E = E_o\left[1 - R\frac{\sigma_a - \sigma_o}{\tilde{\sigma}_a - \sigma_o}\right]^2 \tag{11.49}$$

Duncan and Chang, [30], recognized that in terms of the Mohr-Coulomb failure criteria, Eq. 11.37, the limit load in triaxial conditions can be expressed as

$$\tilde{\sigma}_a - \sigma_o = \frac{2(c\cos\varphi + \sigma_o \sin\varphi)}{1 - \sin\varphi} \tag{11.50}$$

Substituting Eq. 11.50 into Eq. 11.49 and restoring the correct sign convention we obtain the tangent Young modulus expression in Eq. 11.40.

Finally, the law of variation of E_o with the confining pressure σ_o is in accordance with that proposed by Jambu, [58].

11.9.2 Law of variation of the Poisson Ratio

Kulhawy et al., [57], concluded from the examination of many triaxial tests on soils that, within the usual range of variation of the compressive strain, the experimental curve ϵ_a vs. ϵ_o may be approximated by a Kondner type hyperbolic function, that is

$$\epsilon_a = -\frac{\epsilon_o}{\nu_o - D\epsilon_o} \tag{11.51}$$

Accordingly, the tangent Poisson ratio results to be given by

$$\nu = -\frac{d\epsilon_o}{d\epsilon_a} = \frac{\nu_o}{(1 - D\epsilon_a)^2} \tag{11.52}$$

where, always according to Kulhawy, ν_o varies with the confining pressure as reported in Section 11.9 We recall that the inverse of Eq. 11.44 is given by, Eq. 11.27,

$$\epsilon = \frac{a\sigma}{1 - b\sigma} = \frac{1}{E_o}\frac{\sigma}{\left[1 - \dfrac{\sigma}{\overline{\sigma}}\right]}$$

from which, setting the triaxial position in Eqs. 11.46-11.48 we obtain,

$$\epsilon_a = \frac{1}{E_o}\frac{\sigma_a - \sigma_o}{\left[1 - R\dfrac{\sigma_a - \sigma_o}{\overline{\sigma}_a - \sigma_o}\right]} = \frac{\sigma_a - \sigma_o}{E}\left[1 - R\frac{\sigma_a - \sigma_o}{\overline{\sigma}_a - \sigma_o}\right] \tag{11.53}$$

Substituting Eqs. 11.50 and 11.53 into Eq. 11.52 and restoring the correct sign convention we obtain the tangent Poisson ration expression in Eq. 11.41.

11.9.3 Law of variation of the volumetric strain

It is of interest to notice that, the inversion of Eq. 11.51 gives

$$\epsilon_o = -\frac{\nu_o\epsilon_a}{1 - D\epsilon_a}$$

Accordingly, the volumetric strain results to be equal to

$$\epsilon_v = \epsilon_a + 2\epsilon_o = \epsilon_a\left(1 - \frac{2\nu_o}{1 - D\epsilon_a}\right) \tag{11.54}$$

and

$$\frac{d\epsilon_v}{d\epsilon_a} = 1 - \frac{2\nu_o}{(1 - D\epsilon_a)^2} \tag{11.55}$$

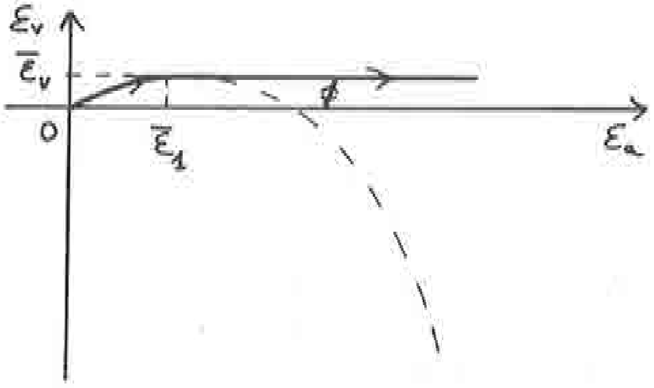


Figure 11.11: Volumetric strain prediction

Figure 11.11 reports the type of ϵ_v vs. ϵ_a curve represented by Eq. 11.54. It is immediate to verify from Eq. 11.55 that the maximum of this curve is at

$$\epsilon_a = \frac{1 - \sqrt{2\nu_o}}{D}$$

which, substituted in Eq. 11.54, gives

$$\epsilon_v = \frac{(1 - \sqrt{2\nu_o})^2}{D}$$

11.9.4 Yield surface

According to Eq. 11.14, the Mohr-Coulomb failure criteria in Eq. 11.36 can be alternatively written as

$$Q = q + Mp - N = 0 \qquad (11.56)$$

where

$$\begin{aligned} M &= M(c, \varphi, \theta) \\ N &= N(c, \varphi, \theta) \end{aligned}$$

Interestingly, the loading criteria in Section 11.9 implicitly assumes the existence of a Mohr-Coulomb yield surface of equation

$$G = q + M'p - N' = 0 \tag{11.57}$$

where

$$\begin{aligned} M' &= M(c', \varphi', \theta) \\ N' &= N(c', \varphi', \theta) \end{aligned}$$

and

$$\varphi' = \arcsin\left[\frac{k \sin\varphi}{1 + (k-1)\sin\varphi}\right] \tag{11.58}$$

$$c' = c\frac{\tan\varphi'}{\tan\varphi} \tag{11.59}$$

$$k = \frac{(\sigma_1 - \sigma_3)(1 - \sin\varphi)}{2(c\cos\varphi - \sigma_1 \sin\varphi)} = \frac{\sigma_1 - \sigma_3}{\sigma_1 - \tilde{\sigma}_3} \tag{11.60}$$

being k the highest stress level ever measured.

In fact, we can see from the Mohr circle drawn in Fig. 11.12, that the

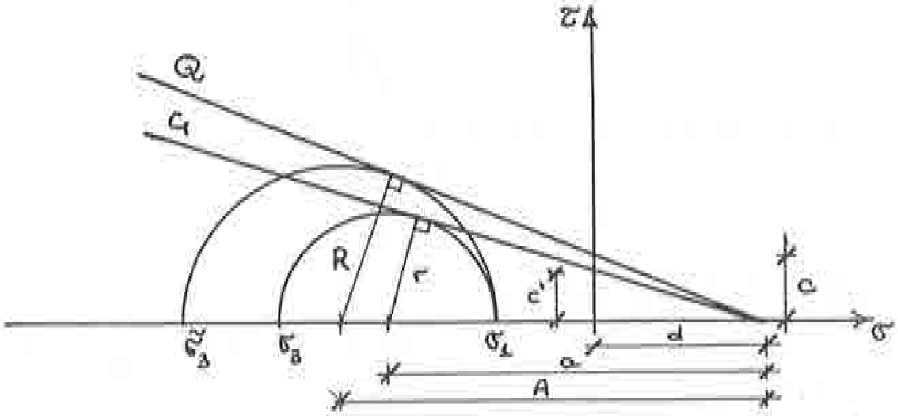


Figure 11.12: Failure and yield surface

stress level k can be expressed as

$$k = \frac{\sigma_1 - \sigma_3}{\sigma_1 - \tilde{\sigma}_3} = \frac{r}{R}$$

from which we can establish that

$$\sin\varphi = \frac{R}{A} = \frac{r}{k(a-r)+r} \tag{11.61}$$

since

$$\begin{aligned} R &= r/k \\ A &= (a-r)+R \end{aligned}$$

Solving Eq. 11.61 in terms of r we obtain

$$r = \frac{ak\sin\varphi}{1+(k-1)\sin\varphi}$$

from which it follows that

$$\sin\varphi' = \frac{r}{a} = \frac{k\sin\varphi}{1+(k-1)\sin\varphi} \tag{11.62}$$

proving therefore Eq. 11.58. The relationship in Eq. 11.59 follows from the fact that

$$d = \frac{c'}{\tan\varphi'} = \frac{c}{\tan\varphi} \tag{11.63}$$

Chapter 12

The Incremental Theory of Plasticity

12.1 Introduction

At the present, the most popular constitutive theory to account for irreversible deformation seems to be the so-called *Incremental Theory of Plasticity* whose first formulation may be traced back to Melan, [67]. The main assumptions of this theory may be summarized as follows, Section 12.2:

- small deformations;
- instantaneous material response;
- The variation of the total strain can be expressed as

$$\delta\epsilon = \delta\epsilon^{(e)} + \delta\epsilon^{(p)}$$

  where $\delta\epsilon^{(e)}$ is the elastic (recoverable) part while $\delta\epsilon^{(p)}$ is the plastic (non recoverable) part.

The elastic component $\delta\epsilon^{(e)}$ is assumed to be always active and fully recoverable. The plastic deformation $\delta\epsilon^{(p)}$, instead, is assumed to occur only during loading processes and above a certain threshold value known as *elastic limit* or *yield stress*, determined on a *yield surface*, Section 12.3.

After initial yielding, the level at which further plastic deformation occurs, is usually assumed to depend on the degree of some hardening parameters $\mathbf{k}$, Section 12.4. The yield surface may then be mathematically represented as $F(\boldsymbol{\sigma}, \mathbf{k})$. Several different physical meanings for $\mathbf{k}$ have been suggested and some of them are presented later. In any case, it represents mathematically the controlling parameter for the evolution of F in the stress space.

This theory of plasticity also assumes the existence of a *plastic function* $G(\boldsymbol{\sigma}, \mathbf{k})$ whose gradient in the stress space determines the direction of the plastic strain increment $\delta\boldsymbol{\epsilon}^{(p)}$. The magnitude of $\delta\boldsymbol{\epsilon}^{(p)}$ is controlled by a scalar parameter $\delta\lambda$, known as the *plastic multiplier*, Section 12.5.

From this general theoretical frame work, it is possible to establish the necessary criteria for predicting the type of material response to be expected from a stress or a strain increment as well as the general form for the relative constitutive equation, Sections 12.6 12.7 and 12.8. However, the theory does not explicitly give the forms of the yield and potential function nor does it identify the hardening parameters. Some restrictions on the possible form of F and G are given for the so-called *stable materials*, Section 12.9.

In the last ten years, an impressive number of alternative proposals for G, F and $\mathbf{k}$ have been formulated. In this Chapter, however, we only report the simplest ones concerning isotropic materials. Finally, we report the extension of the Incremental Theory to account for time dependence of the plastic response, Section 12.17.

12.2 Elasto-Plastic Strain Definition

According to the Incremental Theory of Plasticity, an infinitesimal stress increment $\delta\boldsymbol{\sigma}$ causes an infinitesimal strain increment $\delta\boldsymbol{\epsilon}$, which is the sum of an elastic (fully recoverable) component $\delta\boldsymbol{\epsilon}^{(e)}$ and a plastic (irreversible) component $\delta\boldsymbol{\epsilon}^{(p)}$, namely

$$\delta\boldsymbol{\epsilon} = \delta\boldsymbol{\epsilon}^{(e)} + \delta\boldsymbol{\epsilon}^{(p)} \tag{12.1}$$

The elastic strain may be calculated according to an incremental constitutive relationship of the type

$$\delta\boldsymbol{\epsilon}^{(e)} = \left(\mathbf{C}^{(e)}\right)^{-1} \delta\boldsymbol{\sigma} \tag{12.2}$$

where $\mathbf{C}^{(e)}$, the elastic stiffness matrix, may be function of the current $\boldsymbol{\sigma}$. The plastic strain instead is defined as

$$\delta\boldsymbol{\epsilon}^{(p)} = \delta\lambda\mathbf{b} \tag{12.3}$$

where

$$\mathbf{b} = \frac{\partial G}{\partial \boldsymbol{\sigma}} = \left\{ \frac{\partial G}{\partial \sigma_{11}}, \frac{\partial G}{\partial \sigma_{22}}, \frac{\partial G}{\partial \sigma_{33}}, \frac{\partial G}{\partial \sigma_{12}}, \frac{\partial G}{\partial \sigma_{13}}, \frac{\partial G}{\partial \sigma_{21}}, \frac{\partial G}{\partial \sigma_{23}}, \frac{\partial G}{\partial \sigma_{31}}, \frac{\partial G}{\partial \sigma_{32}} \right\}^T$$

and

$$G = G(\boldsymbol{\sigma}, \mathbf{k})$$

is a scalar function known as the *Potential Function*. The vector $\mathbf{k}$ collects n *hardening parameters* k_i functions of the stress history, Section 12.4. Finally, $\delta\lambda$, known as the *plastic multiplier*, is a non negative scalar, Section 12.5.

Notice that, since the stress $\boldsymbol{\sigma}$ can be expressed as,

$$\boldsymbol{\sigma} = \mathbf{s} + \widehat{\mathbf{m}} p$$

the potential function may be written in the form

$$G = G(p, \mathbf{s}, \mathbf{k})$$

According to the plastic strain definition in Eq. 12.3, it is possible to prove that the plastic deviatoric strain and plastic strain invariants result can be calculated as, Appendix A,

$$\delta \mathbf{e}^{(p)} = \delta\lambda \nabla_s G \tag{12.4}$$

$$\delta \epsilon_v^{(p)} = \delta\lambda \frac{\partial G}{\partial p} \tag{12.5}$$

$$\delta \epsilon_s^{(p)} = \delta\lambda \sqrt{\frac{2}{3}} \|\nabla_s G\| \tag{12.6}$$

where

$$\nabla_s G = \left\{ \frac{\partial G}{\partial s_{kl}} \frac{\partial s_{kl}}{\partial \sigma_{ij}} \right\}^T = \left\{ \frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3} \frac{\partial G}{\partial s_{kk}} \right\}^T$$

$$\|\nabla_s G\| = (\nabla_s G^T \nabla_s G)^{1/2} = \left[\frac{\partial G}{\partial s_{ij}} \frac{\partial G}{\partial s_{ij}} - \frac{1}{3} \left(\frac{\partial G}{\partial s_{kk}} \right)^2 \right]^{1/2}$$

12.3 The Yield Function

The Incremental Theory of Plasticity assumes that the type of material response is controlled by a so called *Yield Surface* of equation

$$F = F(\boldsymbol{\sigma}, \mathbf{k}) \tag{12.7}$$

where k, Section 12.4, collects the same hardening parameters of the potential function G. Sometimes F is assumed to coincide with G, in which case the constitutive model is said to obey an *associative flow rule*.

The yield surface is assumed to bound always the location of each current stress point, so that, if $(\sigma, \mathbf{k})$ represents the current material state, the only admissible alternative conditions are

$$\begin{cases} F(\sigma, \mathbf{k}) = 0; & \sigma \text{ lies on } F \\ \\ F(\sigma, \mathbf{k}) < 0; & \sigma \text{ lies inside } F \end{cases} \tag{12.8}$$

while

$$F(\sigma, \mathbf{k}) > 0$$

is not admissible. Consider a loading process of an infinitesimal increment $(\delta\sigma, \delta\mathbf{k})$, starting from a current configuration $(\sigma, \mathbf{k})$. By definition:

- Elasto-plastic deformations occur if, and only if, during the loading process the current stress point always lies on the yield surface, that is

$$F(\sigma, \mathbf{k}) = 0 \qquad \text{and} \qquad F(\sigma + \delta\sigma, \mathbf{k} + \delta\mathbf{k}) = 0 \tag{12.9}$$

  Thus, only in these cases, $\delta\lambda \geq 0$.

- In any other case, that is

$$F(\sigma, \mathbf{k}) < 0 \qquad \text{and} \qquad F(\sigma + \delta\sigma, \mathbf{k} + \delta\mathbf{k}) \leq 0 \tag{12.10}$$

$$F(\sigma, \mathbf{k}) = 0 \qquad \text{and} \qquad F(\sigma + \delta\sigma, \mathbf{k} + \delta\mathbf{k}) < 0 \tag{12.11}$$

  the material response can be only purely elastic and, consequently, $\delta\lambda = 0$.

It is immediate to verify that the rule in Eq. 12.9 for establishing elasto-plastic conditions is *equivalent* to requiring

$$\begin{cases} F(\sigma, \mathbf{k}) & = & 0 \\ \\ \delta F(\sigma, \mathbf{k}) & = & \mathbf{a}^T \delta\sigma + \dfrac{\partial F}{\partial k_i} \delta k_i = 0 \end{cases} \tag{12.12}$$

where $i = 1, 2, \ldots, n$ and

$$\mathbf{a} = \frac{\partial F}{\partial \sigma} = \left\{ \frac{\partial F}{\partial \sigma_{11}}, \frac{\partial F}{\partial \sigma_{22}}, \frac{\partial F}{\partial \sigma_{33}}, \frac{\partial F}{\partial \sigma_{12}}, \frac{\partial F}{\partial \sigma_{13}}, \frac{\partial F}{\partial \sigma_{23}}, \frac{\partial F}{\partial \sigma_{21}}, \frac{\partial F}{\partial \sigma_{31}}, \frac{\partial F}{\partial \sigma_{32}} \right\}^T$$

The second equation, known as the *Consistency Equation*, can be derived expressing $F(\sigma + \delta\sigma, \mathbf{k} + \delta\mathbf{k})$ into the *exact* Taylor series

$$F(\sigma + \delta\sigma, \mathbf{k} + \delta\mathbf{k}) = F(\sigma, \mathbf{k}) + \delta F(\sigma, \mathbf{k})$$

12.4 The Hardening Parameters

The hardening parameters collected in **k** have to represent the past plastic history of the material point. In general, therefore, they must depend on the total plastic deformation. However, the incremental theory of plasticity does not necessarily need an explicit functional relationship for **k**; in fact, we will see that it is sufficient to establish incremental relationships of the type

$$\delta k_i = \frac{\partial k_i}{\partial h_j} \delta h_j \tag{12.13}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, where:

- the partial variations $\partial k_i / \partial h_j$ and the initial values of k_i have to be known quantities;
- the infinitesimal increments δh_j of the *internal* variables h_j *must* be expressible as

$$\delta h_j = f_j(\delta \epsilon^{(p)}) = c_j \delta \lambda \tag{12.14}$$

  for $j = 1, 2, \ldots, m$, where c_j are m scalar values. The notation $f_j(\delta \epsilon^{(p)})$ indicates any scalar function of the nine scalar variables $\delta \epsilon^p_{rs}$ $(r, s = 1, 2, 3)$.

From Eqs. 12.13 and 12.14 we obtain

$$\delta k_i = \frac{\partial k_i}{\partial h_j} \delta h_j = \frac{\partial k_i}{\partial h_j} c_j \delta \lambda \tag{12.15}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Notice that this incremental relationship assures that during elastic processes, where by definition $\delta \lambda = 0$, the hardening parameters do not vary; consequently, F and G remain fixed in the stress space.

Many formulations of elasto-plastic constitutive models assign directly the hardening parameters **k** without introducing the internal variables h_j. In our notation, this is equivalent to assume

$$h_j \equiv k_i$$

for $j = i$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, with $m = n$. This implies that

$$\frac{\partial k_i}{\partial h_j} = \delta_{ij}$$

The most common choices for k_i are reported in the next Sections.

12.5 The Plastic Multiplier

By definition, the value of the plastic multiplier $\delta\lambda$ is never negative and in a purely elastic process

$$\delta\lambda = 0$$

In an elasto-plastic process, instead, its value may be calculated as follows:

- if $\delta\boldsymbol{\sigma}$ is assigned

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T\delta\boldsymbol{\sigma}}{A}; & \text{if } A \neq 0. \\ \text{indeterminate;} & \text{if } A = 0. \end{cases} \tag{12.16}$$

- if $\delta\boldsymbol{\epsilon}$ is assigned

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)}}{A + \mathbf{a}^T\mathbf{c}^{(\sigma)}}; & \text{if } A \neq -\mathbf{a}^T\mathbf{c}^{(\sigma)}. \\ \text{indeterminate;} & \text{if } A = -\mathbf{a}^T\mathbf{c}^{(\sigma)}. \end{cases} \tag{12.17}$$

where

$$\delta\boldsymbol{\sigma}^{(e)} = \mathbf{C}^{(e)}\delta\boldsymbol{\epsilon} \tag{12.18}$$

$$\mathbf{c}^{(\sigma)} = \mathbf{C}^{(e)}\mathbf{b} \tag{12.19}$$

and the scalar quantity A, known as the *Plastic Modulus*, is defined as

$$A = -\frac{1}{\delta\lambda}\frac{\partial F}{\partial k_i}\frac{\partial k_i}{\partial h_j}\delta h_j = -\frac{\partial F}{\partial k_i}\frac{\partial k_i}{\partial h_j}c_j \tag{12.20}$$

for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. In general, the value of the plastic modulus A, calculated as above, may result negative, null or positive.

In fact, according to the position in Eq. 12.20, the consistency equation in Eq. 12.12 takes on the form

$$\mathbf{a}^T\delta\boldsymbol{\sigma} - A\delta\lambda = 0 \tag{12.21}$$

from which we can verify the statement in Eq. 12.16. On the other hand, according to Eqs 12.2, 12.1 and 12.3, we can express

$$\delta\boldsymbol{\sigma} = \mathbf{C}^{(e)}\delta\boldsymbol{\epsilon}^{(e)} = \mathbf{C}^{(e)}\left(\delta\boldsymbol{\epsilon} - \delta\boldsymbol{\epsilon}^{(p)}\right) = \mathbf{C}^{(e)}\delta\boldsymbol{\epsilon} - \delta\lambda\mathbf{C}^{(e)}\mathbf{b}$$

that is, Eqs. 12.18 and 12.19,

$$\delta\boldsymbol{\sigma} = \delta\boldsymbol{\sigma}^{(e)} - \delta\lambda\mathbf{c}^{(\sigma)}$$

Substituting this expression of $\delta\boldsymbol{\sigma}$ in Eq. 12.21 we have the following alternative form for the consistency equation

$$\mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)} - \left(\mathbf{a}^T\mathbf{c}^{(\sigma)} + A\right)\delta\lambda = 0 \tag{12.22}$$

from which we can verify the statement in Eq. 12.17.

12.6 Stress Based Elasto-Plastic Criterion

By definition, Section 12.3, if

$$F(\boldsymbol{\sigma}, \mathbf{k}) < 0 \tag{12.23}$$

the material response is always elastic, regardless of the applied stress increment $\delta\boldsymbol{\sigma}$. Elasto-plastic response can occur if, and only if, Eqs. 12.12 and 12.21,

$$\begin{cases} F(\boldsymbol{\sigma}, \mathbf{k}) = 0 \\ \delta F = \mathbf{a}^T\delta\boldsymbol{\sigma} - A\delta\lambda = 0 \end{cases} \tag{12.24}$$

Accordingly, we can establish the following *stress based criterion* which predicts the material response for any given stress increment $\delta\boldsymbol{\sigma}$ applied on any material state $(\boldsymbol{\sigma}, \mathbf{k})$, [23]:

1. Elasto-plastic response occurs if

   $$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

   and

   $$\begin{aligned} A > 0 \quad &; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} \geq 0 \\ A = 0 \quad &; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} = 0 \\ A < 0 \quad &; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} = 0 \end{aligned}$$

   The case in which

   $$A > 0 \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} > 0$$

   takes the name of *hardening* since, being the stress increment $\delta\boldsymbol{\sigma}$ directed outside F, it denotes a subsequent expansion of F.

2. Elastic response occurs if

$$F(\sigma, \mathbf{k}) < 0$$

   or

$$F(\sigma, \mathbf{k}) = 0$$

   and

$$A \geq 0 \quad ; \quad \mathbf{a}^T \delta\sigma < 0$$

3. Either elastic or elasto-plastic response may occur if

$$F(\sigma, \mathbf{k}) = 0$$

   and

$$A < 0 \quad ; \quad \mathbf{a}^T \delta\sigma < 0$$

   This is the only ambiguous situation which the classical theory of plasticity does not solve by itself. If plasticity occurs, this case takes the name of *softening* since, being the stress increment $\delta\sigma$ directed inside F, it denotes a subsequent contraction of F.

4. Stress increments by which

$$F(\sigma, \mathbf{k}) = 0$$

   and

$$A \leq 0 \quad ; \quad \mathbf{a}^T \delta\sigma > 0$$

   are not admissible; that is, according to the theory, the material cannot sustain such type of stress increment.

In fact, the elastic case for $F(\sigma, \mathbf{k}) < 0$ is trivially verified. Then, for $F(\sigma, \mathbf{k}) = 0$, elasto-plastic response occurs if, and only if, also the second equation in Eq. 12.24 is satisfied. Since by definition $\delta\lambda \geq 0$, it follows that elasto-plastic response occurs if, and only if,

$$A > 0 \quad ; \quad \mathbf{a}^T \delta\sigma \geq 0 \tag{12.25}$$

$$A = 0 \quad ; \quad \mathbf{a}^T \delta\sigma = 0 \tag{12.26}$$

$$A < 0 \quad ; \quad \mathbf{a}^T \delta\sigma \leq 0 \tag{12.27}$$

Consequently, elasto-plastic deformations cannot occur for

$$A > 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} < 0 \tag{12.28}$$

$$A = 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} \gtrless 0 \tag{12.29}$$

$$A < 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} > 0 \tag{12.30}$$

These situations may indicate either purely elastic or inadmissible material responses. In particular, by definition, in the case of a purely elastic response $\delta\lambda = 0$; consequently, from Eq. 12.15,

$$\delta k_i = 0 \tag{12.31}$$

for $i = 1, 2, \ldots, n$. Moreover, according to Eq. 12.11,

$$F(\boldsymbol{\sigma} + \delta\boldsymbol{\sigma}, \mathbf{k} + \delta\mathbf{k}) < 0$$

which, expressing F into a Taylor series, yields to the conclusion that, in a purely elastic response,

$$F(\boldsymbol{\sigma}, \mathbf{k}) + \mathbf{a}^T \delta\boldsymbol{\sigma} + \frac{\partial F}{\partial k_i} \delta k_i < 0 \tag{12.32}$$

for $i = 1, 2, \ldots, n$. Being $F(\boldsymbol{\sigma}, \mathbf{k}) = 0$ and $\delta k_i = 0$, we obtain

$$\mathbf{a}^T \delta\boldsymbol{\sigma} < 0 \tag{12.33}$$

Hence, the situations

$$A = 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} > 0$$
$$A < 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} > 0$$

in Eqs. 12.29 and 12.30 cannot be elastic; therefore, they are not admissible, as stated in item 4. The uniqueness of these inadmissible conditions is trivially verified.

Then, consider the possible elasto-plastic material response predicted in Eq. 12.27. According to Eq. 12.33, the case

$$A < 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} < 0$$

may also represent a purely elastic material response. This justifies therefore the ambiguity stated in item 3.

With the above observations it becomes easy to verify the statements in items 1 and 2.

12.7 Strain Based Elasto-Plastic Criterion

By definition, Section 12.3, if

$$F(\boldsymbol{\sigma}, \mathbf{k}) < 0 \tag{12.34}$$

the material response is always elastic, regardless of the applied strain increment $\delta\epsilon$. Elasto-plastic response can occur if, and only if, Eqs. 12.12 and 12.22,

$$\begin{cases} F(\boldsymbol{\sigma}, \mathbf{k}) = 0 \\ \delta F = \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} - \left(\mathbf{a}^T \mathbf{c}^{(\sigma)} + A\right) \delta\lambda = 0 \end{cases} \tag{12.35}$$

Accordingly, we can establish the following *strain based criterion* which predict the material response for any given strain increment $\delta\epsilon$ applied on any material state $(\boldsymbol{\sigma}, \mathbf{k})$, [23]:

1. Elasto-plastic response occurs if

$$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

and

$$\begin{array}{lll} A > -\mathbf{a}^T \mathbf{c}^{(\sigma)} & ; & \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} \geq 0 \\ A = -\mathbf{a}^T \mathbf{c}^{(\sigma)} & ; & \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} = 0 \\ A < -\mathbf{a}^T \mathbf{c}^{(\sigma)} & ; & \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} = 0 \end{array}$$

2. Elastic response occurs if

$$F(\boldsymbol{\sigma}, \mathbf{k}) < 0$$

or

$$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

and

$$A \geq -\mathbf{a}^T \mathbf{c}^{(\sigma)} \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} < 0$$

3. Either elastic or elasto-plastic response may occur if

$$F(\boldsymbol{\sigma}, \mathbf{k}) = 0$$

and

$$A < -\mathbf{a}^T\mathbf{c}^{(G)} \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)} < 0$$

This is the only ambiguous situation which the classical theory of plasticity does not solve by itself.

4. Strain increments by which

$$F(\boldsymbol{\sigma}, \mathrm{k}) = 0$$

and

$$A \leq -\mathbf{a}^T\mathbf{c}^{(G)} \quad ; \quad \mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)} > 0$$

are not admissible; that is, according to the theory, the material cannot sustain such type of strain increment.

The proof of the above statements is analogous to that reported in Section 12.6 for the stress based criterion. It is only important to take also into account that, in the case of purely elastic response,

$$\delta\boldsymbol{\sigma}^{(e)} = \mathbf{C}^{(e)}\delta\boldsymbol{\epsilon} = \delta\boldsymbol{\sigma}$$

It is interesting to observe that, in the case of associative flow rule and $\mathbf{C}^{(e)}$ positive definite matrix, it results

$$\mathbf{a}^T\mathbf{c}^{(G)} = \mathbf{a}^T\mathbf{C}^{(e)}\mathbf{a} > 0$$

This implies that the ambiguous and the not admissible situations in items 3 and 4, respectively, may occur only when $A < 0$. Moreover, the condition

$$A \leq -\mathbf{a}^T\mathbf{c}^{(G)} = -\mathbf{a}^T\mathbf{C}^{(e)}\mathbf{a}$$

is obviously more restrictive than the condition

$$A \leq 0$$

which, when $\delta\boldsymbol{\sigma}$ is assigned, may cause ambiguous or not admissible situations, Section 12.6. Then, in practice, it is generally more convenient to assign $\delta\boldsymbol{\epsilon}$.

12.8 The Elasto-Plastic Constitutive Equation

The mathematical developments reported in the previous Sections lead to the conclusion that the strain increment $\delta\epsilon$ resulting from a stress increment $\delta\boldsymbol{\sigma}$, starting from a material state $(\boldsymbol{\sigma}, \mathbf{k})$, can be calculated as

$$\delta\epsilon = \left(\mathbf{C}^{(e)}\right)^{-1}\delta\boldsymbol{\sigma} + \delta\lambda\mathbf{b} \tag{12.36}$$

where, if a purely elastic response occurs,

$$\delta\lambda = 0$$

while, if plasticity develops,

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T\delta\boldsymbol{\sigma}}{A} & \text{if } A \neq 0 \\ \text{indeterminate} & \text{if } A = 0 \end{cases}$$

The type of mechanical response is established according to the stress based criterion, Section 12.6. We remind that, when $A \leq 0$, some $\delta\boldsymbol{\sigma}$ may be not admissible or cause ambiguous situations.

The inversion of Eq. 12.36 gives

$$\delta\boldsymbol{\sigma} = \mathbf{C}^{(e)}\left(\delta\epsilon - \delta\lambda\mathbf{b}\right)$$

where $\delta\lambda$ may be expressed as in Eq. 12.17. Hence, the stress increment $\delta\boldsymbol{\sigma}$ resulting from strain increment $\delta\epsilon$, starting from a material state $(\boldsymbol{\sigma}, \mathbf{k})$, can be calculated as

$$\delta\boldsymbol{\sigma} = \mathbf{C}\delta\epsilon \tag{12.37}$$

where, if a purely elastic response occurs,

$$\mathbf{C} = \mathbf{C}^{(e)}$$

while, if plasticity develops,

$$\begin{aligned} \mathbf{C} &= \mathbf{C}^{(e)} - \mathbf{C}^{(p)} \\ \mathbf{C}^{(p)} &= \begin{cases} \dfrac{\mathbf{c}^{(G)}\mathbf{c}^{(F)T}}{A + \mathbf{a}^T\mathbf{c}^{(G)}}; & \text{for } A \neq -\mathbf{a}^T\mathbf{c}^{(G)}; \\ \text{indeterminate} & \text{for } A = -\mathbf{a}^T\mathbf{c}^{(G)}; \end{cases} \\ \mathbf{c}^{(F)} &= \mathbf{C}^{(e)T}\mathbf{a} \end{aligned}$$

The type of mechanical response is established according to the strain based criterion, Section 12.7. We remind that, when $A \leq -\mathbf{a}^T\mathbf{c}^{(G)}$, some $\delta\epsilon$ may be not admissible, or cause ambiguous situations.

It is interesting to remark that:

- If $A = 0$ and elasto-plastic response occurs, $\mathbf{C}$ is singular. In fact, in this case, the strain increment $\delta\epsilon$ mobilized by any admissible $\delta\sigma$ is indeterminate; hence, the matrix $\mathbf{C}$ in Eq. 12.37 cannot be inverted.
- If $\mathbf{C}^{(e)}$ is symmetric and the flow rule is associative, $\mathbf{C}$ is symmetric.

12.9 Stable Materials

We recall that according to Drucker, Section 10.3, a *stable material* is that respecting the following two conditions:

- *Stability in small.* The work done by an added stress set $\delta\sigma$ on the corresponding change of plastic strain $\delta\epsilon$ is positive, that is

$$\delta\sigma^T\delta\epsilon > 0 \tag{12.38}$$

- *Stability in cycle.* The net work done over a complete cycle of application and removal done by an added stress set $\delta\sigma$ on the corresponding change of strain $\delta\epsilon$ is non negative, that is

$$\oint \delta\sigma^T\delta\epsilon \geq 0 \tag{12.39}$$

Provided that the elastic work is fully recoverable, the above requirements lead to the conclusion that in an elasto-plastic material:

- The plastic work done by an added stress set $\delta\sigma$ on the corresponding change of plastic strain $\delta\epsilon^{(p)}$ is not negative, that is

$$\delta\sigma^T\delta\epsilon^{(p)} \geq 0 \tag{12.40}$$

- The yield surface F must be convex;
- The yield and the plastic potential functions coincide, that is $F \equiv G$;
- the plastic modulus A has a not negative value .

- the elasto-plastic tangential constitutive matrix $\mathbf{C}$ in Eq. 12.37 is symmetric and positive definite.

In fact, consider a material point P in a continuum $\mathcal{M}$ and let $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(o)}$ be its stress at time t=0. Suppose that $\boldsymbol{\sigma}^{(o)}$ is inside the yield surface F and that an external agency brings the stress value to make the following cycle:

- at time $t = t_1$ the stress reaches the value $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(1)}$ on F.
- at time $t = t_2 = t_1 + \delta t_1 =$ the stress reaches the value $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(2)} = \boldsymbol{\sigma}^{(1)} + \delta\boldsymbol{\sigma}^{(1)}$ outside or on the initial yield surface.
- at time $t = t_3 = t_2 + \delta t_2$ the stress returns back to the initial value $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{(o)} = \boldsymbol{\sigma}^{(2)} + \delta\boldsymbol{\sigma}^{(2)}$ inside the initial yield surface.

According to Eq. 12.1,

$$\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}^{(e)} + \dot{\boldsymbol{\epsilon}}^{(p)}$$

and, consequently,

$$\begin{aligned} dw^{(tot)} &= \oint \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}} dt = \oint \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(e)} dt + \oint \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(p)} dt = \\ &= \oint \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(e)} dt + \int_{t_o}^{t_1} \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(p)} dt + \int_{t_1}^{t_2} \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(p)} dt + \int_{t_2}^{t_3} \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(p)} dt = \\ &= \int_{t_1}^{t_2} \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(p)} dt \end{aligned}$$

since, in a cycle, the net elastic work

$$\oint \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(e)} dt = 0$$

and the loading path inside the yield surface as well as the unloading path do not mobilize plastic deformation, so that

$$\begin{aligned} \int_{t_o}^{t_1} \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(p)} dt &= 0 \\ \int_{t_1}^{t_3} \boldsymbol{\sigma}^T \dot{\boldsymbol{\epsilon}}^{(p)} dt &= 0 \end{aligned}$$

Similarly, we can prove that

$$dw^{(o)} = \oint \boldsymbol{\sigma}^{(o)T} \dot{\boldsymbol{\epsilon}} dt = \int_{t_1}^{t_2} \boldsymbol{\sigma}^{(o)T} \dot{\boldsymbol{\epsilon}}^{(p)} dt$$

Thus, the net external work in the cycle results to be equal to

$$đw^{(ext)} = đw^{(tot)} - đw^{(o)} = \int_{t_1}^{t_2} (\sigma - \sigma^{(o)})^T \dot{\epsilon}^{(p)} dt \tag{12.41}$$

According to Drucker's postulate in Eq. 12.39, $đw^{(ext)} \geq 0$ for arbitrary dt; this implies that

$$(\sigma - \sigma^{(o)})^T \delta\epsilon^{(p)} \geq 0 \tag{12.42}$$

Then, setting $(\sigma - \sigma^{(o)}) = \delta\sigma$, we can prove the inequality in Eq. 12.40.

Moreover, if $\delta\epsilon^{(p)}$ is the plastic strain mobilized at σ on the yield surface, then Eq. 12.42 states that the all the initial $\sigma^{(o)}$ inside F lie on one side of the hyperplane (dashed line in Fig. 12.1) orthogonal to $\delta\epsilon^{(p)}$. Hence, through

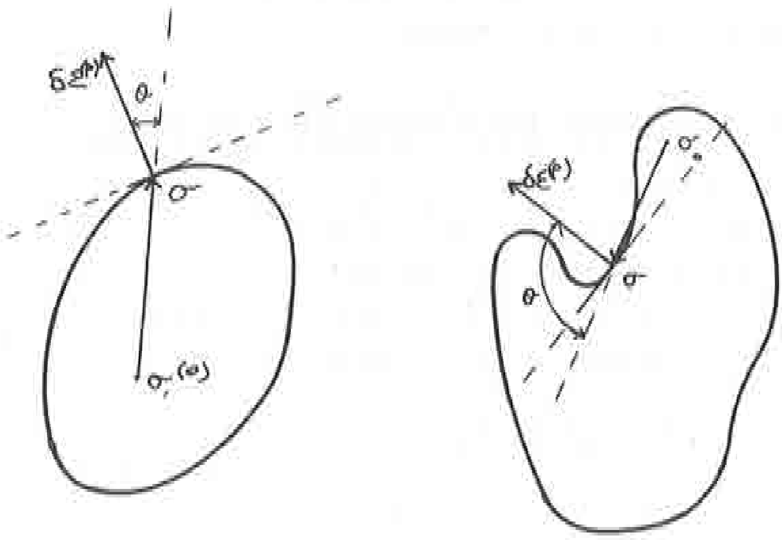


Figure 12.1: A convex and not convex yield surfaces in the stress space

every point σ on F, there is a plane such that all points inside F lie on the same side of the plane, that is F is a convex surface.

Conversely, assume that F is convex. Then, according to Eq. 12.42, the plastic strain vector $\delta\epsilon^{(p)}$ mobilized at σ on the yield surface must make an not obtuse angle with the vector $(\sigma - \sigma^{(o)})$ for all initial $\sigma^{(o)}$ inside F. Hence, since the only direction for $\delta\epsilon^{(p)}$ satisfying this requirement is the outer normal to F, we can conclude that F is a plastic potential.

Assuming $F \equiv G$, it follows that, Eqs. 12.3 and 12.16,

$$\delta\sigma^T\delta\epsilon^{(p)} = \delta\sigma^T\delta\lambda\mathbf{a} = \frac{(\delta\sigma^T\mathbf{a})^2}{A}$$

According to Eq. 12.40,

$$\delta\sigma^T\delta\epsilon^{(p)} = \frac{(\delta\sigma^T\mathbf{a})^2}{A} \geq 0 \tag{12.43}$$

which implies that A must be a not negative value.

Always assuming $F \equiv G$, it follows that the plastic constitutive matrix $\mathbf{C}^{(p)}$ is a symmetric matrix and, consequently, also the elasto-plastic stiffness matrix $\mathbf{C}$ results to be a symmetric matrix. Finally, the requirement in Eq. 12.38 leads to the conclusion that

$$\delta\sigma^T\delta\epsilon = \delta\epsilon^T\mathbf{C}\delta\epsilon > 0 \tag{12.44}$$

that is $\mathbf{C}$ symmetric positive definite matrix.

12.10 Isotropic Elasto-Plastic Materials

By definition, Section 10.6, an isotropic material is a material whose mechanical response at a point is identical in all directions. A suitable form of elasto-plastic constitutive equations for such type of materials is given by Eq. 12.36 or, conversely, by Eq. 12.37 specialized as follows:

1. the elastic stiffness matrix is of the type, Eq. 11.24,

$$\mathbf{C}^{(e)} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & . & 0 \\ \nu & (1-\nu) & \nu & 0 & . & 0 \\ \nu & \nu & (1-\nu) & 0 & . & 0 \\ 0 & 0 & 0 & (1-2\nu) & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & (1-2\nu) \end{bmatrix}_{(9\times 9)} \tag{12.45}$$

   where the *Young Modulus* E and the *Poisson Coefficient* ν may be a function of invariant quantities (p, q, θ).

2. The yield and the potential functions can be expressed in the form, Section 10.6,

$$\begin{aligned} F &= F(p, q, \theta, \mathbf{k}) \\ G &= G(p, q, \theta, \mathbf{k}) \end{aligned}$$

where $\mathbf{k}$ may be eventually expressed in terms of strain invariant quantities.

Notice that the assumption in items 2 implies that the quantities c_j and $\partial k_i/\partial h_j$ defined in Section 12.4 as well as the plastic modulus A in Eq. 12.20 can be eventually expressed in terms of invariants.

It is possible to prove that the gradient vector $\mathbf{b}$ to a potential function of the type $G = G(p, q, \theta, \mathbf{k})$ and its octahedral components can be respectively calculated as, Appendix A,

$$\mathbf{b} = \left\{\frac{\partial G}{\partial \sigma_{ij}}\right\}^T = c_1\widehat{\mathbf{m}} + c_2\mathbf{s} + c_3\mathbf{v} \tag{12.46}$$

$$\nabla_p G = \left\{\frac{\partial G}{\partial p}\frac{\partial p}{\partial \sigma_{ij}}\right\}^T = c_1\widehat{\mathbf{m}} \tag{12.47}$$

$$\nabla_s G = \left\{\frac{\partial G}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial \sigma_{ij}}\right\}^T = c_2\mathbf{s} + c_3\mathbf{v} \tag{12.48}$$

where

$$\begin{aligned} c_1 &= \frac{1}{3}\frac{\partial G}{\partial p} \\ c_2 &= \frac{3}{2q}\left(\frac{\partial G}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial G}{\partial \theta}\right) \\ c_3 &= -\frac{9}{2q^3\cos 3\theta}\frac{\partial G}{\partial \theta} \end{aligned}$$

and the element of $\mathbf{v}$ are given by

$$v_{ij} = w_{ij} - \frac{\delta_{ij}}{3}w_{kk} \tag{12.49}$$

where

$$[w_{ij}] = \left[\frac{\partial J_3}{\partial s_{ij}}\right] = \begin{bmatrix} (s_{22}s_{33} - s_{23}^2) & (s_{13}s_{23} - s_{33}s_{12}) & (s_{12}s_{23} - s_{22}s_{13}) \\ & (s_{11}s_{33} - s_{13}^2) & (s_{12}s_{13} - s_{11}s_{23}) \\ \text{Symmetric} & & (s_{11}s_{22} - s_{12}^2) \end{bmatrix}$$

$$w_{kk} = \frac{\partial J_3}{\partial s_{kk}} = -J_2 = -\frac{q^2}{3}$$

Similarly, we can also express the gradient vector $\mathbf{a}$ to a yield function of the type $F = F(p, q, \theta, \mathbf{k})$ and its octahedral components.

It follows that the plastic strain, its deviatoric component and invariants defined in Section 12.2 can equivalently be expressed as

$$\delta\boldsymbol{\epsilon}^{(p)} = \delta\lambda(c_1\widehat{\mathbf{m}} + c_2\mathbf{s} + c_3\mathbf{v}) \tag{12.50}$$

$$\delta\mathbf{e}^{(p)} = \delta\lambda(c_2\mathbf{s} + c_3\mathbf{v}) \tag{12.51}$$

$$\delta\epsilon_v^{(p)} = \delta\lambda\frac{\partial G}{\partial p} \tag{12.52}$$

$$\delta\epsilon_s^{(p)} = \delta\lambda\left[\left(\frac{\partial G}{\partial q}\right)^2 + \frac{1}{q^2}\left(\frac{\partial G}{\partial \theta}\right)^2\right]^{1/2} \tag{12.53}$$

Finally, it is simple to verify

$$\mathbf{a}^T\delta\boldsymbol{\sigma} = \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial q}\delta q + \frac{\partial F}{\partial \theta}\delta\theta \tag{12.54}$$

12.11 Perfect Plasticity

A material whose yield surface $F = F(\boldsymbol{\sigma}, k)$ coincides always with its failure surface Q is called *perfectly plastic material*. In this case, since $F \equiv Q$ and Q, by definition, remains fix in space and size, we can establish that $k = const$ and, consequently,

$$\delta k = 0 \tag{12.55}$$

$$A = -\frac{1}{\delta\lambda}\frac{\partial F}{\partial k}\delta k = 0 \tag{12.56}$$

Figure 12.2 reports the triaxial stress-strain response for a linear elastic perfectly plastic material with failure represented by a Mohr-Coulomb surface type.

Much of the classical plastic theory have been based on the assumption of elastic perfectly-plastic materials with associated flow rule and yielding state represented by Mohr-Coulomb surface type. The theoretical importance of these ideal materials is related to the fact that it is possible to bound analytically the range of loading conditions in which the collapse of the elasto-plastic structure is to be expected. These analytical solutions, given in the form of upper and lower bounds, are provided by the so-called *limit analysis theory*, [17].

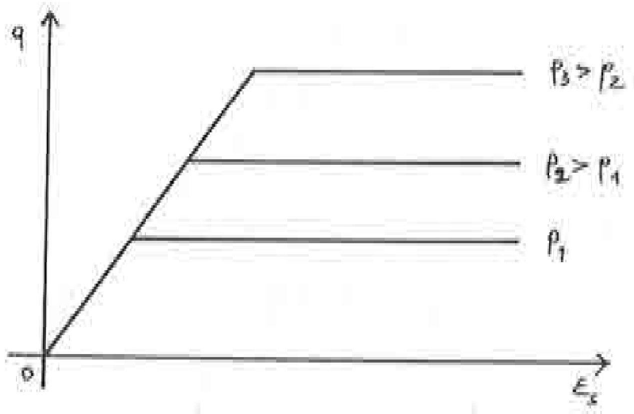


Figure 12.2: Triaxial response of a perfectly plastic Mohr-Coulomb material

12.12 Work Hardening Materials

Apparently, within the common range of engineering applications, the yielding in metals is independent of the mean pressure p and of the θ stress invariant. Accordingly, Hill proposed an incremental constitutive model for isotropic metals whose main hypotheses may be reformulated as follows, [52]:

1. State of failure described by a Von Mises surface type, namely

$$Q = q - N = 0 \tag{12.57}$$

2. State of yielding represented by a Von Mises surface type of equation

$$F = q - N(k) = 0 \tag{12.58}$$

3. Hardening parameter k identified as the plastic work, namely

$$dk = \sigma^T d\epsilon^{(p)} \tag{12.59}$$

4. Associative flow rule.

The above hypotheses yield to the conclusion that stress-strain relationship can be represented by an incremental constitutive equation for isotropic material of the type described in Section 12.10 where

- Because of the hypothesis of associative flow rule, $F \equiv G$.
- Plastic modulus calculated according to

$$A = \frac{dq}{d\epsilon_s^{(p)}} \tag{12.60}$$

where the law of variation

$$q = q(\epsilon_s^{(p)}) \tag{12.61}$$

has to be determined experimentally.

- The variation of the plastic surface can be calculated as

$$\delta N = A\delta\epsilon_s^{(p)} \tag{12.62}$$

In fact, according to Eqs. 12.51-12.53,

$$\delta\mathbf{e}^{(p)} = \delta\lambda\frac{3}{2q}\mathbf{s} \tag{12.63}$$

$$\delta\epsilon_v^{(p)} = 0 \tag{12.64}$$

$$\delta\epsilon_s^{(p)} = \delta\lambda \tag{12.65}$$

and

$$\frac{\partial F}{\partial k}\delta k = -\delta q \tag{12.66}$$

Hence,

$$\begin{aligned} \delta k &= \boldsymbol{\sigma}^T\boldsymbol{\epsilon}^{(p)} = (p\widehat{\mathbf{m}} + \mathbf{s})^T(\delta\epsilon_v^{(p)} + \delta\mathbf{e}^{(p)}) = \\ &= \mathbf{s}^T\delta\mathbf{e}^{(p)} = q\delta\lambda = q\delta\epsilon_s^{(p)} \end{aligned}$$

From Eqs. 12.20 and 12.66, it follows that

$$A = -\frac{1}{\delta\lambda}\frac{\partial F}{\partial k}\delta k = \frac{\delta q}{\delta\epsilon_s^{(p)}}$$

proving therefore Eq. 12.60. Finally, since

$$\frac{\partial F}{\partial k} = \frac{\partial F}{\partial N}\frac{\delta N}{\delta k} = -\frac{\delta N}{\delta k}$$

from Eq. 12.20 it results

$$A = -\frac{1}{\delta\lambda}\frac{\partial F}{\partial k}\delta k = \frac{1}{\delta\lambda}\frac{\partial N}{\partial k}\delta k$$

and, consequently, we can establish that

$$\delta N = A\delta\lambda = A\delta\epsilon_s^{(p)}$$

It should be noticed that, since the plastic response of the model does not depend on the mean pressure, the law of variation of A can be determined from simple uniaxial tests; in this case, it can be proved that

$$A = \frac{d\sigma_a}{d\epsilon_a^{(p)}} \tag{12.67}$$

In fact, in an unixial test, the only stress value non equal to zero is the axial stress σ_a, so that

$$dq = d\sigma_a$$

Moreover, according to Eq. 12.64, we have that hoop and axial plastic strain are related as

$$2\delta\epsilon_o^{(p)} = -\delta\epsilon_a^{(p)}$$

from which it follows that

$$\delta\epsilon_s^{(p)} = \delta\epsilon_a^{(p)}$$

Simple, nevertheless sufficiently accurate, approximations of the uniaxial stress vs. plastic strain relationship can be obtained with the hyperbolic equations reported in Section 11.8.

12.13 Strain Hardening Models

For an isotropic material, we can simplify the general functional relationship for the hardening parameter k reported in Section 12.4 into

$$k = k(\epsilon_v^{(p)}, \epsilon_s^{(p)}) \tag{12.68}$$

in Section 12.4. The relative plastic modulus results to be given by, Eqs. 12.20, 12.52 and 12.53,

$$\begin{aligned} A &= -\frac{1}{\delta\lambda}\frac{\partial F}{\partial k}\delta k = -\frac{1}{\delta\lambda}\frac{\partial F}{\partial k}\left\{\frac{\partial k}{\partial\epsilon_v^{(p)}}d\epsilon_v^{(p)} + \frac{\partial k}{\partial\epsilon_s^{(p)}}d\epsilon_s^{(p)}\right\} = \\ &= -\frac{\partial F}{\partial k}\left\{\frac{\partial k}{\partial\epsilon_v^{(p)}}\frac{\partial G}{\partial p} + \frac{\partial k}{\partial\epsilon_s^{(p)}}\left[\left(\frac{\partial G}{\partial q}\right)^2 + \frac{1}{q^2}\left(\frac{\partial G}{\partial\theta}\right)^2\right]^{\frac{1}{2}}\right\} \end{aligned} \tag{12.69}$$

As an example, consider the following plastic model for isotropic material, Fig. 12.3:

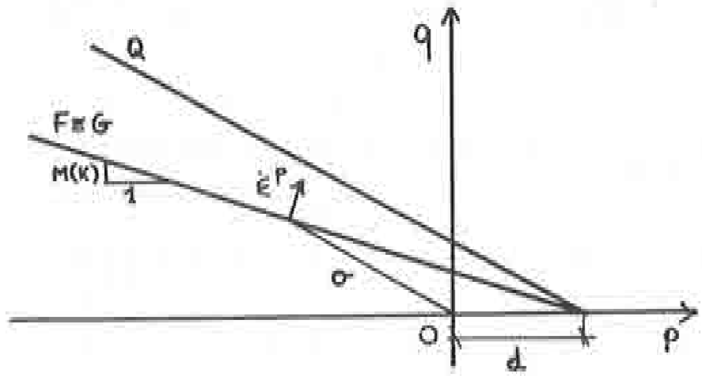


Figure 12.3: Strain hardening Drucker-Pragher elasto-plastic model

1. State of failure described by a Drucker-Pragher surface type, Eq. 11.14, namely
$$Q = q - M(d - p) = 0 \tag{12.70}$$
where
$$d = \frac{N}{M} = \frac{c}{\tan\varphi}$$

2. State of yielding represented by a Drucker-Pragher surface type of equation
$$F = q - M(d - p) = 0 \tag{12.71}$$
where
$$\begin{aligned} d &= \frac{N}{M} = \frac{c}{\tan\varphi} \\ M &= M(k) \end{aligned}$$
and the law of variation of $M(k)$ has to be determined experimentally.

3. Hardening parameter k identify as the plastic shear stress invariant, namely

$$\delta k = \delta \epsilon_s^{(p)} \tag{12.72}$$

4. Associative flow rule.

The above hypotheses yield to the conclusion that stres-strain relationship can be represented by an incremental constitutive equation for isotropic material as described in Section 12.10 where

- Because of the hypothesis of associative flow rule, $F \equiv G$.
- Plastic modulus calculated according to

$$A = (d-p)\frac{dM(\epsilon_s^{(p)})}{d\epsilon_s^{(p)}} = (d-p)\frac{d}{d\epsilon_s^{(p)}}\left(\frac{q}{d-p}\right) \tag{12.73}$$

  where the law of variation of M with respect to $\epsilon_s^{(p)}$ has to be determined experimentally.

In fact, according to the hypotheses of the model,

$$\begin{aligned} \delta k &= \delta \epsilon_s^{(p)} = \delta \lambda \frac{\partial F}{\partial q} \\ A &= -\frac{\partial F}{\partial k}\frac{\partial k}{\partial \epsilon_s^{(p)}}\frac{\partial G}{\partial q} = -\frac{\partial F}{\partial \epsilon_s^{(p)}}\frac{\partial F}{\partial q} = (d-p)\frac{dM}{d\epsilon_s^{(p)}} = \\ &= (d-p)\frac{d}{d\epsilon_s^{(p)}}\left(\frac{q}{d-p}\right) \end{aligned}$$

12.14 Capped Plastic Models

For a number of engineering materials and especially for soils, experimental evidence shows that plasticity takes place even under pure isotropic compression. Therefore, the yield and the potential functions cannot be represented by a Mohr-Coulomb based model; they should have, instead, a convex shape and cross the hydrostatic axis.

Roscoe et al., [79], indicated that at failure, or as they called it, *Critical State*, the volumetric strain tends to be a constant value. Hence, Eq. 12.5, the potential function should present a maximum with respect to p, at the intersection with the failure surface.

Apparently, Drucker et al., [33], were the first to suggest that soil might be modeled as an elasto-plastic hardening material. They proposed that successive yield surfaces might resemble Von Mises's cones with convex end caps. Drucker, [34], discussed this concept in a later paper in which he suggested that failure surface may not be the yield surface. This point was further emphasized by Drucker, [35], who noted that successive loading surfaces or yields do not approach the failure surface.

Roscoe et al., [79], published a paper which contained the basis for a number of subsequent hardening models for soils. The paper was concerned primarily with the behavior of soils in a triaxial test and contained the so-called *State Boundary Surface* (called yield surface in that paper) and the *Critical State Line* postulates. The primary merit of this work lies in the attempt to explain the behavior of soil in a global way, giving new light to the early intuitions of Rendulic, [77, 78], Hvorslev, [53], and Terzaghi, [89].

Roscoe and Poorooshab, [81], used the ideas contained in the early paper of Roscoe in order to develop a stress-strain theory for clay which, however, was not based upon the theory of plasticity. Calladine, [14], suggested an alternative interpretation of this theory using concepts from strain hardening theory.

Subsequently, Roscoe et al., [80], adopted the strain hardening theory of plasticity in order to formulate a complete stress-strain model for *normally consolidated* or *lightly pre-consolidated* clay in triaxial test. This model has since become known as the Cam-Clay model, [86].

Burland, [13], suggested a modified version of the Cam-Clay model which was eventually extended to a general 3D stress state by Roscoe and Burland, [82]. The complete mathematical formulation of this model is reported in the next Section.

12.15 Cam-Clay Model

According to Roscoe and Burland, [82], the following postulates are valid for an isotropic soil:

1. There exists a *Critical State Condition* where indeterminate shear strain ϵ_s occurs with no change in the stress $\boldsymbol{\sigma}$ and in the specific volume
$$v = \frac{\delta\Omega}{\delta\Omega_m} \tag{12.74}$$

where $\delta\Omega$ is the total volume of the soil (porous) particle while $\delta\Omega_m$ is the part of the volume $\delta\Omega$ occupied by the mineral grains only.

2. The locus in the $\boldsymbol{\sigma}$ space of all the stress levels corresponding to critical state conditions is called *Critical State Surface* and it is represented by a Drucker-Pragher cohesionless surface, that is

$$Q = Q(p,q) = q - Mp = 0 \tag{12.75}$$

where M is function of the friction angle φ only.

3. There exists a yield-potential function for plastic strain which, in the stress space $\boldsymbol{\sigma}$, is represented by ellipsoid of rotation about the diagonal space, Fig. 12.4a, that is

$$F = F(p,q,\overline{p}_y) = p^2 - p\overline{p}_y + \frac{q^2}{M^2} = 0 \tag{12.76}$$

Soils whose stress state is on F are called *Normally Consolidated Soils*, NCS. Conversely, soils whose stress state is inside F are called *Over Consolidated Soils*, OCS.

4. Under cycles of all-round pressure, the soil behaves as reported in Fig. 12.4b, that is

   - Isotropic loading on NCS, elasto-plastic deformations,

$$v = v_\lambda - \lambda \ln \frac{p}{p_\lambda} \tag{12.77}$$

   - Isotropic loading and unloading on OCS, elastic deformations,

$$v = v_\chi - \chi \ln \frac{p}{p_\chi} \tag{12.78}$$

5. The variation of the plastic specific volume in a NCS is only dependent on the variation of the $\overline{p}_y$ and on the current v.

The above listed hypotheses lead to the conclusion that the stress-strain relation an isotropic soil may be described by an incremental elasto-plastic constitutive equation of the type described in Section 12.10 where:

- $F \equiv G$, associative flow rule.

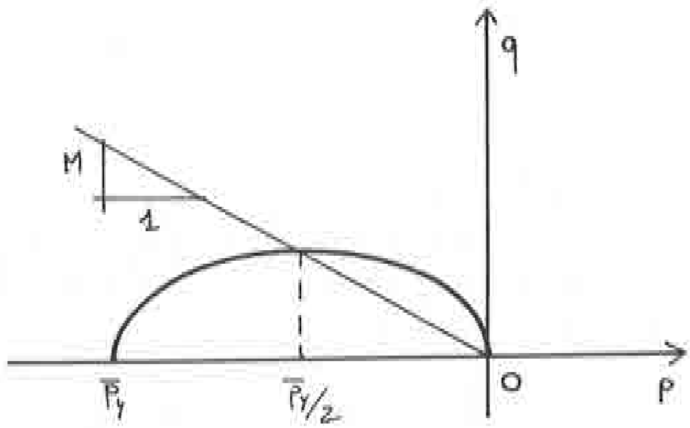


Figure 12.4: Cam-Clay model

- Nonlinear elastic deformations where the tangent Young Modulus is set to be equal to

$$E = -\frac{3(1-2\nu)}{\chi} vp \tag{12.79}$$

- The plastic modulus is given by

$$A = \frac{v\overline{p}_y}{\lambda - \chi} \frac{\partial F}{\partial \overline{p}_y} \frac{\partial F}{\partial p} \tag{12.80}$$

where

$$\begin{aligned} \frac{\partial F}{\partial \overline{p}_y} &= -p \\ \frac{\partial F}{\partial p} &= 2p - \overline{p}_y \end{aligned}$$

In fact, we preliminarily observe that according to the definition of specific volume in Eq. 12.74 it results that the volumetric strain increment can be calculated as, Section 15.2,

$$\delta\epsilon_v = \frac{\delta v}{v} \tag{12.81}$$

from which it follows that

$$\delta\epsilon_v = \delta\epsilon_v^{(e)} + \delta\epsilon_v^{(p)} \tag{12.82}$$

where

$$\begin{aligned} \delta\epsilon_v^{(e)} &= \frac{\delta v^{(e)}}{v} \\ \delta\epsilon_v^{(p)} &= \frac{\delta v^{(p)}}{v} \end{aligned}$$

According to Eqs. 12.77 and 12.78, in a NCS subjected to an isotropic compression test, the total variation of the specific volume and the variation of its elastic part can be respectively calculated as

$$\delta v = -\lambda\frac{\delta p}{p} \tag{12.83}$$

$$\delta v^{(e)} = -\chi\frac{\delta p}{p} \tag{12.84}$$

from which if follows that the plastic part can be calculated as

$$\delta v^{(p)} = \delta v - \delta v^{(e)} = (\chi - \lambda)\frac{\delta p}{p} \tag{12.85}$$

We recall that in an isotropic soil, the elastic volumetric strain rate can be calculated as

$$\delta\epsilon_v^{(e)} = \frac{\delta p}{B}$$

where

$$B = \frac{E}{3(1-2\nu)}$$

Consequently,

$$B = \frac{\delta p}{\delta\epsilon_v^{(e)}} = \frac{v\delta p}{\delta v^{(e)}} = -\frac{vp}{\chi}$$

from which it follows the Young modulus expression in Eq. 12.79.

According to Eqs. 12.82 and 12.85,

$$\delta\epsilon_v^{(p)} = \frac{\delta v^{(p)}}{v} = \frac{(\chi-\lambda)}{v}\frac{\delta p}{p}$$

which implies that there exists a function for isotropic compression on NCS such that

$$\epsilon_v^{(p)} = \epsilon_v^{(p)}(v, p)$$

We notice that in a NCS subjected to isotropic loading conditions $p = \overline{p}_y$. Thus, according to the hypothesis in item 5, we can state that for any loading path the volumetric strain can be calculated as

$$\epsilon_v^{(p)} = \epsilon_v^{(p)}(v, \overline{p}_y) \tag{12.86}$$

and its rate of variation is given by

$$\delta\epsilon_v^{(p)} = \frac{(\chi - \lambda)}{v}\frac{\delta\overline{p}_y}{\overline{p}_y} \tag{12.87}$$

Conversely, we can also state that there exists a function

$$\overline{p}_y = \overline{p}_y(\epsilon_v^{(p)}) \tag{12.88}$$

whose rate of variation is given by

$$\delta\overline{p}_y = \frac{v\overline{p}_y}{\chi - \lambda}\delta\epsilon_v^{(p)} \tag{12.89}$$

Identifying the hardening parameter k as $\overline{p}_y$, we have that the plastic modulus expression in Eq. 12.20 can be specialized into

$$\begin{aligned} A &= -\frac{1}{\delta\lambda}\frac{\partial F}{\partial\overline{p}_y}\delta\overline{p}_y = -\frac{1}{\delta\lambda}\frac{\partial F}{\partial\overline{p}_y}\frac{v\overline{p}_y}{\chi - \lambda}\delta\epsilon_v^{(p)} = \\ &= -\frac{\partial F}{\partial\overline{p}_y}\frac{v\overline{p}_y}{\chi - \lambda}\frac{\partial F}{\partial p} \end{aligned}$$

since, in an isotropic material with associative flow rule,

$$\delta\epsilon_v^{(p)} = \delta\lambda\frac{\partial F}{\partial p}$$

This proves the plastic modulus expression in Eq. 12.80.

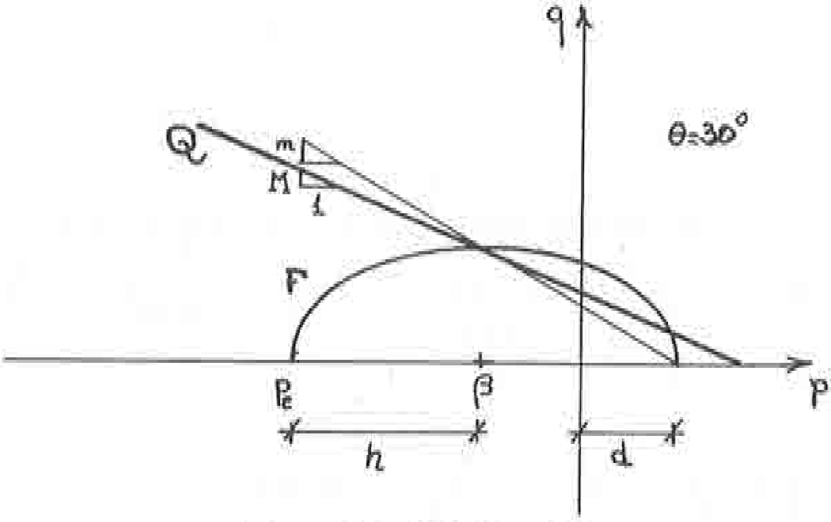


Figure 12.5: ECAM model

12.16 ECAM Model

ECAM model, [37], is an extension of the Cam-Clay model as described in the previous Section in order to account for limit states described by the Mohr-Coulomb surface type. Moreover, in ECAM, the major ellipse semi-axis ratio of the yield surface, Fig. 12.5b, may be independently controlled. In particular, the major differences with the Cam-Clay model may be summarized as follows:

- The Critical State Function is represented by a Mohr-Coulomb surface type of equation

$$Q = q + Mp - N = 0 \tag{12.90}$$

  where M and N may be set either independent or dependent functions of the θ-invariant, Eq. 11.14.

- The yield-potential function for plastic strains is represented by an (irregular) ellipsoid of rotation about the diagonal space, Fig. 12.5b, of equation

$$F = (p - \beta)^2 + \frac{q^2}{m^2} - h^2 = 0 \tag{12.91}$$

where

$$\begin{aligned} \beta &= d - rh \\ d &= c/\tan\varphi \\ m &= rM \end{aligned}$$

and

$$r = (m/M)_{\theta=30^\circ}$$

is a given parameter controlling the ellipse semi-axis ratio.

- The hardening parameter of the potential surface is identified by the *equivalent preconsolidation pressure* which is the intersection of F with the hydrostatic axis, that is

$$\overline{p}_y = \beta - h = d - (r+1)h \tag{12.92}$$

- The plastic modulus results to be equal to

$$A = \frac{v\overline{p}_y}{\lambda - \chi}\frac{\partial F}{\partial \overline{p}_y}\frac{\partial F}{\partial p} \tag{12.93}$$

where

$$\begin{aligned} \frac{\partial F}{\partial \overline{p}_y} &= \frac{2[(p-\beta)r - h]}{r+1} \\ \frac{\partial F}{\partial p} &= 2(p-\beta) \end{aligned}$$

It can be proved that, Chapter 16, for a given soil state (σ, v) the equivalent mean pressure $\overline{p}_y$ can be calculated solving the equation

$$(\chi - \lambda)\ln\frac{\overline{p}_y}{p_\lambda} = \chi\ln\frac{p}{p_\lambda} + v - v_\lambda$$

Hence, (σ, v) may be considered as the state variables of the model.

12.17 Extension to Visco-Plasticity

The Incremental Theory of Plasticity presented in this Chapter does not account for any time dependence and plastic strains are assumed to be immediately mobilized. Conversely, experimental evidence shows that for many real materials the permanent (irreversible) deformations are not instantaneously mobilized. A generalization of the Incremental Theory of Plasticity to account for strain rate dependence of the material response, has been proposed by Perzyna, [74, 72], on the basis of the pioneer work of Bingham, [10].

Analogously to the Incremental Theory of Plasticity, it is assumed that, under *loading conditions*, the strain rate $\dot{\epsilon}$ results to be the sum of an elastic (fully recoverable) component $\dot{\epsilon}^{(e)}$ and a visco-plastic (irreversible) component $\dot{\epsilon}^{(vp)}$, namely

$$\dot{\epsilon} = \dot{\epsilon}^{(e)} + \dot{\epsilon}^{(vp)} \tag{12.94}$$

These two strain vectors are respectively defined as

$$\dot{\epsilon}^{(e)} = \left(\mathbf{C}^{(e)}\right)^{-1} \dot{\sigma} \tag{12.95}$$

$$\dot{\epsilon}^{(vp)} = \dot{\eta}\mathbf{b} \tag{12.96}$$

where $\mathbf{C}^{(e)}$ is the elastic (symmetric) constitutive matrix,

$$\dot{\eta} = \begin{cases} \gamma < \phi(a) >; & \text{for } F(\sigma, k) > 0. \\ 0; & \text{for } F(\sigma, k) \leq 0. \end{cases}$$

is a *fluid parameter* controlling the plastic flow rate and

$$\mathbf{b} = \left\{ \frac{\partial G}{\partial \sigma_{11}}, \frac{\partial G}{\partial \sigma_{22}}, \frac{\partial G}{\partial \sigma_{33}}, \frac{\partial G}{\partial \sigma_{12}}, \frac{\partial G}{\partial \sigma_{13}}, \frac{\partial G}{\partial \sigma_{21}}, \frac{\partial G}{\partial \sigma_{23}}, \frac{\partial G}{\partial \sigma_{31}}, 2\frac{\partial G}{\partial \sigma_{32}} \right\}^T$$

In general, the parameter γ is a function of time, temperature and of the total creep strain, [36, 59, 60]. Finally, the functions

$$F = F(\sigma, k)$$
$$G = G(\sigma, k)$$

are the usual *yield* and *potential* functions of the Incremental Theory of Plasticity. If $G \equiv F$ the constitutive model is said to obey an *associative flow rule*.

According to the strain increment definition reported in Eq. 12.94, the stress increment result to be defined as

$$\begin{aligned} \dot{\sigma} &= \mathbf{C}^{(e)}\dot{\epsilon}^{(e)} = \mathbf{C}^{(e)}\left(\dot{\epsilon} - \dot{\epsilon}^{(vp)}\right) = \\ &= \mathbf{C}^{(e)}\left(\dot{\epsilon} - \dot{\eta}\mathbf{b}\right) \end{aligned} \tag{12.97}$$

definition of the (viscous) plastic multiplier $\dot{\eta}$. In this case, $\dot{\eta}$ is given as an explicit function of some distance from the yield, while the value of the plastic multiplier $\dot{\lambda}$ is found in order to satisfy the consistent equation.

Among the various possibilities, the two most common expressions for ϕ are:

$$\phi(a) = -1 + e^{\xi a} \tag{12.98}$$

$$\phi(a) = a^{\xi} \tag{12.99}$$

where ξ is an arbitrary prescribed constant.

The *distance* a has been defined in various alternative ways. For yield functions which can be expressed as

$$F(\sigma, k) = f(\sigma, k) - h(k) \tag{12.100}$$

it has been proposed to calculate this distance as

$$a = \frac{F(\sigma, k)}{h(k)} \tag{12.101}$$

In this way we obtain an adimensional value which is representative of the distance of the stress point from the yield surface and, at the same time, we account for size effects of the yield surface.

Chapter 13

A Generalized Theory of Plasticity

13.1 Introduction

Experimental evidence [1] shows that many real materials present progressive accumulation of plastic (irreversible) deformations under cyclic loading. In some cases this progressive accumulation may cause serious effects; for example, in a saturated soil, it may eventually cause the *liquefaction* of the solid skeleton.

The *standard* Incremental Theory of Plasticity presented in Chapter 12 is not able to describe such phenomena. For example, consider an isotropic material subjected to a triaxial cyclic loading test in which the deviatoric stress invariant value q is confined within the range $[0, q^*]$. Assume for simplicity that the yield surface of this material is represented by a Von Mises surface type, that is, Section 11.3,

$$F(\sigma, k) = q - N(k)$$

where the initial value of the hardening parameter k is equal to zero. During the first loading phase, the yield surface expands following the stress point and, consequently, plastic deformations are mobilized and the hardening parameter increases its value reaching eventually a limit value k^* such that

$$F(\sigma, k^*) = q^* - N(k^*) = 0$$

[1] Chapter written in collaboration with A. De Crescenzo

According to the standard incremental theory, the space region bounded by this yield surface is the new elastic region. Thus, all next loading phases cannot mobilize any plastic deformation, being the relative stress path always confined within the elastic stress region.

In the last 20 years a number of theoretical works have aimed to overcome the above described limitation of the standard theory. Some of them assume the existence of three, generally, distinct stress surfaces, [21, 22, 56, 68, 69, 73]:

- a *yield surface* F which is assumed to follow always the stress point, even during unloading conditions;
- a *bounding surface* $\overline{F}$;
- an *elastic surface* $\hat{F}$;

and

$$F \subseteq \overline{F}$$
$$\hat{F} \subseteq \overline{F}$$

Loading conditions whose relative stress point increments are directed outside F may mobilize plastic deformations, unless the current stress point lies inside the elastic region bounded by $\hat{F}$. The size of the plastic deformations results to be a function of a relative distance of the current stress point from $\overline{F}$.

As a rule, the above proposed modification of the standard theory allows to describe the accumulation of plastic deformations during cyclic loading. However, we still have to identify the law of variation of the size of the plastic deformation and the interrelationship between the evolution of these three stress surfaces.

In this Chapter we present a possible theoretical framework which allows to answer the above two questions, [23]. The practical use of this theory is explained by means of a small example. A more complex elasto-plastic model is proposed in Chapter 16.

13.2 The Main Hypotheses

In this Section we present the general theoretical framework by which it is possible to extend any classical elasto-plastic model in order to account for possible plastic deformations at any stress level. The basic assumptions are:

1. There exists a *Bounding Surface*

$$\overline{F} = \overline{F}(\sigma, \overline{\mathrm{k}}) \tag{13.1}$$

which, as the name indicates, bounds always the location of the current stress point. Accordingly, if $(\sigma, \overline{\mathrm{k}})$ represents the current material state, the only admissible alternative conditions are

$$\begin{cases} \overline{F}(\sigma, \overline{\mathrm{k}}) = 0; & \text{i.e. } \boldsymbol{\sigma} \text{ lies on } \overline{F}. \\ \overline{F}(\sigma, \overline{\mathrm{k}}) < 0; & \text{i.e } \sigma \text{ lies inside } \overline{F} \end{cases}$$

while

$$\overline{F}(\sigma, \overline{\mathrm{k}}) > 0$$

is not admissible.

2. The vector $\overline{\mathrm{k}}$ collects n hardening parameters $\overline{k}_i$ whose incremental variation is controlled by $\overline{m}$ *internal variables* $\overline{h}_{\overline{j}}$, namely

$$\delta \overline{k}_i = \frac{\partial \overline{k}_i}{\partial \overline{h}_{\overline{j}}} \delta \overline{h}_{\overline{j}} \tag{13.2}$$

for $i = 1, 2, \ldots, n$ and $\overline{j} = 1, 2, \ldots, \overline{m}$. It is required that the infinitesimal increments $\delta \overline{h}_{\overline{j}}$ *must* be expressible as, Eq. 12.14,

$$\delta \overline{h}_{\overline{j}} = \overline{f}_{\overline{j}}(\delta \epsilon^{(p)}) = \overline{c}_{\overline{j}} \delta \lambda \tag{13.3}$$

where $\delta \epsilon^{(p)}$ and $\delta \lambda$ are defined as reported in item 9.

3. There exists a *new* type of *Yield Surface*

$$F = F(\sigma, \overline{\mathrm{k}}, \mathrm{k}) \tag{13.4}$$

which always follows the location of the current stress point. Accordingly, the only admissible material condition is

$$F(\sigma, \overline{\mathrm{k}}, \mathrm{k}) = 0$$

while

$$F(\sigma, \overline{\mathrm{k}}, \mathrm{k}) \neq 0$$

is not admissible.

4. The vectors $\overline{\mathbf{k}}$ and $\mathbf{k}$ are of equal size n and

   (a) There exist n scalar functions of the type

   $$k_i = k_i(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \tag{13.5}$$

   for $i = 1, 2, \ldots, n$, so that for any given material state $(\boldsymbol{\sigma}, \overline{\mathbf{k}})$, the current location of the yield surface $F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k})$ can be uniquely determined.

   (b) If plasticity occurs, then the incremental variation of the hardening parameters k_i is controlled by the same $\overline{m}$ internal variables $\overline{h}_{\overline{j}}$ of $\overline{k}_i$ plus other m internal variables h_j, namely

   $$\delta k_i = \frac{\partial k_i}{\partial \overline{h}_{\overline{j}}} \delta \overline{h}_{\overline{j}} + \frac{\partial k_i}{\partial h_j} \delta h_j \tag{13.6}$$

   for $i = 1, 2, \ldots, n$, $\overline{j} = 1, 2, \ldots, \overline{m}$ and $j = 1, 2, \ldots, m$. Analogously to the hypothesis on $\overline{h}_{\overline{j}}$ in item 2, it is required that the infinitesimal increments δh_j *must* be expressible as

   $$\delta h_j = f_j(\delta \boldsymbol{\epsilon}) = c_j \delta \lambda \tag{13.7}$$

5. The space region bounded by F is a subspace of $\overline{F}$, that is

   $$F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \subseteq \overline{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \tag{13.8}$$

   If the current stress point $\boldsymbol{\sigma}$ lies on $\overline{F}$, then F and $\overline{F}$ must coincide, that is

   $$F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \equiv \overline{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \tag{13.9}$$

   and, in particular,

   $$k_i = \overline{k}_i \tag{13.10}$$
   $$\frac{\partial k_i}{\partial \overline{h}_{\overline{j}}} = \frac{\partial \overline{k}_i}{\partial \overline{h}_{\overline{j}}} \tag{13.11}$$
   $$\frac{\partial k_i}{\partial h_j} = 0 \tag{13.12}$$

   for all $i = 1, 2, \ldots, n$, $\overline{j} = 1, 2, \ldots, \overline{m}$ and $j = 1, 2, \ldots, m$.

6. There exists an *Elastic Surface*

$$\hat{F} = \hat{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \tag{13.13}$$

defined as

$$\hat{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}) \equiv F(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k} = \hat{\mathbf{k}})$$

where $\hat{\mathbf{k}} = \hat{\mathbf{k}}(\overline{\mathbf{k}})$ collects the n known scalar functions

$$\hat{k}_i = \hat{k}_i(\overline{\mathbf{k}}) \tag{13.14}$$

where $i = 1, 2, \ldots, n$. Notice that the above definition implies that if the current stress point lies on $\hat{F}$, then F coincides with $\hat{F}$.

7. In general, for $k_i \to \overline{k}_i$,

$$\begin{aligned} \frac{\partial k_i}{\partial h_{\overline{j}}} &\to \infty \\ \frac{\partial k_i}{\partial h_j} &\to \infty \\ A &\to +\infty \end{aligned}$$

for all $i = 1, 2, \ldots, n$; $\overline{j} = 1, 2, \ldots, \overline{m}$ and $j = 1, 2, \ldots, m$. The definition of the *Plastic Modulus* A is postponed to item 9. If $\hat{F}$ and $\overline{F}$ always coincide, then the above assumptions do not apply.

8. There exists a *Potential Function* for plastic deformations, of the form

$$G = G(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \tag{13.15}$$

9. Analogously to the Standard Incremental Theory of Plasticity, it is assumed that an infinitesimal strain increment $\delta\boldsymbol{\epsilon}$ can be expressed as, Eq. 12.1,

$$\delta\boldsymbol{\epsilon} = \delta\boldsymbol{\epsilon}^{(e)} + \delta\boldsymbol{\epsilon}^{(p)} \tag{13.16}$$

where:

- $\delta\boldsymbol{\epsilon}^{(e)}$ represents the elastic (fully recoverable) component which may be calculated according to the Generalized Hooke's law, Section 11.7,

$$\delta\boldsymbol{\epsilon}^{(e)} = \left(\mathbf{C}^{(e)}\right)^{-1} \delta\boldsymbol{\sigma} \tag{13.17}$$

where $\mathbf{C}^{(e)}$, the tangential elastic symmetric stiffness matrix, may be function of the current $\boldsymbol{\sigma}$.

- $\delta\epsilon^{(p)}$ is the plastic (irreversible) component, defined as

$$\delta\epsilon^{(p)} = \delta\lambda\mathbf{b} \tag{13.18}$$

where

$$\delta\lambda \quad \begin{cases} \geq 0, & \text{if elasto-plastic response occurs.} \\ = 0, & \text{if elastic response occurs.} \end{cases}$$

$$\begin{aligned} \mathbf{b} &= \frac{\partial G}{\partial \boldsymbol{\sigma}} = \\ &= \left\{ \frac{\partial G}{\partial \sigma_{11}}, \frac{\partial G}{\partial \sigma_{22}}, \frac{\partial G}{\partial \sigma_{33}}, \frac{\partial G}{\partial \sigma_{12}}, \frac{\partial G}{\partial \sigma_{13}}, \frac{\partial G}{\partial \sigma_{21}}, \frac{\partial G}{\partial \sigma_{23}}, \frac{\partial G}{\partial \sigma_{31}}, \frac{\partial G}{\partial \sigma_{32}} \right\}^T \end{aligned}$$

The respect of the consistency equation on the yield surface $F(\boldsymbol{\sigma}, \mathbf{k}, \overline{\mathbf{k}})$ yields to the conclusion that, item 4 Section 13.3, the value plastic multiplier $\delta\lambda$ can be calculated as:

- If $\delta\boldsymbol{\sigma}$ is assigned, Eq. 12.16,

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T\delta\boldsymbol{\sigma}}{A}, & \text{if } A \neq 0. \\ \text{indeterminate}, & \text{if } A = 0. \end{cases} \tag{13.19}$$

- if $\delta\epsilon$ is assigned, Eq. 12.17,

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)}}{A + \mathbf{a}^T\mathbf{c}^{(\sigma)}}, & \text{if } A \neq -\mathbf{a}^T\mathbf{c}^{(\sigma)}, \\ \text{indeterminate}, & \text{if } A = -\mathbf{a}^T\mathbf{c}^{(\sigma)}. \end{cases} \tag{13.20}$$

where

$$\begin{aligned} \delta\boldsymbol{\sigma}^{(e)} &= \mathbf{C}^{(e)}\delta\epsilon \\ \mathbf{c}^{(\sigma)} &= \mathbf{C}^{(e)}\mathbf{b} \end{aligned}$$

and the *Plastic Modulus* A is defined as

$$\begin{aligned} A &= -\frac{1}{\delta\lambda}\left[\frac{\partial F}{\partial \overline{k}_i}\delta\overline{k}_i + \frac{\partial F}{\partial k_i}\delta k_i\right] = \\ &= -\left[\frac{\partial F}{\partial \overline{k}_i}\frac{\partial \overline{k}_i}{\partial \overline{h}_j}\overline{c}_j + \frac{\partial F}{\partial k_i}\frac{\partial k_i}{\partial \overline{h}_j}\overline{c}_j + \frac{\partial F}{\partial k_i}\frac{\partial k_i}{\partial h_j}c_j\right] \end{aligned} \tag{13.21}$$

where $i = 1, 2, \ldots, n$; $\overline{j} = 1, 2, \ldots, \overline{m}$ and $j = 1, 2, \ldots, m$.

According to the above definition of elasto-plastic deformation, the stress increment $\delta\boldsymbol{\sigma}$ resulting from strain increment $\delta\boldsymbol{\epsilon}$, starting from a material state $(\boldsymbol{\sigma}, \mathbf{k})$, can be calculated as, Eq. 12.37,

$$\delta\boldsymbol{\sigma} = \mathbf{C}\delta\boldsymbol{\epsilon} \tag{13.22}$$

where

$$\mathbf{C} = \begin{cases} \mathbf{C}^{(e)}; & \text{if a purely elastic response occurs,} \\ \mathbf{C}^{(e)} - \mathbf{C}^{(p)}; & \text{if plasticity develops.} \end{cases}$$

and

$$\mathbf{C}^{(p)} = \begin{cases} \dfrac{\mathbf{c}^{(G)}\mathbf{c}^{(F)T}}{A + \mathbf{a}^T\mathbf{c}^{(G)}}; & \text{for } A \neq -\mathbf{a}^T\mathbf{c}^{(G)}, \\ \text{indeterminate;} & \text{for } A = -\mathbf{a}^T\mathbf{c}^{(G)}, \end{cases}$$

in which

$$\mathbf{c}^{(F)} = \mathbf{C}^{(e)T}\mathbf{a}$$

The type of mechanical response is established according to the following criterion.

10. By definition, if

$$\hat{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) < 0 \tag{13.23}$$

the material response is always elastic, regardless of the applied stress increment $\delta\boldsymbol{\sigma}$ or strain increment $\delta\boldsymbol{\epsilon}$. If, instead,

$$\hat{F}(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k}) \geq 0 \tag{13.24}$$

the type of material response is established as follows, see remark in item 5, Section 13.3:

- *Stress Based Criterion.* Let $\delta\boldsymbol{\sigma}$ be a stress increment applied on any material state $(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k})$, then:

  (a) Elasto-plastic response occurs if

$$\begin{aligned} A > 0 \quad &; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} \geq 0 \\ A = 0 \quad &; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} = 0 \\ A < 0 \quad &; \quad \mathbf{a}^T\delta\boldsymbol{\sigma} = 0 \end{aligned}$$

(b) Elastic response occurs if

$$A \geq 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} < 0$$

(c) Either elastic or elasto-plastic response may occur if

$$A < 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} < 0$$

This is the only ambiguous situation which this theory of plasticity does not solve by itself.

(d) Stress increments by which

$$A \leq 0 \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma} > 0$$

are not admissible; that is, according to the theory, the material cannot sustain such type of stress increment.

- *Strain Based Criterion.* Let $\delta\boldsymbol{\epsilon}$ be a strain increment applied on any material state $(\boldsymbol{\sigma}, \overline{\mathbf{k}}, \mathbf{k})$, then:

(a) Elasto-plastic response occurs if

$$\begin{aligned} A &> -\mathbf{a}^T \mathbf{c}^{(\sigma)} \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} \geq 0 \\ A &= -\mathbf{a}^T \mathbf{c}^{(\sigma)} \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} = 0 \\ A &< -\mathbf{a}^T \mathbf{c}^{(\sigma)} \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} = 0 \end{aligned}$$

(b) Elastic response occurs if

$$A \geq -\mathbf{a}^T \mathbf{c}^{(\sigma)} \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} < 0$$

(c) Either elastic or elasto-plastic response may occur if

$$A < -\mathbf{a}^T \mathbf{c}^{(\sigma)} \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} < 0$$

This is the only ambiguous situation which this theory of plasticity does not solve by itself.

(d) Strain increments by which

$$A \leq -\mathbf{a}^T \mathbf{c}^{(\sigma)} \quad ; \quad \mathbf{a}^T \delta\boldsymbol{\sigma}^{(e)} > 0$$

are not admissible; that is, according to the theory, the material cannot sustain such a type of strain increment.

13.3 Remarks

It is important to make the following remarks on the main hypotheses listed in Section 13.2:

1. When purely elastic response occurs, the bounding surface $\overline{F}$ remains fixed in the stress space, that is

$$\overline{k}_i = \text{ constant}$$

for all $i = 1, 2, \ldots, n$. In fact, by definition, in a purely elastic response $\delta\lambda = 0$; consequently, according to the hypothesis in item 2 in Section 13.2,

$$\delta\overline{k}_i = 0$$

for all $i = 1, 2, \ldots, n$.

2. The requirements on the partial derivatives of k_i in Eqs. 13.11 and 13.12 are consequences of Eq. 13.10. In fact, when the current stress point $\boldsymbol{\sigma}$ lies on $\overline{F}$, from Eq. 13.10 it results that

$$\delta k_i = \delta\overline{k}_i \tag{13.25}$$

Hence, if plasticity occurs, Eqs. 13.6 and 13.2,

$$\frac{\partial k_i}{\partial \overline{h}_{\overline{j}}}\delta\overline{h}_{\overline{j}} + \frac{\partial k_i}{\partial h_j}\delta h_j = \frac{\partial \overline{k}_i}{\partial \overline{h}_{\overline{j}}}\delta\overline{h}_{\overline{j}}$$

Since this equality must be true for any arbitrary value of $\delta\overline{h}_{\overline{j}}$ and δh_j, the requirements in Eqs. 13.11 and 13.12 immediately follow.

3. The requirement in item 7 on the limit value for A guarantees that, if $\hat{F}$ does not always coincide with $\overline{F}$ and the stress point lies on the elastic surface, then:

   - the infinity value of A makes $\delta\lambda = 0$, Eqs. 13.19 and 13.20, and, consequently $\delta\boldsymbol{\epsilon}^{(p)} = \delta\lambda\mathbf{b} = \mathbf{o}$, Eq. 13.18. This assures the continuity of any deformative process in which the stress point crosses the elastic surface.
   - The positiveness of the A value makes admissible any stress increments applied on a material state lying on $\hat{F}$. In fact, according to the stress based criterion in item 10 in Section 13.2, the only not admissible situation may occur if $A \leq 0$, case (d).

4. The expressions of $\delta\lambda$ and A in Eqs. 13.19-13.21 may be proved noticing that the expansion of the yield surface F defined in Eq. 13.4 into the *exact* Taylor series about a material state $(\sigma, \overline{\mathbf{k}}, \mathbf{k})$ yields to

$$F(\sigma + \delta\sigma, \overline{\mathbf{k}} + \delta\overline{\mathbf{k}}, \mathbf{k} + \delta\mathbf{k}) = F(\sigma, \overline{\mathbf{k}}, \mathbf{k}) + \delta F(\sigma, \overline{\mathbf{k}}, \mathbf{k})$$

   Since by definition the material state always satisfies the yield surface F, then

$$\begin{aligned} F(\sigma + \delta\sigma, \overline{\mathbf{k}} + \delta\overline{\mathbf{k}}, \mathbf{k} + \delta\mathbf{k}) &= 0 \\ F(\sigma, \overline{\mathbf{k}}, \mathbf{k}) &= 0 \end{aligned}$$

   and, consequently, we can identify the following *consistency equation*

$$\delta F(\sigma, \overline{\mathbf{k}}, \mathbf{k}) = \mathbf{a}^T \delta\sigma + \frac{\partial F}{\partial \overline{k}_i} \delta\overline{k}_i + \frac{\partial F}{\partial k_i} \delta k_i = 0 \qquad (13.26)$$

   that is

$$\mathbf{a}^T \delta\sigma - A\delta\lambda = 0 \qquad (13.27)$$

   where

$$A = -\frac{1}{\delta\lambda}\left[\frac{\partial F}{\partial \overline{k}_i}\delta\overline{k}_i + \frac{\partial F}{\partial k_i}\delta k_i\right] \qquad (13.28)$$

   The consistency equation in Eq. 13.27 is formally identical with that reported in Eq. 12.21. Then, proceeding as in Section 12.5, we eventually obtain the expressions for $\delta\lambda$ in Eqs. 13.19 and 13.20.

   In the case of plastic deformation, $\delta\overline{k}_i$ and δk_i may be expressed as Eqs. 13.2 and 13.6, respectively. Thus, the expression of A in Eq. 13.28 takes the form reported in Eq. 13.21.

5. The stress and strain based criteria in item 10 are identical with those derived in Sections 12.6 and 12.7 for the case of the standard incremental theory of plasticity. However, in this generalized incremental theory, these criteria are given as *definitions*. In fact, according to item 3 in Section 13.2, the current stress point lies always on the yield surface F; consequently, the mathematical procedure followed in Sections 12.6 and 12.7 for deriving the stress and strain criteria cannot be applied.

13.4 Recovery of the Classical Theory

It is easy to verify that the generalized theory of plasticity proposed in Section 13.2 recovers the standard incremental theory if:

- The bounding surface $\overline{F}$ coincides with the yield surface of the standard theory.
- The elastic and the bounding surfaces coincide, i.e.

  $$\hat{F}(\sigma, \overline{\mathbf{k}}) \equiv \overline{\mathbf{F}}(\sigma, \overline{\mathbf{k}})$$

  According to items 5 and 6 in Section 13.2, this is assured with the position:

  $$\hat{\mathbf{k}} = \overline{\mathbf{k}}$$

- When the current stress point lies on $\overline{F}$, the potential surface G is made to coincide with that of the standard theory.

In fact, we have that:

- Being $\hat{F} \equiv \overline{F}$, plasticity may occur only when the stress point σ lies on $\overline{F}$.
- In general, if σ lies on $\overline{F}$, it can be proved that the plastic modulus expression in Eq. 13.21 reduces to the standard form

  $$A = -\frac{1}{\delta\lambda}\frac{\partial\overline{F}}{\partial\overline{k}_i}\delta\overline{k}_i = -\frac{\partial\overline{F}}{\partial\overline{k}_i}\frac{\partial\overline{k}_i}{\partial\overline{h}_j}\overline{c}_j \tag{13.29}$$

Consequently, the elasto-plastic strain definition and the loading criteria in items 9 and 10 of Section 13.2 result to be identical with those of the standard theory.

The reduced form of A in Eq. 13.29 can be proved noticing that, when σ lies on $\overline{F}$, Eq. 13.25,

$$\delta k_i = \delta\overline{k}_i$$

Hence, we can simplify the expression of Eq. 13.21 into

$$A = -\frac{1}{\delta\lambda}\left[\frac{\partial F}{\partial\overline{k}_i} + \frac{\partial F}{\partial k_i}\right]\delta\overline{k}_i = -\left[\frac{\partial F}{\partial\overline{k}_i} + \frac{\partial F}{\partial k_i}\right]\frac{\partial\overline{k}_i}{\partial\overline{h}_j}\overline{c}_j \tag{13.30}$$

During a loading process in which the current stress point does not leave the bounding surface $\overline{F} \equiv F$, we have that

$$\delta F = \delta \overline{F}$$

that is

$$\mathbf{a}^T \delta \boldsymbol{\sigma} + \frac{\partial F}{\partial \overline{k}_i} \delta \overline{k}_i + \frac{\partial F}{\partial k_i} \delta k_i = \overline{\mathbf{a}}^T \delta \boldsymbol{\sigma} + \frac{\partial \overline{F}}{\partial \overline{k}_i} \delta \overline{k}_i$$

from which, being $\mathbf{a} \equiv \overline{\mathbf{a}}$ and $\delta k_i = \delta \overline{k}_i$, we can establish that

$$\left[\left(\frac{\partial F}{\partial \overline{k}_i} + \frac{\partial F}{\partial k_i} \right) - \frac{\partial \overline{F}}{\partial \overline{k}_i} \right] \delta \overline{k}_i = 0$$

Since this equality must be true for any arbitrary value of the n increments $\delta \overline{k}_i$, it follows that

$$\frac{\partial F}{\partial \overline{k}_i} + \frac{\partial F}{\partial k_i} = \frac{\partial \overline{F}}{\partial \overline{k}_i} \tag{13.31}$$

for each $i = 1, 2, \ldots, n$. Substituting Eq. 13.31 into Eq. 13.30 we obtain the expression of A in Eq. 13.29.

13.5 A Generalized Von-Mises Strain Hardening Model

Consider a material whose uniaxial response is of the type shown in Fig. 13.1. Assume that the behavior of this material under monotonically increasing loading conditions can be modeled with an isotropic elasto-plastic constitutive equation of the type:

- Elastic strain obeying the Hooke law for isotropic linear elastic material.
- Failure condition represented by a Von-Mises surface of equation

$$Q = q - N = 0 \tag{13.32}$$

- Plastic strain obeying an associative flow rule with yield-potential function represented by a Von-Mises surface of equation

$$\overline{F} = q - \overline{q}_y = 0 \tag{13.33}$$

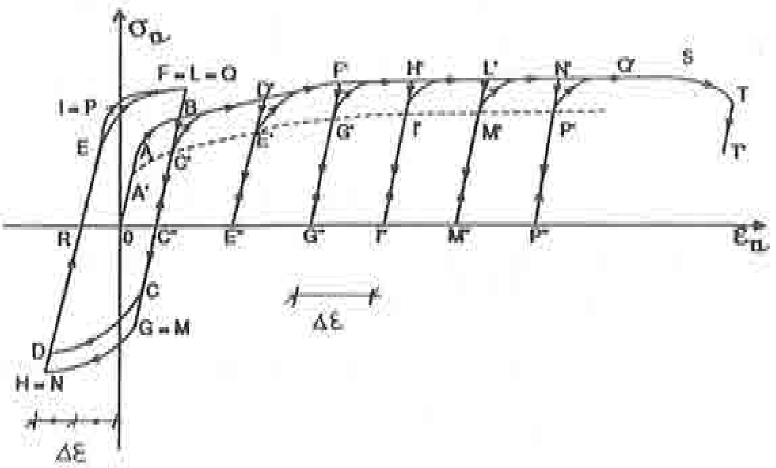


Figure 13.1: Material response under uniaxial cyclic loading

where $\overline{q}_y$ is a hardening parameter whose incremental variation is controlled by the total plastic shear strain $\epsilon_s^{(p)}$ only, that is

$$\delta\overline{q}_y = \frac{d\overline{q}_y}{d\epsilon_s^{(p)}}\delta\epsilon_s^{(p)} \tag{13.34}$$

where

$$\begin{aligned}\epsilon_s^{(p)} &= \int \delta\epsilon_s^{(p)} \\ \frac{d\overline{q}_y}{d\epsilon_s^{(p)}} &= \overline{a}_y(\epsilon_s^{(p)})\end{aligned}$$

and $\overline{a}_y(\epsilon_s^{(p)})$, which has the dimension of a stress, is a function to be determined experimentally.

In the following Section we present the extension of this simple Von-Mises strain hardening elasto-plastic model to account for the uniaxial response under cyclic load of the type in Fig. 13.1.

13.5.1 The constitutive equation

According to the recipe listed in Section 13.2, a generalization of the Von-Mises strain hardening model to account for plastic accumulation can be obtained as follows:

1. The bounding surface has equation

$$\overline{F} = \overline{F}(q, \overline{q}_y) = q - \overline{q}_y \tag{13.35}$$

   where $\overline{q}_y$ is the hardening parameter whose value is assumed to be always a positive quantity.

2. The incremental variation of the hardening parameter $\overline{q}_y$ is controlled by the total plastic shear strain $\epsilon_s^{(p)}$ only, that is

$$\delta\overline{q}_y = \frac{d\overline{q}_y}{d\epsilon_s^{(p)}}\delta\epsilon_s^{(p)} \tag{13.36}$$

   where

$$\epsilon_s^{(p)} = \int \delta\epsilon_s^{(p)} \tag{13.37}$$

$$\frac{d\overline{q}_y}{d\epsilon_s^{(p)}} = \overline{a}_y(\epsilon_s^{(p)}) \tag{13.38}$$

   and $\overline{a}_y(\epsilon_s^{(p)})$, which has the dimension of a stress, is a function to be determined experimentally.

3. The yielding surface has equation

$$F = F(q, \overline{q}_y, q_y) = q - t\overline{q}_y - q_y = 0 \tag{13.39}$$

   with the conditions

$$t = 0$$
$$0 \leq q_y \leq \overline{q}_y$$

   In practice, F does not depend on $\overline{q}_y$.

4. With regard to the hardening parameter q_y, we have that:

- From the above defined yield function equation we find that q_y can be expressed as

$$q_y = q_y(q, \overline{q}_y) = q - t\overline{q}_y \qquad (13.40)$$

where $t = 0$, that is

$$q_y = q \qquad (13.41)$$

- If plasticity occurs, the incremental variation of q_y is controlled by the variable $\epsilon_s^{(p)}$ only, that is

$$\delta q_y = \frac{dq_y}{d\epsilon_s^{(p)}} \delta\epsilon_s^{(p)} \qquad (13.42)$$

where

$$\frac{dq_y}{d\epsilon_s^{(p)}} = a_y(q_y, \epsilon_s^{(p)}) \qquad (13.43)$$

and $a_y(q_y, \epsilon_s^{(p)})$, which has the dimension of a stress, is a function to be determined experimentally. However, the mathematical framework of the model requires that, items 5 and 7 in Section 13.2,

$$a_y(q_y, \epsilon_s^{(p)}) \begin{cases} = \overline{a}_y(\epsilon_s^{(p)}); & \text{for } q_y = \overline{q}_y. \\ \to \infty; & \text{for } q_y \to \hat{q}_y. \end{cases} \qquad (13.44)$$

The parameter $\hat{q}_y$ is defined in item 6.

5. It is easy to verify that:

- the space region bounded by F is a subspace of $\overline{F}$, that is

$$F(q, \overline{q}_y, q_y) \subseteq \overline{F}(q, \overline{q}_y) \qquad (13.45)$$

- if the current stress point σ lies on $\overline{F}$, then F coincides with $\overline{F}$, that is

$$F(q, \overline{q}_y, q_y) \equiv \overline{F}(q, \overline{q}_y) \qquad (13.46)$$

and

$$\begin{aligned} q_y &= \overline{q}_y \\ \frac{dq_y}{d\epsilon_s^{(p)}} &= \frac{d\overline{q}_y}{d\epsilon_s^{(p)}} \end{aligned}$$

6. The elastic surface $\widehat{F}$ has equation

$$\widehat{F} = \widehat{F}(q, \overline{q}_y) = q - \widehat{q}_y \tag{13.47}$$

where

$$\widehat{q}_y = \widehat{a}_y(\overline{q}_y) \tag{13.48}$$

and $\widehat{a}_y(\widehat{q}_y)$, which has the dimension of a stress, is a function to be determined experimentally. Notice that, according to item 6 in Section 13.2, $\widehat{F}$ is derived from F as

$$\widehat{F}(q, \overline{q}_y) \equiv F(q, \overline{q}_y, q_y = \widehat{q}_y)$$

Moreover, we require

$$\widehat{a}_y(\overline{q}_y) > 0$$

for any $\overline{q}_y$, so that for $\boldsymbol{\sigma} = \mathbf{o}$ the mechanical response is purely elastic.

7. It will be proved later in this Section that, according to the definition of F and assuming associative flow rule as stated in the next item 8, the plastic modulus results to be given by

$$A = \frac{dq_y}{d\epsilon_s^{(p)}} \tag{13.49}$$

Note that, if the experimental function in Eq. 13.43 respect the conditions in Eq. 13.44, then for $q_y \to \widehat{q}_y$

$$A \to +\infty$$

as required in item 7 in Section 13.2.

8. The potential function for plastic deformations coincides with the yield surface, associative flow rule, that is

$$G \equiv F(q, \overline{q}_y, q_y) = q - t\overline{q}_y - q_y = 0 \tag{13.50}$$

with $t = 0$; in practice G does not depend on $\overline{q}_y$.

9. According to the definition in item 9 in Section 13.2, the infinitesimal strain increment $\delta\epsilon$ is given by

$$\delta\epsilon = \delta\epsilon^{(e)} + \delta\epsilon^{(p)} \tag{13.51}$$

where:

- the elastic (fully recoverable) strain increment can be calculated according to the Hooke Law for Isotropic material, Eq. 10.68,

$$\delta\epsilon^{(e)} = \left(\mathbf{C}^{(e)}\right)^{-1}\delta\sigma \tag{13.52}$$

where the elastic stiffness matrix $\mathbf{C}^{(e)}$ is a constant symmetric positive definite matrix function of two constants E and ν to be determined experimentally;

- the plastic (irreversible) strain increment can be calculated as

$$\delta\epsilon^{(p)} = \delta\lambda\mathbf{b} \tag{13.53}$$

where, according to the above listed hypotheses, it results that

$$\mathbf{b} \equiv \mathbf{a} = \frac{\partial F}{\partial \sigma} = \frac{3}{2q}\mathbf{s} \tag{13.54}$$

$$A = a_y(q_y, \epsilon_s^{(p)}) \tag{13.55}$$

and the value of $\delta\lambda$ can then be calculated as reported in item 9 in Section 13.2.

According to the above definition of elasto-plastic deformation, the stress increment $\delta\sigma$ resulting from strain increment $\delta\epsilon$, starting from a material state $(\sigma, \mathbf{k})$, can be calculated as, Eq. 13.22,

$$\delta\sigma = \mathbf{C}\delta\epsilon \tag{13.56}$$

where

$$\mathbf{C} = \begin{cases} \mathbf{C}^{(e)}; & \text{if a purely elastic response occurs.} \\ \mathbf{C}^{(e)} - \mathbf{C}^{(p)}; & \text{if plasticity develops.} \end{cases}$$

and

$$\mathbf{C}^{(p)} = \begin{cases} \dfrac{\mathbf{c}^{(F)}\mathbf{c}^{(F)T}}{A + \mathbf{a}^T\mathbf{c}^{(F)}}; & \text{for } A \neq -\mathbf{a}^T\mathbf{c}^{(F)}; \\ \text{indeterminate} & \text{for } A = -\mathbf{a}^T\mathbf{c}^{(F)}; \end{cases}$$

in which

$$\mathbf{c}^{(F)} = \mathbf{C}^{(e)}\mathbf{a}$$

10. If the current stress point $\boldsymbol{\sigma}$ lies inside $\bar{F}$, that is

$$q < \bar{q}_y$$

the material response is purely elastic. Otherwise the type of mechanical response is determined according to the criteria in item 10 in Section 13.2.

The expression for $\mathbf{b}$ in Eq. 13.54 can be easily obtained by specializing the general expression of the gradient to stress invariant function in Eq. 12.46. From Eqs. 12.50-12.53, it follows that the plastic strain, its deviatoric component and invariants associated with the above described model can be calculated as

$$\begin{aligned} \delta\boldsymbol{\epsilon}^{(p)} &= \frac{3}{2q}\delta\lambda\mathbf{s} && (13.57)\\ \delta\mathbf{e}^{(p)} &= \delta\boldsymbol{\epsilon}^{(p)} && (13.58)\\ \delta\epsilon_v^{(p)} &= 0 && (13.59)\\ \delta\epsilon_s^{(p)} &= \delta\lambda && (13.60) \end{aligned}$$

and, specializing Eq. 12.54, we find that

$$\mathbf{a}^T\delta\boldsymbol{\sigma} = \delta q \qquad (13.61)$$

The expression for A in Eq. 13.55 can be then easily obtained specializing the general expression in Eq. 13.21 as

$$A = -\frac{1}{\delta\lambda}\frac{\partial F}{\partial q_y}\delta q_y = \frac{1}{\delta\lambda}\frac{dq_y}{d\epsilon_s^{(p)}}\delta\epsilon_s^{(p)} = \frac{dq_y}{d\epsilon_s^{(p)}} \qquad (13.62)$$

A complete, incremental calculation can be performed knowing the initial values of the stresses and of the hardening parameter $\bar{q}_y$. The initial value of $\epsilon_s^{(p)}$ is by definition zero. Then, at $t = 0$,

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}_o\\ \bar{q}_y &= \bar{q}_{yo}\\ \epsilon_s^{(p)} &= 0 \end{aligned}$$

At the initial time, the mechanical response due to a given stress increment $\delta\boldsymbol{\sigma}$ is calculated as indicated in item 9. Hence $\delta\epsilon_s^{(p)}$ is known and, from Eq. 13.36, the value of $\delta\bar{q}_y$ can be calculated. This allows to update $\boldsymbol{\sigma}$, $\epsilon_s^{(p)}$ and $\bar{q}_y$.

13.5.2 The uniaxial response of the model

In an uniaxial stress test, the material sample is subjected to a vertical uniform stress field σ_a. In this case, therefore, the stress vector field is given by

$$\sigma = \sigma_a \{1,0,0,0,0,0,0,0,0\}^T \qquad (13.63)$$

from which it follows that

$$s = \sigma - \widehat{m}\frac{\sigma_{kk}}{3} = \frac{\sigma_a}{3}\{2,-1,-1,0,0,0,0,0,0\}^T \qquad (13.64)$$

$$q = \left(\frac{3}{2}s^T s\right)^{1/2} = |\sigma_a| \qquad (13.65)$$

Under a uniform axial stress field, the model presented in Section 13.5.1 predicts a strain vector field

$$\delta\epsilon = \delta\epsilon^{(e)} + \delta\epsilon^{(p)} \qquad (13.66)$$

where, Eqs. 13.52 and 13.57,

$$\delta\epsilon^{(e)} = \left(C^{(e)}\right)^{-1}\delta\sigma = \frac{\delta\sigma_a}{E}\{1,-\nu,-\nu,0,0,0,0,0,0\}^T \qquad (13.67)$$

$$\delta\epsilon^{(p)} = \frac{3}{2q}\delta\lambda s = \begin{cases} \mathbf{o}; & \text{for pure elastic response} \\ \frac{\delta\sigma_a}{A}\left\{1,-\frac{1}{2},-\frac{1}{2},0,0,0,0,0,0\right\}^T; & \text{if } A \neq 0 \\ \text{indeterminate}; & \text{if A} = 0 \end{cases} \qquad (13.68)$$

and,

$$A = a_y(|\sigma_a|, \epsilon_s^{(p)})$$
$$\epsilon_s^{(p)} = \int \left|\delta\epsilon_a^{(p)}\right| \qquad (13.69)$$

In fact, according to Eqs. 13.19, 13.61 and 13.65, for $A \neq 0$ and $\sigma_a \neq 0$,

$$\delta\lambda = \frac{a^T\delta\sigma}{A} = \frac{\delta q}{A} = \frac{\sigma_a}{|\sigma_a|}\frac{\delta\sigma_a}{A} \qquad (13.70)$$

Moreover A is given by, Eqs. 13.55, 13.41, and 13.65,

$$A = a_y(q, \epsilon_s^{(p)}) = a_y(|\sigma_a|, \epsilon_s^{(p)})$$

In order to prove the expression for $\epsilon_s^{(p)}$ in Eq. 13.69, we observe that,

$$\delta \mathbf{e}^{(p)} = \delta \boldsymbol{\epsilon}^{(p)} - \frac{\delta \epsilon_v^{(p)}}{3} = \delta \boldsymbol{\epsilon}^{(p)}$$

being, Eq. 13.59, $\delta \epsilon_v^{(p)} = 0$. Then,

$$\delta \epsilon_s^{(p)} = \left(\frac{2}{3} \delta \mathbf{e}^{(p)T} \delta \mathbf{e}^{(p)} \right)^{1/2} = \left(\frac{2}{3} \delta \boldsymbol{\epsilon}^{(p)T} \delta \boldsymbol{\epsilon}^{(p)} \right)^{1/2}$$

which specialized according to the uniaxial response in Eq. 13.68 yields to

$$\delta \epsilon_s^{(p)} = |\delta \epsilon_a^{(p)}| \tag{13.71}$$

Finally, note that, when plasticity occurs, it results, Eq. 13.68,

$$A = a_y(|\sigma_a|, \epsilon_s^{(p)}) = \frac{\delta \sigma_a}{\delta \epsilon_a^{(p)}} \tag{13.72}$$

13.5.3 Experimental determination of the material properties

The constitutive model in Section 13.5.2 requires the experimental determination of the following material properties:

- the constant elastic parameters E and ν;
- the plastic functional relationships, Eqs. 13.38, 13.43 and 13.48,

$$\begin{aligned} \overline{a}_y(\epsilon_s^{(p)}) &= \frac{d\overline{q}_y}{d\epsilon_s^{(p)}} \\ a_y(q_y, \epsilon_s^{(p)}) &= \frac{dq_y}{d\epsilon_s^{(p)}} \\ \hat{a}_y(\overline{q}_y) &= \hat{q}_y \end{aligned}$$

All these material properties can be determined from the simple uniaxial test shown in Fig. 13.1 as follows:

1. We identify in the linear paths in Fig. 13.1 below the dashed line A'C'E'G'I'M'P' the purely elastic response of the material.

2. With reference to these linear elastic paths and the elastic relationships in Eq. 13.67, we can calculate the elastic material constants as

$$E = \frac{\delta\sigma_a}{\delta\epsilon_a}$$
$$\nu = \frac{1}{2}\left(1 - E\frac{\delta\epsilon_v}{\delta\sigma_a}\right)$$

3. We draw the diagram σ_a vs. $\epsilon_a^{(p)}$ in Fig. 13.2 scaling the abscissa in Fig. 13.1 by the elastic strain, that is

$$\epsilon_a^{(p)} = \epsilon_a - \epsilon_a^{(e)} = \epsilon_a - \frac{\sigma_a}{E}$$

We note that, according to Eq. 13.69, for $\sigma_a > 0$, it results, Fig. 13.2,

$$\epsilon_s^{(p)} = \int \left|\delta\epsilon_a^{(p)}\right| = \epsilon_a^{(p)}$$

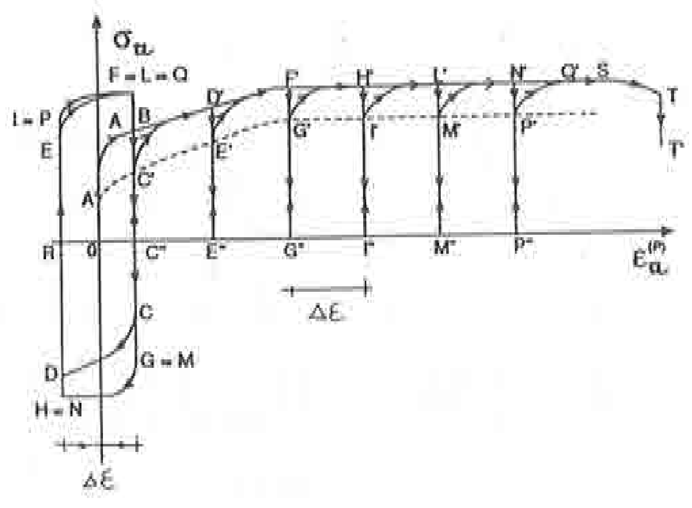


Figure 13.2: The uniaxial material response scaled by the elastic strain

4. We identify in the stress path ABD'F'H'L'N'Q'ST in Fig. 13.2 the plastic response of the material when the stress point lies on the bounding surface; then:

   - According to Eq. 13.35, along this path $\sigma_a = |\sigma_a| = q = \overline{q}_y$.
   - According to Eqs. 13.44 and 13.72, the functional relationship $\overline{a}_y(\epsilon_s^{(p)})$ can be identified by interpolating, with an opportune function, various pairs

   $$(\overline{a}_y(\epsilon_s^{(p)}) = \sigma', \epsilon_s^{(p)} = \epsilon_a^{(p)})$$

   where
   $$\sigma' = \frac{d\sigma_a}{d\epsilon_a^{(p)}}$$
   is the slope of the stress path measured at $\epsilon_a^{(p)}$.

5. We identify in the dashed line A'C'E'G'I'M'P' in Fig. 13.2 the boundary for pure elastic deformation; the functional relationship $\widehat{a}_y(\overline{q}_y)$ can be identified by interpolating, with an opportune function, the pairs

$$(\widehat{a}_y(\overline{q}_y) = \sigma_a^{C'}, \overline{q}_y = \sigma_a^{B})$$
$$(\widehat{a}_y(\overline{q}_y) = \sigma_a^{E'}, \overline{q}_y = \sigma_a^{D'})$$
$$\vdots$$

6. We identify in the stress paths within the continuous line ABD'F'H'L'N'Q'ST and the dashed line A'C'E'G'I'M'P' in Fig. 13.2 the plastic response of the material when the stress point lies inside the bounding surface; then:

   - according to Eqs. 13.41 and 13.65 along this path $q_y = q = |\sigma_a| = \sigma_a$;
   - the functional relationship $a_y(q_y, \epsilon_s^{(p)})$ can be identified by interpolating, with an opportune function, various pairs

   $$(a_y(q_y, \epsilon_s^{(p)}) = \sigma', q_y = \sigma_a, \epsilon_s^{(p)} = \epsilon_a^{(p)})$$

   where
   $$\sigma' = \frac{d\sigma_a}{d\epsilon_a^{(p)}}$$
   is the slope of the stress path, for example C'D', measured at $\epsilon_a^{(p)}$.

13.5.4 Note on the hysteretic loop

We recall that the σ_a vs. $\epsilon_a^{(p)}$ graph shown in Fig. 13.2 is obtained scaling the elastic strain component from the assumed material behavior in Fig. 13.1, item 3 in Subsection 13.5.3.

In particular, the hysteretic loop sketched in Fig. 13.2 presents the following geometrical characteristics:

- The paths C"CD, C"GH and C"MN result symmetric to the paths C"C'D', G"G'H' and M"M'N', with respect to the $\epsilon_a^{(p)}$ axis.
- The paths REF,RIL and RPQ are identical to the paths E"E'F', I"I'L' and P"P'Q', with an opportune shift.

It is possible to prove that the proposed model is able to reproduce exactly such type of response.

In fact, we note that, according to Eq. 13.72, the slope of the material plastic response depends only on the absolute value of the uniaxial stress σ_a and plastic strain $\epsilon_a^{(p)}$, namely

$$\frac{\delta\sigma_a}{\delta\epsilon_a^{(p)}} = A = a_y(q_y, \epsilon_s^{(p)})$$

where

$$\begin{aligned} q_y &= q = |\sigma_a| \\ \epsilon_s^{(p)} &= \int \delta\epsilon_s^{(p)} = \int |\delta\epsilon_a^{(p)}| \end{aligned}$$

Thus, for example, the q_y and $\epsilon_s^{(p)}$ values at the point C' are equal to those at the point C. This implies that the slope at the point C and C' coincide and consequently the path CD has to result symmetric to the path C'D'.

Chapter 14

Field Equations for Solids and Fluids

14.1 Introduction

The final aim of the continuum mechanics theory is to establish a complete set of equations, known as *field equations*, which allows to describe the motion and the deformation of a continuum subjected to a set of external agents. The diagram in Fig. 14.1 sketches the main lines of the continuum mechanic mathematical developments presented in the previous Chapters:

- Chapter 6 describes the motion and the deformation mobilized by an applied displacement field;
- Chapter 7 describes the stress mobilized by an applied set of external forces;
- Chapter 8 describes the overall mechanical principles relating the input power due to the external forces and the induced stress power in the interior of the continuum;
- Chapter 9 presents the thermodynamic principles that a real continuum has been experimented to obey;
- finally, Chapters 10-13 report a number of alternative constitutive equations relating stress and strain which may used to describe the mechanical behavior of different types of continuum.

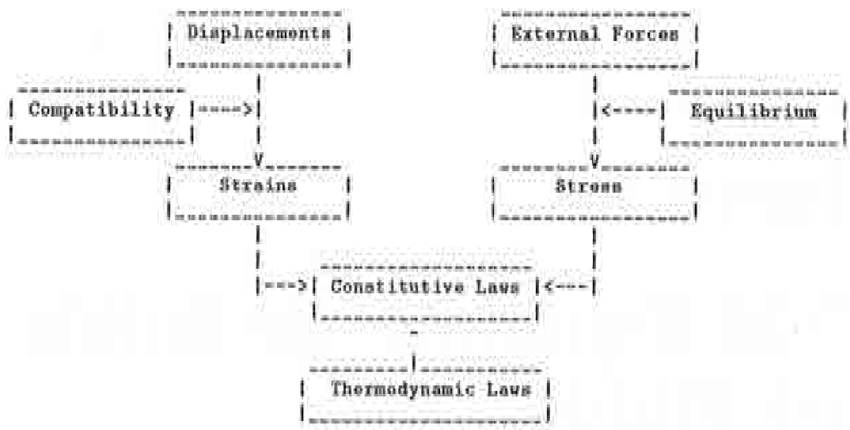


Figure 14.1: Interrelationships of variables in the solution of a continuum mechanic problem

In this Chapter we collect from each of the above listed Chapters the necessary relationships for setting up a complete system of equations for the solution of a one-phase solid or fluid continuum mechanic problem.

14.2 Field Equations for Solids

It is customary to solve solid problems by means of the Lagrangian formulation. In terms of such type of formulation, the complete set of field equations governing the purely mechanical respond of a solid is given by:

1. the 3 Cauchy Equations of Motion in the undeformed configuration, Eq. 7.62,

$$\sigma^{(o)}_{ji,j} + \varrho_o b^{(o)}_i = \varrho_o \frac{\partial^2 x_i}{\partial t^2} \tag{14.1}$$

2. The 9 Geometric Equations relating the 1st and 2nd Piola-Kirchhoff stress tensor components, Eq. 7.39,

$$\sigma^{(o)}_{ji} = \tilde{\sigma}_{jk} x_{i,k} \tag{14.2}$$

3. the 6 Constitutive Equations for Solids, Chapter 10,

$$\tilde{\sigma}_{ij} = f(\tilde{\epsilon}_{ij}) \tag{14.3}$$

Note that the above equation indicates 9 relationships but only 6 are linear independent because of the symmetry in $\tilde{\sigma}_{ij}$ and $\tilde{\epsilon}_{ij}$.

4. the 6 Geometric Equations for the Finite Lagrange Strain tensor components, Section 6.3.1,

$$\tilde{\epsilon}_{ij} = \frac{1}{2}(x_{k,i}x_{k,j} - \delta_{ij}) \tag{14.4}$$

Note that the above equation indicates 9 relationships but only 6 are linear independent because of the symmetry.

This set of equations represents a complete system of 24 equations into the following 24 unknowns:

- the current position components x_i, 3 unknowns;
- the finite Lagrange strain tensor components $\tilde{\epsilon}_{ij}$, 6 unknowns;
- the 1st Piola-Kirchhoff stress tensor components $\sigma_{ij}^{(o)}$, 9 unknowns;
- the 2st Piola-Kirchhoff stress tensor components $\tilde{\sigma}_{ij}$, 6 unknowns.

14.2.1 Linearized theory of elasticity

The Linearized Theory of Elasticity assumes that:

- The deformation process may be approximated by means of the Small Deformation Theory, Section 6.6, according to which we may set

$$\begin{aligned} \varrho_o &\approx \varrho \\ b_i^{(o)} &\approx b_i \\ \sigma_{ij}^{(o)} &\approx \tilde{\sigma}_{ij} \approx \sigma_{ij} \\ \tilde{\epsilon}_{ij} &\approx \epsilon_{ij} \end{aligned}$$

  where σ_{ij} is a component of the symmetric Cauchy stress tensor and ϵ_{ij} is a component of the symmetric linear Lagrange strain tensor.

- The solid behaves according to the Elastic Hooke's Law, Section 10.9.

Under the above hypothesis and recalling that

$$x_i = X_i + u_i$$

where X_i is the initial location and u_i is the deformation function, we can reduce the system of equations in Eqs. 14.1-14.4 into the following system of equations:

1. the 3 Cauchy Equations of Motion

$$\sigma_{ji,j} + \varrho b_i = \varrho \frac{\partial^2 u_i}{\partial t^2} \tag{14.5}$$

2. the 6 Hooks Law for Elastic Solids

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \tag{14.6}$$

   Note that the above equation indicates 9 relationships but only 6 are linear independent because of the symmetry in σ_{ij} and ϵ_{ij}.

3. the 6 Geometric Equations for the Linear Lagrange Strain Tensor components

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{14.7}$$

   Note that the above equation indicates 9 relationships but only 6 are linear independent because of the symmetry.

This set of equations represents a complete system of 15 equations into the following 15 unknowns:

- the displacement components u_i, 3 unknowns;
- the linear Lagrange strain tensor components ϵ_{ij}, 6 unknowns;
- the Cauchy stress tensor components σ_{ij}, 6 unknowns.

It is easily verified that combining the above set of equations we obtain the following 3 *Navier Equations of Motion* for linearized elasticity

$$(\lambda + \mu) u_{k,ik} + \mu u_{i,kk} + \varrho b_i = \varrho \frac{\partial^2 u}{\partial t^2} \tag{14.8}$$

in the only three displacement unknowns u_i.

14.2.2 Nonlinear solid and small deformation theory

If the deformation process can still be approximated according to the Small Deformation Theory, Section 6.6, but the material behaves according to a nonlinear hypoelastic or elasto-plastic constitutive relationship, Chapters 11-13, then the relative solution can be obtained from the following modified form of the set of equations in Eqs. 14.5-14.7

1. the 3 (incremental) Cauchy Equations of Motion

$$\delta\sigma_{ji,j} + \varrho\delta b_i = \varrho\frac{\partial^2 \delta u_i}{\partial t^2} \tag{14.9}$$

2. the 6 Incremental Constitutive Relationships

$$\delta\sigma_{ij} = f(\delta\epsilon_{ij}) \tag{14.10}$$

   Note that the above equation indicates 9 relationships but only 6 are linear independent because of the symmetry in σ_{ij} and ϵ_{ij}.

3. the 6 Geometric Equations for the Linear Lagrange Strain Tensor components

$$\delta\epsilon_{ij} = \frac{1}{2}(\delta u_{i,j} - \delta u_{j,i}) \tag{14.11}$$

   Note that the above equation indicates 9 relationships but only 6 are linear independent because of the symmetry.

This set of equations represents a complete system of 15 equations into the following 15 incremental unknowns:

- the displacement components δu_i, 3 unknowns;
- the linear Lagrange strain tensor components $\delta\epsilon_{ij}$, 6 unknowns;
- the Cauchy stress tensor components $\delta\sigma_{ij}$, 6 unknowns.

14.3 Field equations for Newtonian Fluids

It is customary to solve fluid problems by means of the Eulerian formulation. According to such formulation, the complete set of field equations governing the flow of a Newtonian fluid, Section 10.10, is given by:

1. the 3 Cauchy Equations of Motion, Eq. 7.55,

$$\sigma_{ji,j} + \varrho b_i = \varrho \frac{dv_i}{dt} \tag{14.12}$$

2. the 6 Constitutive Equations for Newtonian Fluid, Eq. 10.89,

$$\sigma_{ij} = -\pi\delta_{ij} + \lambda d_{kk}\delta_{ij} + 2\mu d_{ij} \tag{14.13}$$

   Note that the above equation indicates 9 relationships but only 6 are linear independent because of the symmetry in σ_{ij} and d_{ij};

3. the 6 Geometric Equations for the Euler Strain Rate Tensor components, Section 6.4.2,

$$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}) \tag{14.14}$$

   Note that the above equation indicates 9 relationships but only 6 are linear independent because of the symmetry;

4. the Euler form of the Continuity Equation, Eq. 7.11,

$$\frac{d\varrho}{dt} + \varrho v_{k,k} = 0 \tag{14.15}$$

5. the Kinetic Equation of State, Eq. 9.142,

$$\pi = \pi(\varrho, \vartheta) \tag{14.16}$$

6. the Caloric Equation of State, Eq. 9.143,

$$u = u(\varrho, \vartheta) \tag{14.17}$$

7. the Energy Equation, Eq. 9.12,

$$\varrho \frac{du}{dt} = \dot{w} - c_{i,i} + \varrho r \tag{14.18}$$

   where

$$\dot{w} = \sigma_{ij} d_{ij}$$

8. the 3 Fourier Relationships of Heat Conduction, Eq. 9.5,

$$c_i = -K\vartheta_{,i} \tag{14.19}$$

This set of equations represents a complete system of 22 equations into the following 22 unknowns:

- the velocity components v_i, 3 unknowns;
- the Euler strain rate stress tensor components d_{ij}, 6 unknowns;
- the Cauchy stress tensor components σ_{ij}, 6 unknowns;
- the thermodynamic pressure π, 1 unknown;
- the density ϱ, 1 unknown;
- the internal energy u, 1 unknown;
- the heat flux components c_i, 3 unknowns;
- the absolute temperature ϑ, 1 unknown.

It can be immediately verified that the above set of equations can be equivalently reduced to the following set of equations:

1. the 3 Generalized Navier-Stokes Equations of Motion for (Newtonian) Fluids
$$-\pi_{,j} + (\lambda + \mu)v_{i,ij} + \mu v_{j,ii} + \varrho b_j = \varrho \frac{dv_j}{dt} \tag{14.20}$$

2. the Euler form of the Continuity Equation
$$\frac{d\varrho}{dt} + \varrho v_{k,k} = 0 \tag{14.21}$$

3. the Kinetic Equation of State
$$\pi = \pi(\varrho, \vartheta) \tag{14.22}$$

4. the Caloric Equation of State
$$u = u(\varrho, \vartheta) \tag{14.23}$$

5. the Energy Equation
$$\varrho \frac{du}{dt} = \dot{w} + K\vartheta_{,ii} + \varrho r \tag{14.24}$$

where, Eq. 10.92,

$$\dot{w} = -\pi d_{kk} + \kappa d_{kk}^2 + 2\mu e_{ij} e_{ij}$$

and

$$\begin{aligned} \kappa &= \lambda + \frac{2}{3}\mu \\ d_{ij} &= \frac{1}{2}(v_{i,j} + v_{j,i}) \\ e_{ij} &= d_{ij} - \delta_{ij}\frac{d_{kk}}{3} \end{aligned}$$

This set represents a complete system of 7 equations into the following 7 unknowns:

- the velocity components v_i, 3 unknowns;
- the thermodynamic pressure π, 1 unknown;
- the density ϱ, 1 unknown;
- the internal energy u, 1 unknown;
- the absolute temperature ϑ, 1 unknown.

We recall that according to the Eulerian formulation, Section 6.4.7, the acceleration term may be alternatively expressed as

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \mathcal{V}\mathbf{v} = \tag{14.25}$$

$$= \frac{\partial \mathbf{v}}{\partial t} + 2\mathcal{W}\mathbf{v} + \nabla\frac{v^2}{2} = \tag{14.26}$$

$$= \frac{\partial \mathbf{v}}{\partial t} + 2\mathbf{q} \times \mathbf{v} + \nabla\frac{v^2}{2} \tag{14.27}$$

where

$$\begin{aligned} \mathcal{V} &= [v_{i,j}] \\ 2\mathcal{W} &= [v_{i,j} - v_{j,i}] \\ q_j &= -\frac{1}{4}\epsilon_{ijk}(v_{j,k} - v_{k,j}) = -\frac{1}{2}\epsilon_{ijk}\mathcal{W}_{jk} \end{aligned}$$

14.3.1 Steady Flow

A flow is said to be *steady* if all flow variables are independent of time, so that

$$\frac{\partial f}{\partial t} = 0 \tag{14.28}$$

and, consequently,

$$\frac{df}{dt} = f_{,k} v_k \tag{14.29}$$

where f represents, for example, a velocity component v_i, the density ϱ or the internal energy u. It is clear that under the above condition of steady flow, the set of equations in Eqs. 14.20-14.24 can be equivalently reduced to the following set of equations:

1. the 3 Generalized Navier-Stokes Equations of Motion for (Newtonian) Fluids

$$-\pi_{,j} + (\lambda + \mu) v_{i,ij} + \mu v_{j,ii} + \varrho b_j = \varrho v_{j,k} v_k \tag{14.30}$$

2. the Euler form of the Continuity Equation

$$\varrho_{,k} v_k + \varrho v_{k,k} = 0 \tag{14.31}$$

that is

$$(\varrho v_k)_{,k} = 0 \tag{14.32}$$

3. the Kinetic Equation of State

$$\pi = \pi(\varrho, \vartheta) \tag{14.33}$$

4. the Caloric Equation of State

$$u = u(\varrho, \vartheta) \tag{14.34}$$

5. the Energy Equation

$$\varrho u_{,k} v_k = \dot{w} + K \vartheta_{,ii} + \varrho r \tag{14.35}$$

where

$$\dot{w} = -\pi d_{kk} + \kappa d_{kk}^2 + 2\mu e_{ij} e_{ij}$$

and

$$\begin{aligned} \kappa &= \lambda + \frac{2}{3}\mu \\ d_{ij} &= \frac{1}{2}(v_{i,j} + v_{j,i}) \\ e_{ij} &= d_{ij} - \delta_{ij}\frac{d_{kk}}{3} \end{aligned}$$

This set represents a complete system of 7 equations into the following 7 unknowns:

- the velocity components v_i, 3 unknowns;
- the thermodynamic pressure π, 1 unknown;
- the density ρ, 1 unknown;
- the internal energy u, 1 unknown;
- the absolute temperature ϑ, 1 unknown.

14.3.2 Irrotational Flow

A flow is said to be *irrotational* if there exists a potential function Φ for $\mathbf{v}$ such that

$$v_i = -\Phi_{,i} \tag{14.36}$$

and, consequently, the acceleration term in Eq. 14.20 reduces to, Eq. 14.27,

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \nabla \frac{v^2}{2} \tag{14.37}$$

since in this case

$$\mathcal{W} = [v_{i,j} - v_{j,i}] = [-\Phi_{,ij} + \Phi_{,ji}] = 0$$

Moreover, if the body forces are either neglegible or conservative, i.e. if they may be also expressed as the gradient of a potential function, it is possible to reduce to 5 the number of unknowns and of equations in Eqs. 14.20-14.24. This is because the 3 Equations of Motions may be reduced in a scalar function and the 3 unknowns v_i may be replaced by the only unknown Φ. Examples of such a type of reduction are reported in the next Sections.

14.3.3 Barotropic flows

A flow is said to be *barotropic* if the Kinetic Equation of State is independent of the temperature, that is

$$\pi = \pi(\varrho) \tag{14.38}$$

or, conversely,

$$\varrho = \varrho(\pi) \tag{14.39}$$

Example of Barotropic Equations of State can be found in the following thermodynamic processes:

- Isothermal flow where, by definition, $\theta = const$.
- Incompressible flow where, Section 14.5, it results $\varrho = \varrho_o$.
- Reversible adiabatic flow in a perfect gas where, it may be shown that,

$$\frac{p}{\varrho^\gamma} = const \tag{14.40}$$

  where

$$\gamma = \frac{c_p}{c_v} = 1 + \frac{R}{c_v}$$

  and c_p and c_v are the specific heats at constant pressure and constant volume, respectively, while R is the gas constant for the particular gas. In an adiabatic expansion of dry air $\gamma \approx 1.4$.

It is easy to verify that the barotropic condition allows to decouple the set of equations in Eqs. 14.20-14.24 into the following two independent sets of equations:

1. The first set includes only mechanical quantities and is given by:

   (a) the 3 Generalized Navier-Stokes Equations of Motion for (Newtonian) Fluids

$$-\pi_{,j} + (\lambda + \mu)v_{i,ij} + \mu v_{j,ii} + \varrho b_j = \varrho \frac{dv_j}{dt} \tag{14.41}$$

   (b) the Euler form of the Continuity Equation

$$\frac{d\varrho}{dt} + \varrho v_{k,k} = 0 \tag{14.42}$$

(c) the Kinetic Equation of State

$$\pi = \pi(\varrho) \tag{14.43}$$

This set represents a complete system of 5 equations into the following 5 unknowns:

- the velocity components v_i, 3 unknowns;
- the thermodynamic pressure π, 1 unknown;
- the density ϱ, 1 unknown.

2. The second set is given by:

(a) the Caloric Equation of State

$$u = u(\varrho, \vartheta) \tag{14.44}$$

(b) the Energy Equation

$$\varrho \frac{du}{dt} = \dot{w} + K\vartheta_{,kk} + \varrho r \tag{14.45}$$

where

$$\dot{w} = -\pi d_{kk} + \kappa d_{kk}^2 + 2\mu e_{ij} e_{ij}$$

and

$$\begin{aligned} \kappa &= \lambda + \frac{2}{3}\mu \\ d_{ij} &= \frac{1}{2}(v_{i,j} + v_{j,i}) \\ e_{ij} &= d_{ij} - \delta_{ij}\frac{d_{kk}}{3} \end{aligned}$$

Once the first set is solved in terms of $\mathbf{v}$, π and ϱ, the second set represents a complete system of 2 equations into the following 2 unknowns:

- the internal energy u, 1 unknown;
- the absolute temperature ϑ, 1 unknown.

14.4 Perfect Fluids

We recall that, Section 10.10, that a fluid is said to be *perfect* if

$$\lambda = \mu = 0 \tag{14.46}$$

and, under this conditions, it results

$$\sigma_{ij} = \sigma_{ij}^{(n)} = -\delta_{ij}\pi \tag{14.47}$$

$$p = -\frac{\sigma_{kk}}{3} = -\pi \tag{14.48}$$

$$w = -\pi d_{kk} \tag{14.49}$$

It can be verified that for a perfect fluid the set of equations in Eqs. 14.20-14.24 can be reduced to the following set of equations:

1. the 3 Euler Equations of Motion for (Ideal) Fluids

$$-\pi_{,j} + \varrho b_j = \varrho \frac{dv_j}{dt} \tag{14.50}$$

2. the Euler form of the Continuity Equation

$$\frac{d\varrho}{dt} + \varrho v_{k,k} = 0 \tag{14.51}$$

3. the Kinetic Equation of State

$$\pi = \pi(\varrho, \vartheta) \tag{14.52}$$

4. the Caloric Equation of State

$$u = u(\varrho, \vartheta) \tag{14.53}$$

5. the Energy Equation

$$\varrho \frac{du}{dt} = -\pi v_{k,k} + K\vartheta_{,ii} + \varrho r \tag{14.54}$$

This set represents a complete system of 7 equations into the following 7 unknowns:

- the velocity components v_i, 3 unknowns;
- the thermodynamic pressure π, 1 unknown;
- the density ϱ, 1 unknown;
- the internal energy u, 1 unknown;
- the absolute temperature ϑ, 1 unknown.

14.4.1 Barotropic flows

In the case of barotropic flow, Section 14.3.3, the set of equations in Eqs. 14.50-14.54 governing the flow of a perfect fluid can be equivalently decoupled into the following two independent sets of equations:

1. The first one includes only mechanical quantities and is given by:

   (a) the 3 Euler Equation of Motion for (Ideal) Fluids

   $$-\pi_{,j} + \varrho b_j = \varrho \frac{dv_j}{dt} \qquad (14.55)$$

   (b) the Euler form of the Continuity Equation

   $$\frac{d\varrho}{dt} + \varrho v_{k,k} = 0 \qquad (14.56)$$

   (c) the Kinetic Equation of State

   $$\pi = \pi(\varrho) \qquad (14.57)$$

   This set represents a complete system of 5 equations into the following 5 unknowns:

   - the velocity components v_i, 3 unknowns;
   - the thermodynamic pressure π, 1 unknown;
   - the density ϱ, 1 unknown.

2. The second set is given by:

   (a) the Caloric Equation of State

   $$u = u(\varrho, \vartheta) \qquad (14.58)$$

   (b) the Energy Equation

   $$\varrho \frac{du}{dt} = -\pi v_{k,k} + K\vartheta_{,kk} + \varrho r \qquad (14.59)$$

   Once the first set is solved in terms of $\mathbf{v}$, π and ϱ, the second set represents a complete system of 2 equations into the following 2 unknowns:

   - the internal energy u, 1 unknown;
   - the absolute temperature ϑ, 1 unknown.

14.4.2 The Bernoulli equation

Consider the flow of perfect fluid in which it is possible to assume that:

- Barotropic flow, Section 14.3.3, so that we can define a pressure function

$$\mathcal{P}(\pi) = \int_{\pi_o}^{\pi} \frac{d\pi}{\varrho(\pi)} \tag{14.60}$$

  such that

$$\pi_{,i} = \varrho \mathcal{P}_{,i} \tag{14.61}$$

- Conservative body forces, i.e. there exists a potential function $\mathcal{B}$ for the body forces $\mathbf{b}$ such that

$$b_i = -\mathcal{B}_{,i} \tag{14.62}$$

It is immediately verified that under the above hypothesis the Euler Equation in Eq. 14.50 results into

$$-(\mathcal{P} + \mathcal{B})_{,i} = \frac{dv_i}{dt} \tag{14.63}$$

which shows that, in this case, the acceleration is a gradient of a potential. The integration of this equation along a streamline yields to, *Bernoulli Equation*,

$$\mathcal{P} + \mathcal{B} + \frac{v^2}{2} + \int \frac{\partial v}{\partial t} dx_i = C(t) \tag{14.64}$$

In fact, let dx_i be the i-th incremental component of the displacement along a streamline. Taking the scalar product of this increment with Eq. 14.63 and integrating, we obtain

$$-\left[\int (\mathcal{P}_{,i} + \mathcal{B}_{,i}) dx_i + C(t)\right] = \int \left(\frac{\partial v_i}{\partial t} + v_j v_{i,i}\right) dx_i$$

Since $\mathcal{P}_{,i} dx_i = d\mathcal{P}$ and $\mathcal{B}_{,i} dx_i = d\mathcal{B}$, the left-hand-side of this equation can be integrated at once. Along a streamline we can express $dx_i = (v_i/v)ds$, where ds is the increment of distance; thus

$$v_j v_{i,j} dx_i = v_j v_{i,j}(v_i/v) ds = v_i v_{i,j}(v_j/v) ds = v_i v_{i,j} dx_j = v_i dv_i$$

and, consequently,

$$\int v_j v_{i,j} dx_i = \int v_i dv_i = \frac{1}{2} v_i v_i = \frac{1}{2} v^2$$

proving therefore Eq. 14.64.

It is obvious that, expressing the acceleration term as in Eq. 14.27, we obtain the following alternative form of the Euler Equation in Eq. 14.63

$$-\nabla\left(\mathcal{P}+\mathcal{B}+\frac{v^2}{2}\right)=\frac{\partial \mathbf{v}}{\partial t}+2\mathbf{q}\times\mathbf{v} \tag{14.65}$$

We notice that this Euler Equation and the Bernoulli Equation in Eq. 14.64 may be further specialized as follows:

- If the flow is steady, so that $\partial\mathbf{v}/\partial t=0$, we have that the Euler and the Bernoulli Equations reduce to

$$-\nabla\left(\mathcal{P}+\mathcal{B}+\frac{v^2}{2}\right) = 2\mathbf{q}\times\mathbf{v} \tag{14.66}$$

$$\mathcal{P}+\mathcal{B}+\frac{v^2}{2} = C \tag{14.67}$$

  where C, known as the *Bernoulli Constant*, is a constant value along a stream line and, in general, it assumes different values along different line.

- If the flow is irrotational as well, i.e. there exists a potential function Φ for the velocity such that $\mathbf{v}=-\nabla\Phi$ and, consequently, $\mathbf{q}\times\mathbf{v}=0$, then the above Euler and Bernoulli Equations can be further reduced to

$$-\nabla\left(\mathcal{P}+\mathcal{B}+\frac{v^2}{2}\right) = 0 \tag{14.68}$$

$$\mathcal{P}+\mathcal{B}+\frac{v^2}{2} = C \tag{14.69}$$

  where, in this case, C is a constant along any streamline.

- If in addition the only body forces present is gravity, so that the potential for body forces is given by

$$\mathcal{B}=gy$$

  where g is gravitational constant and y is the elevation above some reference level, then the above Euler and Bernoulli Equations can be further reduced to

$$\nabla h = 0 \tag{14.70}$$
$$h = const \tag{14.71}$$

where

$$h = y + \int_{\pi_o}^{\pi} \frac{d\pi}{g\varrho(\pi)} + \frac{v^2}{2g}$$

is known as *total head*.

14.4.3 The gas dynamical equation

In the case of:

- steady flow, Section 14.3.1;
- irrotational flow, i.e. there exists a potential Φ for the velocity such that
$$\mathbf{v} = -\nabla\Phi \tag{14.72}$$
- barotropic flow, Section 14.3.3, and there exists a pressure function
$$\mathcal{P}(\pi) = \int_{\pi_o}^{\pi} \frac{d\pi}{\varrho(\pi)} \tag{14.73}$$
such that
$$\pi_{,i} = \varrho\mathcal{P}_{,i} \tag{14.74}$$
- negligible body forces, so that $\mathbf{b} \simeq \mathbf{o}$;

the Bernoulli Equations in Eq. 14.69 reduces to

$$\mathcal{P}(\pi) + \frac{v^2}{2} = C \tag{14.75}$$

where C is the Bernoulli constant. This equation establishes that π is a function of the velocity v and, thus, of the potential Φ. Consequently, the density ϱ, which according to Eq. 14.74 is a function of π, it results also a function of v.

Under the above hypothesis, the first set of equations in Eqs. 14.55-14.57 governing the flow of a barotropic perfect fluid can be equivalently decoupled into the following 3 independent scalar equations:

1. the Gas Dynamical Equation

$$(c^2\delta_{i,j} - \Phi_{,i}\Phi_{,j})\Phi_{,ji} = 0 \tag{14.76}$$

that is, in extended form,

$$\begin{aligned}(c^2 - \Phi_{,x}^2)\Phi_{,xx} + (c^2 - \Phi_{,y}^2)\Phi_{,yy} + (c^2 - \Phi_{,z}^2)\Phi_{,zz} - \\ -2(\Phi_{,x}\Phi_{,y}\Phi_{,xy} + \Phi_{,y}\Phi_{,z}\Phi_{,yz} + \Phi_{,z}\Phi_{,x}\Phi_{,zx}) = 0\end{aligned} \tag{14.77}$$

where c is the *local sound velocity* defined as

$$c = \left(\frac{d\pi}{d\varrho}\right)^{1/2} = c(v) = c(\Phi)$$

This is a nonlinear differential equation for the only unknown Φ. Once this equation is solved, we can calculate the velocity field $\mathbf{v}$ as in Eq. 14.72.

2. The Euler form of the Continuity Equation

$$\varrho_{,k}v_k + \varrho v_{k,k} = 0 \tag{14.78}$$

which, for a given velocity field $\mathbf{v}$, allows to calculate the associated density field ϱ.

3. the Barotropic Equation of State

$$\pi = \pi(\varrho) \tag{14.79}$$

which, for a given density field ϱ, allows to calculate the associated pressure field π.

In fact, the hypothesis of steady flow and negligible body forces allows to reduce the first set of equations in Eqs. 14.55-14.57 as

$$-\pi_{,j} = \varrho v_{j,i}v_i \tag{14.80}$$

$$\varrho_{,k}v_k + \varrho v_{k,k} = 0 \tag{14.81}$$

$$\pi = \pi(\varrho) \tag{14.82}$$

From Eq. 14.82 we obtain

$$\frac{\partial \pi}{\partial x_j} = \frac{\partial \pi}{\partial \varrho}\frac{\partial \varrho}{\partial x_j} = \frac{d\pi}{d\varrho}\frac{\partial \varrho}{\partial x_j}$$

that is

$$\pi_{,j} = c^2 \varrho_{,j}$$

Making use of this relationship in Eq. 14.80, we obtain

$$\varrho_{,k} = \frac{1}{c^2}\pi_{,k} = -\frac{\varrho}{c^2} v_{k,j} v_j$$

which inserted in Eq. 14.81 yields to

$$\begin{aligned} 0 &= \varrho_{,k} v_k + \varrho v_{k,k} = \\ &= \left(-\frac{\varrho}{c^2} v_{k,j} v_j\right) v_k + \varrho \delta_{jk} v_{j,k} = \\ &= \varrho \left(-\frac{1}{c^2} v_j v_k + \delta_{jk}\right) v_{j,k} \end{aligned}$$

Expressing the velocity as in Eq. 14.72, we obtain the gas dynamical equation in Eq. 14.76.

For a two-dimensional irrotational flow where Φ is independent on z direction, Eq. 14.77 takes on the form

$$\mathcal{A}\Phi_{,xx} + \mathcal{B}\Phi_{,xy} + \mathcal{C}\Phi_{,yy} = 0 \tag{14.83}$$

where

$$\begin{aligned} \mathcal{A} &= c^2 - \Phi_{,x}^2 \\ \mathcal{B} &= -2\Phi_{,x}\Phi_{,y} \\ \mathcal{C} &= c^2 - \Phi_{,y}^2 \end{aligned}$$

The character of this quasi–linear, partial differential equation is determined by the sign of the discriminant

$$\mathcal{B}^2 - 4\mathcal{A}\mathcal{C} = 4c^4(M^2 - 1)$$

where

$$M = \frac{v}{c}$$

is the local *Mach number*. Hence, the equation is:

- *elliptic*, for $M < 1$ (subsonic)
- *hyperbolic*, for $M > 1$ (supersonic)

- *parabolic*, for $M = 1$ (transonic)

Notice that the solution methods differ markedly from one classification to another. Completely subsonic flows may be solved with some difficulties. Completely supersonic flows are easier to solve, but shock waves may occur; these are surfaces across which the velocity, pressure and density are discontinuous and on which the differential equation does not hold.

Finally, we notice that if:

- all the velocity components $\Phi_{,x}, \Phi_{,y}$ and $\Phi_{,z}$ are small compared to the sound velocity c,
- the mixed derivatives $\Phi_{,xy}, \Phi_{,yz}$ and $\Phi_{,zx}$ are not an order of magnitude greater than $\Phi_{,xx}, \Phi_{,yy}$ and $\Phi_{,zz}$,

then, the gas dynamic equation in Eq. 14.77 can be reduced to the linear differential equation

$$\Phi_{,kk} = 0 \tag{14.84}$$

We will see in Section 14.5.2 that this equation governs also the flow of an incompressible fluid.

14.4.4 The wave equations

Let $\mathcal{M}$ be a continuum instantaneously bounded by a volume Ω and let us indicate the velocity, the density and the pressure fields at the current time t as

$$\mathbf{v} = \mathbf{v}^{(o)} + \mathbf{v}' \tag{14.85}$$

$$\varrho = \varrho_o + \varrho' \tag{14.86}$$

$$\pi = \pi_o + \pi' \tag{14.87}$$

where ^{o} denotes the initial values at time $t = t_o$ while the primed quantities denote the variation from the initial values. Assume that:

- the continuum $\mathcal{M}$ is a perfect and barotropic fluid;
- neglegible body forces, so that $\mathbf{b} \simeq \mathbf{o}$;
- Statical Equilibrium State at time $t = t_o$, so that, in particular, $\mathbf{v}^{(o)} \equiv \mathbf{o}, \varrho_o$ and π_o constant in Ω_o, Eqs. 14.55 and 14.57.

- Small fluctuation of ϱ and $\mathbf{v}$ about the corresponding values at the equilibrium state in the sense that

$$\varrho_o \gg \varrho' \tag{14.88}$$

$$\frac{\partial \varrho'}{\partial t} \gg |\varrho'_{,k} v'_k| \tag{14.89}$$

$$\frac{\partial v'_i}{\partial t} \gg |v'_{i,k} v'_k| \tag{14.90}$$

- barotropic flow with barotropic equation of state $\pi = \pi(\varrho)$ given by the linear form (1st order Taylor expansion of π about π_o)

$$\pi = \pi_o + c_o^2(\varrho - \varrho_o) \tag{14.91}$$

where

$$c_o^2 = \left(\frac{d\pi}{d\varrho}\right)_o$$

- potential (irrotational) flow, i.e. there exists a potential Φ for the velocity such that

$$\mathbf{v} \equiv \mathbf{v}' = -\nabla\Phi \tag{14.92}$$

- negligible body forces, so that $\mathbf{b} \simeq \mathbf{o}$;

Under the above hypothesis, the first set of equations governing the flow of a barotropic perfect fluid in Eqs. 14.55-14.57 can be decoupled in the following three independent scalar functions:

1. the Wave Equation for the velocity potential function

$$\frac{\partial^2\Phi}{\partial t^2} - c_o^2\Phi_{,kk} = 0 \tag{14.93}$$

Once this (Laplace) linear differential equation is solved for the only unknown Φ, we can calculate the velocity field $\mathbf{v}$ as in Eq. 14.92.

2. the Wave Equation for the density function

$$\frac{\partial^2\varrho}{\partial t^2} - c_o^2\varrho_{,kk} = 0 \tag{14.94}$$

3. the Wave Equation for the thermodynamic pressure function

$$\frac{\partial^2\pi}{\partial t^2} - c_o^2\pi_{,kk} = 0 \tag{14.95}$$

In fact, under the above listed hypothesis, the first set of equations in Eqs. 14.55-14.57 governing the barotropic flow of a perfect fluid can be reduced at once into

$$\begin{aligned}
\frac{\partial \pi}{\partial x_j} &= -\varrho \frac{dv_j}{dt} \\
\frac{d\varrho}{dt} &= -\varrho \frac{\partial v_k}{\partial x_k} \\
\pi &= \pi_o + c_o^2(\varrho - \varrho_o)
\end{aligned}$$

where we can express

$$\begin{aligned}
\frac{\partial \pi}{\partial x_j} &= \frac{\partial \pi}{\partial \varrho}\frac{\partial \varrho}{\partial x_j} = \frac{\partial}{\partial \varrho}[\pi_o + c_o^2(\varrho - \varrho_o)]\frac{\partial}{\partial x_j}(\varrho_o + \varrho') = c_o^2\frac{\partial \varrho'}{\partial x_j} \\
\frac{dv_j}{dt} &= \frac{dv_j'}{dt} = \frac{\partial v_j'}{\partial t} + \frac{\partial v_j'}{\partial x_k}v_k' \approx \frac{\partial v_j'}{\partial t} = -\frac{\partial^2 \Phi}{\partial t \partial x_j} \\
\frac{\partial v_k}{\partial x_k} &= \frac{\partial v_k'}{\partial x_k} = -\frac{\partial^2 \Phi}{\partial x_k \partial x_k} \\
\frac{d\varrho}{dt} &= \frac{d}{dt}(\varrho_o + \varrho') = \frac{d\varrho'}{dt} = \frac{\partial \varrho'}{\partial t} + \frac{\partial \varrho'}{\partial x_k}v_k' \approx \frac{\partial \varrho'}{\partial t}
\end{aligned}$$

and $\varrho \approx \varrho_o$. Hence,

$$\begin{aligned}
c_o^2\frac{\partial \varrho'}{\partial x_j} &= \varrho_o\frac{\partial^2 \Phi}{\partial t \partial x_j} \\
\frac{\partial \varrho'}{\partial t} &= \varrho_o\frac{\partial^2 \Phi}{\partial x_k \partial x_k} \\
\pi' &= c_o^2\varrho'
\end{aligned}$$

and

- Differentiating the first equation by t and the second one by x_j and equalizing them, we obtain

$$\frac{\partial}{\partial x_j}\left[\frac{\partial^2 \Phi}{\partial t^2} - c_o^2\frac{\partial^2 \Phi}{\partial x_k \partial x_k}\right] = 0 \qquad (14.96)$$

  This indicates that the expression in the bracket is independent of the position and, therefore, it is at most function of time. Hence, adding a function of time t does not affect the velocity and, consequently, we loose no generality in writing Eq. 14.93.

- Differentiating the first equation by x_j and the second one by t and equalizing them, we obtain

$$\frac{\partial^2 \varrho'}{\partial t^2} - c_o^2 \frac{\partial^2 \varrho'}{\partial x_j \partial x_j} = 0 \tag{14.97}$$

- Substituting in Eq. 14.97 the barotropic condition in the third equation we obtain

$$\frac{\partial^2 \pi'}{\partial t^2} - c_o^2 \frac{\partial^2 \pi'}{\partial x_j \partial x_j} = 0 \tag{14.98}$$

14.5 Incompressible Newtonian Fluids

We recall that the rate of change of an elementary volume $\delta\Omega$ results to be equal to, Eqs. 6.22 and 7.11,

$$\frac{1}{\delta\Omega}\frac{d(\delta\Omega)}{dt} = \operatorname{div} \mathbf{v} = -\frac{1}{\varrho}\frac{d\varrho}{dt} \tag{14.99}$$

It follows that, if a fluid may be considered incompressible, that is

$$\frac{1}{\delta\Omega}\frac{d(\delta\Omega)}{dt} = 0 \tag{14.100}$$

then:

- the divergence of the velocity vector results to be zero at any time and location, that is

$$\operatorname{div} \mathbf{v} = v_{k,k} = 0 \tag{14.101}$$

  This implies that, in a Newtonian incompressible fluid, the thermodynamic pressure is equal to the mean compressive stress, that is, Eq. 10.96,

$$\pi = p \tag{14.102}$$

- The rate of change of the density results to be zero at any time and location, that is

$$\frac{d\varrho}{dt} = 0 \tag{14.103}$$

  This implies the barotropic condition

$$\varrho = \varrho_o \tag{14.104}$$

It is evident that the incompressibility condition in Eq. 14.101 and the barotropic condition in Eq. 14.104 allow to reduce the set of equations in Eqs. 14.20-14.24 governing the flow of a Newtonian fluid into the following two independent sets of equations:

1. The first set includes only mechanical quantities and is given by:

   (a) the 3 Navier-Stokes Equations of Motion for (Newtonian incompressible) Fluids

   $$-\pi_{,j} + \mu v_{j,ii} + \varrho_o b_j = \varrho_o \frac{dv_j}{dt} \tag{14.105}$$

   (b) the Incompressibility Condition

   $$v_{k,k} = 0 \tag{14.106}$$

2. This set represents a complete system of 4 equations into the following 4 unknowns:

   - the velocity components v_i, 3 unknowns;
   - the thermodynamic pressure π, 1 unknown.

3. The second set is given by:

   (a) the Caloric Equation of State

   $$u = u(\varrho_o, \vartheta) \tag{14.107}$$

   (b) the Energy Equation

   $$\varrho_o \frac{du}{dt} = \dot{w} + K\vartheta_{,kk} + \varrho_o r \tag{14.108}$$

   where

   $$\dot{w} = \dot{w}^{(v)} = 2\mu d_{ij} d_{ij}$$

   and

   $$d_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$$

   Once the first set is solved in terms of **v**, this second set represents a complete system of 2 equations into the following 2 unknowns:

   - the internal energy u, 1 unknown;
   - the absolute temperature ϑ, 1 unknown.

14.5.1 The Poisson and Stokes pressure equations

It is possible to prove that the field variables $(\mathbf{v}, \pi)$ satisfying the first set of equations in Eqs. 14.105-14.106 governing the flow of an incompressible fluid can be equivalently determined by solving the following set of equations:

1. the *Poisson Pressure Equation*

$$\pi_{,jj} = \varrho_o(b_j - v_{j,k}v_k)_{,j} \tag{14.109}$$

2. the 3 Navier-Stokes Equations of Motion for (Incompressible Newtonian) Fluids

$$-\pi_{,j} + \mu v_{j,ii} + \varrho_o b_j = \varrho_o \frac{dv_j}{dt} \tag{14.110}$$

This set represents a complete system of 4 equations into the following 4 unknowns:

- the velocity components v_i, 3 unknowns;
- the thermodynamic pressure π, 1 unknown.

In fact, taking the partial derivative of the NavierStokes Equation in Eq. 14.105 with respect to x_j, we obtain

$$\begin{aligned} 0 &= \left[\varrho_o \frac{dv_j}{dt} + \pi_{,j} - \mu v_{j,ii} - \varrho_o b_j\right]_{,j} = \\ &= \varrho_o \left[\frac{\partial v_j}{\partial t} + v_{j,k}v_k\right]_{,j} + \pi_{,jj} - \mu v_{j,iij} - \varrho_o b_{j,j} = \\ &= \varrho_o \left[\frac{\partial (v_{j,j})}{\partial t} + (v_{j,k}v_k)_{,j}\right] + \pi_{,jj} - \mu(v_{j,j})_{,ii} - \varrho_o b_{j,j} \end{aligned}$$

which, according to the incompressibility condition in Eq. 14.101, can be simplified into

$$0 = \varrho_o(v_{j,k}v_k)_{,j} + \pi_{,jj} - \varrho_o b_{j,j}$$

proving, therefore, the Poisson Equation in Eq. 14.109.

It is immediate to verify that if the convective terms are negligible with respect to the body forces, that is

$$b_j \gg v_{j,k}v_k \tag{14.111}$$

for all $j = 1, 2, 3$, then the above set of equations can be decoupled into the following two independent set of equations:

1. the *Stokes Pressure Equation*

$$\pi_{,jj} = \varrho_o b_{j,j} \tag{14.112}$$

   which represents a linear scalar equation for the only unknown π.

2. the 3 Navier-Stokes Equations of Motion for (Incompressible Newtonian) Fluids

$$\varrho_o \frac{\partial v_j}{\partial t} - \mu v_{j,ii} = -\pi_{,j} + \varrho_o b_j \tag{14.113}$$

   which, for a given pressure field π, represent a complete system of equations for the 3 velocity unknowns v_j.

14.5.2 Incompressible potential flow

In the case of potential flow, i.e. there exists a potential Φ for the velocity such that

$$\mathbf{v} = -\nabla\Phi \tag{14.114}$$

the first set of equations in Eqs. 14.105-14.106 governing the flow of an incompressible Newtonian fluid can be decoupled into:

1. The Laplace Equation

$$\Phi_{,kk} = 0 \tag{14.115}$$

   which is obtained from the incompressibility condition in Eq. 14.106 expressing the velocity as the gradient of the potential Φ. Once this equation is solved for Φ we can then calculate the velocity field as in Eq. 14.114.

2. The Poisson Pressure Equation, Eq. 14.109,

$$\pi_{,jj} = \varrho_o(b_j - v_{j,k}v_k)_{,j} \tag{14.116}$$

   which, for a given velocity field, allows to calculate the associated pressure field π.

We notice that the Laplace Equation in Eq. 14.115 coincide with the linearized form of the Gas Dynamical Equation in Eq. 14.84. This implies that the steady potential flow of a compressible fluid, in which the velocities are small compared to the local sound velocity c, can be approximately described by the Laplace equation governing the steady potential flow of an incompressible fluid.

14.6 Incompressible Inviscid Fluid

Although no fluids are actually incompressible or inviscid, in many important situations the compressibility and the viscosity of the fluid may be safely neglected. Liquids may be considered incompressible for flow studies under usual conditions, and we remarked in Section 14.5.2 that a gas may be treated approximately as incompressible for flow studies at a speed very small compared to the sonic speed.

Supported by similar considerations, engineers solve many practical fluid problems assuming that:

- Inviscid and incompressible fluids, that is

$$\mu = 0 \tag{14.117}$$

$$v_{k,k} = 0 \tag{14.118}$$

- Potential (irrotational) flow, i.e. there exists a potential function Φ for the velocity such that

$$v_i = -\Phi_{,i} \tag{14.119}$$

According to the results presented in Sections 14.5 and 14.5.2, the field equations governing an incompressible potential flows are given by the following 3 uncoupled sets of equations:

1. The Laplace Equation, Eq. 14.115,

$$\Phi_{k,k} = 0 \tag{14.120}$$

   which represents a linear differential equation in the only unknown Φ. Once this equation is solved for Φ, we can then calculate the velocity field as

$$v_j = -\Phi_{,j} \tag{14.121}$$

2. The Poisson Pressure Equation, Eq. 14.116

$$\pi_{,jj} = \varrho_o(b_j - v_{j,k}v_k)_{,j} \tag{14.122}$$

   which, for a given velocity field, allows to calculate the associated pressure field.

3. the third set is given by:

(a) the Caloric Equation of State, Eq. 14.107

$$u = u(\varrho_o, \vartheta) \tag{14.123}$$

(b) the Energy Equation, Eq. 14.108,

$$\varrho_o \frac{du}{dt} = K\vartheta_{,kk} + \varrho_o r \tag{14.124}$$

which represents a complete system of two equations in the 2 unknowns u and ϑ.

We remark that if

- the fluid is inviscid and incompressible,
- the body forces are conservative,

then the Bernoulli equation in Eq. 14.64, that is

$$\mathcal{P} + \mathcal{B} + \frac{v^2}{2} + \int \frac{\partial v}{\partial t} dx_i = C(t) \tag{14.125}$$

holds along any streamline. In fact, under the hypotheses of inviscid and incompressible fluids, the Generalized Navier-Stokes Equation in Eq. 14.20 reduces to the Euler form in Eq. 14.50. We have remarked in Section 14.5 that the hypothesis of incompressibility implies also the barotropic conditions

$$\varrho = \varrho_o \tag{14.126}$$

As a consequence, we can establish that in a incompressible flow there exists a pressure function

$$\mathcal{P}(\pi) = \int_{\pi_o}^{\pi} \frac{d\pi}{\varrho} = \frac{\pi}{\varrho_o} + const \tag{14.127}$$

such that

$$\pi_{,i} = \varrho \mathcal{P}_{,i} \tag{14.128}$$

Thus, following the same mathematical procedure reported in Section 14.4.2 we can derive the Bernoulli equation.

If we assume that the Potential Flow of an Incompressible Inviscid Fluid respects also the condition of:

- steady flow, Section 14.3.1, so that in particular

$$\frac{\partial v_j}{\partial t} = 0 \tag{14.129}$$

- body forces given by a uniform gravitational field, that is

$$b_j = g y_{,j} \tag{14.130}$$

  where g is the gravitational constant and y is the elevation above some reference level;

then, we can reduce the Bernoulli Equation in the following form, Eq. 14.70,

$$h = const \tag{14.131}$$

where h is the *total head* defined as

$$h = y + \frac{\pi}{g\varrho_o} + \frac{v^2}{2g}$$

Chapter 15

Coupled Solid-Fluid Problems

15.1 Introduction

Depending on the scale of investigation, soil can be considered either as a discrete or as a continuous medium. In general, the scale of investigation of a practical geotechnical engineering problem is so large with respect to a single soil grain, that it is possible to consider the whole soil mass to behave as a continuous medium.

Under the hypothesis of continuity, geotechnical analysis can be performed, as any other engineering problem involving solid materials, in the framework of the well established Continuum Mechanics Theory.

The only peculiarity arises when the soil is saturated with water, or with any other fluid. In this case, the soil is considered as a continuous two-phases medium, that is, any field variable relative to each phase, solid and fluid, is assumed to vary continuously within the soil spatial domain. Under this hypothesis, all mathematical functions and their derivatives entering in the theory are continuous and, as a result, they can be defined at each geometrical point. Following this approach, it is possible to take advantage of the whole Continuum Mechanics Theory just by extending some of its aspects, in order to account for the interaction between solid and fluid phases.

The fundamental field equations governing a *Coupled Solid-Fluid Problem* were first established by Biot, [4, 5, 6, 7, 8, 9]. Biot essentially extended Cauchy's Equation of Motion to a fully saturated solid medium by introducing Terzaghi's *Principle of Effective Stress*, Section 15.4. In this way, the equilibrium problem involves the additional field variable represented by the excess pore pressure mobilized by the fluid phase. For the solution of this kind of problems, Biot added the Flow Continuity Equation, Section 15.6, to the traditional set of Cauchy's differential equations.

More recently, the all field of *Mixture Theory* was reviewed by Truesdell, [92, 94], and many other researchers [11, 29, 45, 46, 47]. Though these works have provided new and more general basis for such coupled theory, no major changes have been introduced to Biot's theory.

In addition to the Coupled Solid-Fluid Theory, this Chapter reports the so-called *Uncoupled Theory of Consolidation*, Section 15.12, which is still the current approach with which practitioners use to solve geotechnical problems.

15.2 Preliminary Definitions

Let $\mathcal{M}$ be a porous medium of volume Ω bounded by a surface Γ. If we distinguish with the subscript m and n the quantities related to the solid skeleton and to the pores, respectively, we can express

$$\begin{aligned} \Omega &= \Omega_m + \Omega_n \\ \Gamma &= \Gamma_m + \Gamma_n \end{aligned}$$

In geotechnical engineering, it is customary to refer to volume ratio quantities such as porosity, void ratio and specific volume. We will adopt their definitions in terms of the elemental volumes so that we can utilize them within the framework of the Continuum Mechanics Theory, namely:

- *Porosity*, defined as

$$n = \frac{\delta\Omega_n}{\delta\Omega} \tag{15.1}$$

- *Void Ratio*, defined as

$$e = \frac{\delta\Omega_n}{\delta\Omega_m} \tag{15.2}$$

- *Specific Volume*, defined as

$$v = \frac{\delta\Omega}{\delta\Omega_m} \tag{15.3}$$

It is simple to verify that the above volume ratios are inter-related as

$$n = \frac{e}{1+e} = \frac{v-1}{v} \tag{15.4}$$

$$e = \frac{n}{1-n} = v-1 \tag{15.5}$$

$$v = 1+e = \frac{1}{1-n} \tag{15.6}$$

and that an elemental volume can be alternatively expressed as

$$\begin{aligned}\delta\Omega &= \delta\Omega_m + \delta\Omega_n = \\ &= (1+e)\delta\Omega_m = v\delta\Omega_m = \frac{\delta\Omega_n}{n}\end{aligned} \tag{15.7}$$

since

$$\delta\Omega_m = (1-n)\delta\Omega = \frac{\delta\Omega_n}{e} = \frac{\delta\Omega_n}{v-1} \tag{15.8}$$

$$\delta\Omega_n = n\delta\Omega = e\delta\Omega_m = (v-1)\delta\Omega_m \tag{15.9}$$

Similarly, we can express an elemental surface as

$$\begin{aligned}\delta\Gamma &= \delta\Gamma_m + \delta\Gamma_n = \\ &\simeq (1+e)\delta\Gamma_m = v\delta\Gamma_m = \frac{\delta\Gamma_n}{n}\end{aligned} \tag{15.10}$$

since

$$\delta\Gamma_m \simeq (1-n)\delta\Gamma = \frac{\delta\Gamma_n}{e} = \frac{\delta\Gamma_n}{v-1} \tag{15.11}$$

$$\delta\Gamma_n \simeq n\delta\Gamma = e\delta\Gamma_m = (v-1)\delta\Gamma_m \tag{15.12}$$

If $\mathcal{M}$ is fully saturated, the fluid phase occupies all the volume of the pores. Thus, the mass of the solid-fluid mixture can be expressed as

$$\delta m = \delta m_m + \delta m_f \tag{15.13}$$

where δm_m is the mass of the solid skeleton while δm_f is the mass of the fluid phase. Accordingly, the density of the solid-fluid mixture can be expressed as

$$\varrho = \frac{\delta m}{\delta\Omega} = (1-n)\varrho_m + n\varrho_f \tag{15.14}$$

since

$$\varrho_m = \frac{\delta m_m}{\delta \Omega_m} \tag{15.15}$$

$$\varrho_f = \frac{\delta m_f}{\delta \Omega_n} \tag{15.16}$$

We recall from Sections 6.3.3 and 7.2, that

$$\frac{1}{\delta\Omega}\frac{d}{dt}\delta\Omega = \operatorname{div}\mathbf{v} = -\frac{1}{\varrho}\frac{d\varrho}{dt} \tag{15.17}$$

and that, in the hypothesis of small deformation, Section 6.6.3,

$$\frac{1}{\delta\Omega}\frac{d}{dt}\delta\Omega = \frac{d\epsilon_v}{dt} \tag{15.18}$$

The same relationships are applied in a coupled solid-fluid problem as well, identifying in Ω the volume of the porous medium, in $\mathbf{v}$ the velocity field of the solid skeleton and in ϱ the density of the mixed solid-fluid.

Finally, it is interesting to note that expressing $\delta\Omega$ as in Eq. 15.7, we obtain

$$\frac{1}{\delta\Omega}\frac{d}{dt}\delta\Omega = \frac{1}{v}\frac{dv}{dt} + \frac{1}{\delta\Omega_m}\frac{d}{dt}\delta\Omega_m \tag{15.19}$$

In soils, $\delta\Omega_m$ represents the volume of the mineral grains and its variation is by all means negligible compared to that of the soil skeleton. Hence, we can set with sufficient accuracy

$$\frac{1}{\delta\Omega}\frac{d}{dt}\delta\Omega = \frac{1}{v}\frac{dv}{dt} \tag{15.20}$$

form which, in the hypothesis of small deformation, it follows that

$$\frac{d\epsilon_v}{dt} = \frac{1}{v}\frac{dv}{dt} \tag{15.21}$$

15.3 Seepage Law

In a fully saturated soil, the fluid phase flows through the interconnected pores between the solid particles. Thus, on the microscopic scale the fluid follows a very tortuous path but, at a macroscopic level, the flow path can be considered as linear.

In the engineering practice, it is customary to refer to an average velocity $\overline{v}_s^{(r)}$ obtained by dividing the volume $\dot{\Omega}_\ell$ of fluid flowing out per unit time by the area Γ of a cross section normal to the macroscopic direction of flow

$$\overline{v}_s^{(r)} = \frac{\dot{\Omega}_\ell}{\Gamma} \tag{15.22}$$

Early experimental works done by Darcy (1856) established that, in a 1D steady seepage state through a fully saturated soil, the average velocity $\overline{v}_s^{(r)}$ results to be equal, *Darcy's law of permeability*,

$$\overline{v}_s^{(r)} = -\mathcal{P}_s \frac{\Delta h}{\Delta s} \tag{15.23}$$

where, Fig. 15.1:

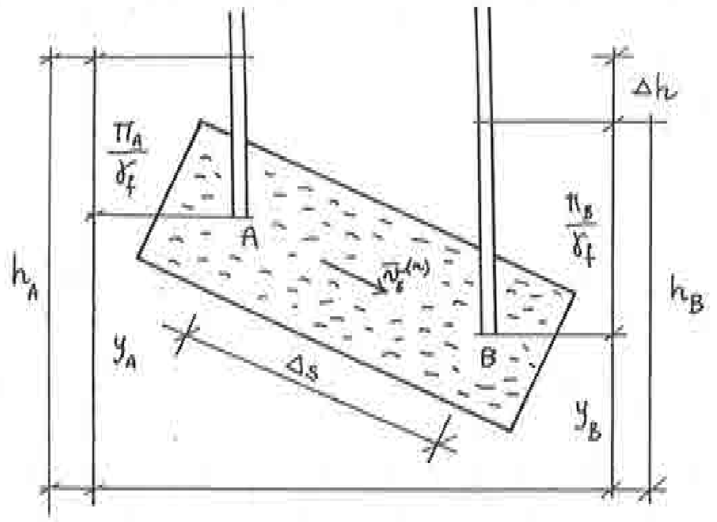


Figure 15.1: Seepage through a porous medium

- $\mathcal{P}_s$ is the *coefficient of permeability* whose value depends on the physical characteristics of the soil. Some typical values are reported in Table 15.1.

In the engineering practice, it is customary to refer to an average velocity $\overline{v}_s^{(r)}$ obtained by dividing the volume $\dot{\Omega}_\ell$ of fluid flowing out per unit time by the area Γ of a cross section normal to the macroscopic direction of flow

$$\overline{v}_s^{(r)} = \frac{\dot{\Omega}_\ell}{\Gamma} \tag{15.22}$$

Early experimental works done by Darcy (1856) established that, in a 1D steady seepage state through a fully saturated soil, the average velocity $\overline{v}_s^{(r)}$ results to be equal, *Darcy's law of permeability*,

$$\overline{v}_s^{(r)} = -\mathcal{P}_s \frac{\Delta h}{\Delta s} \tag{15.23}$$

where, Fig. 15.1:

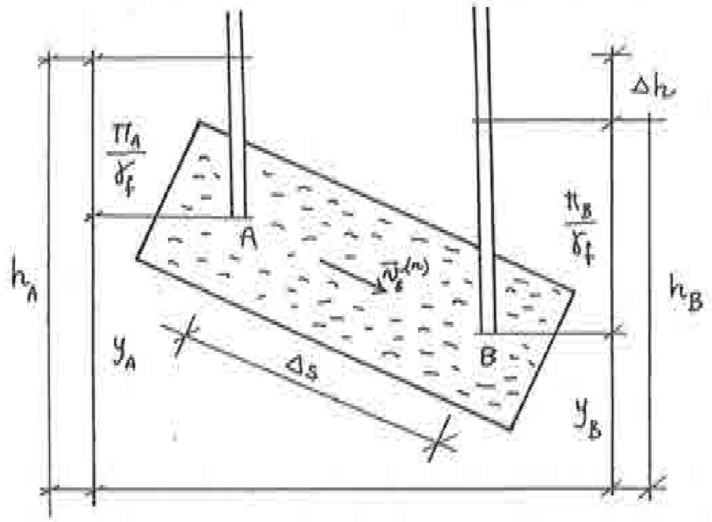


Figure 15.1: Seepage through a porous medium

- $\mathcal{P}_s$ is the *coefficient of permeability* whose value depends on the physical characteristics of the soil. Some typical values are reported in Table 15.1.

Table 15.1: Typical values of the permeability coefficients in soils

Soil type	Coefficient of Permeability $[cm/sec]$
Clean gravel	1.0 and greater
Clean sand (coarse)	1.0-0.01
Sand (mixture)	0.01-0.003
Fine sand	0.05-0.001
Silty sand	0.002-0.0001
Silt	0.0005-0.00001
Clay	0.00001 or smaller

- h represents the *total head pressure* defined as

$$\begin{aligned} h &= y + \frac{\pi}{\gamma_f} + \frac{v^{(f)2}}{2} \simeq \\ &\simeq y + \frac{\pi}{\gamma_f} \end{aligned} \tag{15.24}$$

  where y is the elevation, also called *head pressure*, and π is the *pore pressure*. Finally, $v^{(f)} = \|\mathbf{v}^{(f)}\|$ is the true velocity of the fluid phase which is generally negligible.

- Δh represents the difference of the total head measured at the distance Δs.

Further investigations have confirmed the validity of Darcy's law provided that the flow can be considered as laminar, that is, when

$$R = \frac{\overline{v}_s^{(r)} D \varrho_f}{\mu} \leq 1 \tag{15.25}$$

where

- R is the *Reynolds number*, adimensionless;
- D is average diameter of the soil grains, $[cm]$;

- ϱ_f is the density of the fluid, $[g(mass)/cm^3]$;
- μ is the coefficient of viscosity $[s\ cm/g(mass)]$.

More recently, Darcy's law has been theoretically justified and located into the framework of the Mixture Theory, [45, 48, 44]. Such extensions are based on the following preliminary observations:

- Since the fluid can cross only the area of the pores $\Gamma_n = n\Gamma$, Eq. 15.12, we can define a local true velocity of the fluid as

$$v_s^{(r)} = \frac{\dot{\Omega}_\ell}{\Gamma_n} = \frac{\dot{\Omega}_\ell}{n\Gamma} = \frac{\overline{v}_s^{(r)}}{n} \tag{15.26}$$

- $v_s^{(r)}$ and $\overline{v}_s^{(r)}$ are the projections on the normal $\hat{\mathbf{n}}$ of the cross section Γ of the velocity vectors $\mathbf{v}^{(r)}$ and $\overline{\mathbf{v}}^{(r)}$, respectively, that is

$$\begin{aligned} v_s^{(r)} &= \hat{\mathbf{n}}^T \mathbf{v}^{(r)} \\ \overline{v}_s^{(r)} &= \hat{\mathbf{n}}^T \overline{\mathbf{v}}^{(r)} \end{aligned}$$

  and, according to Eq. 15.26,

$$\overline{\mathbf{v}}^{(r)} = n\mathbf{v}^{(r)} \tag{15.27}$$

- If the solid skeleton is at rest, i.e $\mathbf{v} = \mathbf{o}$, then $\mathbf{v}^{(r)}$ represents the velocity $\mathbf{v}^{(f)}$ of the fluid phase. While, if the solid skeleton undergoes to a deformation process with a velocity field $\mathbf{v} \neq \mathbf{o}$, then $\mathbf{v}^{(r)}$ is the relative velocity between the fluid and the solid phase, that is

$$\mathbf{v}^{(r)} = \mathbf{v}^{(f)} - \mathbf{v} \tag{15.28}$$

For the general case of

- 3D non steady flow;
- dynamic loading conditions;
- solid skeleton undergoing a deformation process;

it has been proposed that, the average relative velocity $\overline{\mathbf{v}}^{(r)}$ of the outflowing fluid from an elementary volume of soil may be calculated according to the following *generalized Darcy's law*,

$$\overline{\mathbf{v}}^{(r)} = -\mathcal{P}\left(\frac{\varrho_f \mathbf{a}^{(f)}}{\gamma_f} + \operatorname{grad} h\right) \tag{15.29}$$

where

$$\begin{aligned} \mathbf{a}^{(f)} &= \dot{\mathbf{v}}^{(f)} = \dot{\mathbf{v}} + \frac{\dot{\overline{\mathbf{v}}}^{(r)}}{n} \\ h &= -\frac{\mathbf{b}^T\mathbf{x}}{g} + \frac{\pi}{\gamma_f} \end{aligned}$$

and

- $\mathbf{x}$ is the coordinate vector of the point at which h is evaluated;
- $\mathbf{b}$ is the gravity vector;
- $\mathcal{P}$ is a permeability tensor, which in a CaORS, is represented by a 3×3 positive definite matrix.

Note that, since $\mathcal{P}$ is positive definite, it has three orthogonal principal permeability directions. In a CaORS coincident with these principal directions, the permeability tensor is represented by a diagonal matrix of the type

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 & 0 & 0 \\ 0 & \mathcal{P}_2 & 0 \\ 0 & 0 & \mathcal{P}_3 \end{bmatrix}$$

where $\mathcal{P}_i$ is the coefficient of permeability in the i-th principal direction. For any other arbitrary choice of CaORS, the relative permeability tensor can be determined through the tensor transformation rule, Section 3.4,

$$\mathcal{P} = \mathbf{A}\mathcal{P}'\mathbf{A}^T$$

where $\mathbf{A}$ is the orthogonal transformation matrix. In an isotropic soil, $\mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P}_3$ and the permeability matrix keeps its diagonal form for any arbitrary choice of CaORS.

The total head pressure definition in Eq. 15.29 is merely a generalization of that one reported in Eq. 15.24. In fact, when one of CaORS axes is chosen to coincide with the gravity direction, say y, so that

$$\begin{aligned} \mathbf{x} &= \{x, y, z\}^T \\ \mathbf{b} &= \{0, -g, 0\}^T \end{aligned}$$

it results

$$-\frac{\mathbf{b}^T\mathbf{x}}{g} = y$$

15.4 Principle of Effective Stress

In a fully saturated soil, because of the fluid component, only part of the total stress σ in equilibrium with the applied external forces is carried by the solid phase (soil skeleton), Fig. 15.2. Since fluid cannot support shear, its

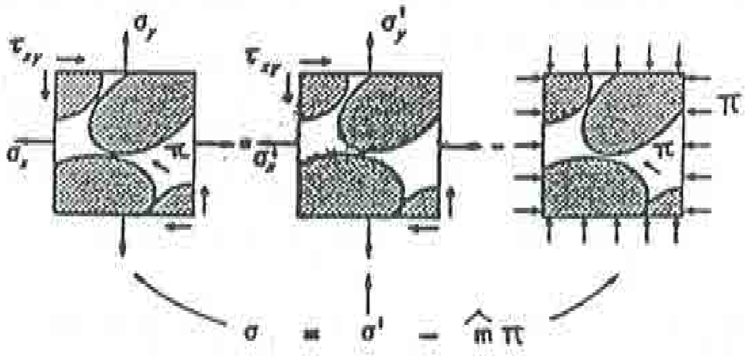


Figure 15.2: Total and effective stress in a porous media

presence affects only the normal stress component. In mathematical form, this concept, known as Terzaghi's *principle of effective stress*, [89, 91], is expressed by

$$\sigma = \sigma' - \widehat{\mathbf{m}}\pi \tag{15.30}$$

where

- σ' is the *effective stress* acting on the solid skeleton,
- π is the *pore pressure* due to the fluid phase. The negative sign has been introduced since π acts in compression.

We recall that in our convention

$$\widehat{\mathrm{m}} = \{1, 1, 1, 0, 0, 0, 0, 0, 0\}^T$$

Lately, this principle has been extended to the more general case of multi-phases medium including partially saturated soil. A fairly complete presentation of the more recent works on this topic can be found in [61].

15.5 Stress-Strain Relationships

It has been proposed to consider the rate of change of the total strain $\dot{\epsilon}$ of the solid skeleton in a saturated soil as the result of three distinct contributions, [96], namely

$$\dot{\epsilon} = \dot{\epsilon}^{(\sigma')} + \dot{\epsilon}^{(\pi)} + \dot{\epsilon}^{(\beta)} \tag{15.31}$$

where:

- $\dot{\epsilon}^{(\sigma')}$ is the strain rate caused by a variation of the effective stress and it may be expressed as

$$\dot{\epsilon}^{(\sigma')} = \mathbf{C}^{-1}\dot{\sigma}' \tag{15.32}$$

  where $\mathbf{C}$ is the tangent (elasto-plastic) stiffness matrix of the solid skeleton. This constitutive relationship is not affected by the fluid phase and, consequently, it may be determined experimentally in dry conditions. Clearly in this latter case, neglecting $\dot{\epsilon}^{(\beta)}$, $\dot{\epsilon}^{(\sigma')} \equiv \dot{\epsilon}$ and $\dot{\sigma}' \equiv \dot{\sigma}$.

- $\dot{\epsilon}^{(\pi)}$ is the strain rate caused by a variation of the pore pressure. Since $\dot{\pi}$ acts on the single grain with an hydrostatic compression, the relative strain rate may be expressed as

$$\dot{\epsilon}^{(\pi)} = -\frac{\dot{\pi}}{3B_m}\widehat{\mathrm{m}} \tag{15.33}$$

  where B_m is the bulk modulus of the mineral particles.

- $\dot{\epsilon}^{(\beta)}$ represents all others strain rates not directly related to stress changes (creeps, swellings, thermal, chemical, etc.). Some time $\dot{\epsilon}^{(\beta)}$ is called *autogeneous strain*.

According to the above definitions and the Principle of Effective Stress in Eq. 15.30, the constitutive relationship for a saturated porous medium results to be of the following forms:

- Effective stress constitutive equations,

$$\dot{\sigma}' = \mathbf{C}\left(\dot{\epsilon} - \dot{\epsilon}^{(\pi)} - \dot{\epsilon}^{(\beta)}\right) = \mathbf{C}\left(\dot{\epsilon} - \dot{\epsilon}^{(\beta)}\right) + \dot{\pi}\mathbf{r} \tag{15.34}$$

  where

$$\mathbf{r} = \frac{\mathbf{C}\widehat{\mathbf{m}}}{3B_m}$$

- Total stress constitutive equations,

$$\dot{\sigma} = \dot{\sigma}' - \dot{\pi}\widehat{\mathbf{m}} = \mathbf{C}\left(\dot{\epsilon} - \dot{\epsilon}^{(\beta)}\right) - \dot{\pi}\mathbf{q} \tag{15.35}$$

  where

$$\mathbf{q} = \widehat{\mathbf{m}} - \mathbf{r}$$

Note that, if the tangent stiffness matrix of the solid skeleton behaves as an incrementally isotropic material with the tangent stiffness matrix defined as

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & \vdots & 0 \\ \nu & (1-\nu) & \nu & 0 & \vdots & 0 \\ \nu & \nu & (1-\nu) & 0 & \vdots & 0 \\ 0 & 0 & 0 & (1-2\nu) & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \vdots & (1-2\nu) \end{bmatrix}_{(9\times 9)} \tag{15.36}$$

then

$$\mathbf{r} = r\widehat{\mathbf{m}} \tag{15.37}$$

$$\mathbf{q} = \left(1 - \frac{B}{B_m}\right)\widehat{\mathbf{m}} \tag{15.38}$$

where

$$r = \frac{B}{B_m}$$

$$q = 1 - r$$

and

$$B = \frac{E}{3(1-2\nu)}$$

is the bulk modulus of the solid skeleton.

In most soils, the bulk modulus of the mineral grains is much greater than that of the solid skeleton, that is $3B_m \gg |\mathbf{C}\widehat{\mathbf{m}}|$, and, consequently,

$$\begin{aligned} \mathbf{r} &\simeq \mathbf{o} \\ \mathbf{q} &\simeq \widehat{\mathbf{m}} \end{aligned}$$

If in addition we neglect the autogeneous strain rate $\dot{\boldsymbol{\epsilon}}^{(\beta)}$, Eqs. 15.34 and 15.35 reduce to

$$\dot{\boldsymbol{\sigma}}' = \mathbf{C}\dot{\boldsymbol{\epsilon}} \tag{15.39}$$

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}\dot{\boldsymbol{\epsilon}} - \widehat{\mathbf{m}}\pi \tag{15.40}$$

The inversion of Eq. 15.39 gives

$$\dot{\boldsymbol{\epsilon}} = \mathbf{C}^{-1}\dot{\boldsymbol{\sigma}}'$$

which justifies the classical geotechnical engineering assumption which states that the deformation of the soil structure depends only on the effective stress. However, it is important to remark that the overall bulk modulus B of lightly fractured rocks can be of the same order of magnitude of the bulk modulus B_m of the rock blocks. In this case, therefore, we cannot neglect the contribution of the $\mathbf{r}$ vector.

15.6 Flow Continuity Relation

According to the definition of relative fluid velocity reported in Section 15.3 and the Gauss Divergence Theorem, Section 2.5.2, we can establish that the rate of change $\dot{\Omega}_\ell$ of fluid volume lost or stored by a finite volume Ω of porous medium is given by

$$\dot{\Omega}_\ell = -\int_\Gamma \bar{\mathbf{n}}^T \overline{\mathbf{v}}^{(r)}\, d\Gamma = -\int_\Omega \operatorname{div} \overline{\mathbf{v}}^{(r)}\, d\Omega \tag{15.41}$$

where, according to Eq. 15.28, the divergence of the relative velocity $\overline{\mathbf{v}}^{(r)}$ can be expressed as

$$-\operatorname{div} \overline{\mathbf{v}}^{(r)} = n(\operatorname{div}\mathbf{v} - \operatorname{div}\mathbf{v}^{(f)}) \tag{15.42}$$

We recall that in the small deformation theory, Eq. 6.109,

$$\operatorname{div} \mathbf{v} = \dot{\epsilon}_v \tag{15.43}$$

Mostly based on physical intuitions, it has been proposed that, [95], in the hypothesis of small deformation and fully saturated porous medium, the rate of change of fluid storage volume per unit elementary volume can be calculated as

$$-\operatorname{div} \overline{\mathbf{v}}^{(r)} = \dot{\epsilon}_v^{(\ell)} \tag{15.44}$$

where

$$\dot{\epsilon}_v^{(\ell)} = \dot{\epsilon}_v - (\dot{\epsilon}_v^{(m)} + \dot{\epsilon}_v^{(n)}) \tag{15.45}$$

and

- $\dot{\epsilon}_v$ is the volumetric rate of change of the solid skeleton, that is

$$\dot{\epsilon}_v = \widehat{\mathbf{m}}^T \dot{\boldsymbol{\epsilon}} \tag{15.46}$$

- $\dot{\epsilon}_v^{(m)}$ is the volumetric rate of change of the mineral grains caused by a change of both effective mean pressure and pore pressure, namely

$$\dot{\epsilon}_v^{(m)} = \frac{1}{\mathcal{B}_m}\left[\frac{\widehat{\mathbf{m}}^T \dot{\boldsymbol{\sigma}}'}{3} - (1-n)\dot{\pi}\right] \tag{15.47}$$

  where, we recall, $\mathcal{B}_m$ is the bulk modulus of the soil mineral grains.

- $\dot{\epsilon}_v^{(f)}$ is the volumetric rate of change of the fluid caused by a change of pore, namely

$$\dot{\epsilon}_v^{(n)} = -n\frac{\dot{\pi}}{\mathcal{B}_f} \tag{15.48}$$

  where $\mathcal{B}_f$ is the bulk modulus of the fluid phase.

From the above definition and the constitutive relationship in Eq. 15.35, we can eventually derive the following *Flow Continuity equations*

$$-\operatorname{div} \overline{\mathbf{v}}^{(r)} = \mathbf{q}^T \dot{\boldsymbol{\epsilon}} + \mathbf{r}^T \dot{\boldsymbol{\epsilon}}^{(\beta)} + \frac{\dot{\pi}}{\mathcal{B}_\pi} \tag{15.49}$$

where $\overline{\mathbf{v}}^{(r)}$ can be calculated according to the generalized Darcy Law in Eq. 15.29. Assuming $\mathbf{C}$ symmetric matrix,

$$\begin{aligned}
\mathbf{r} &= \frac{\mathbf{C}\widehat{\mathbf{m}}}{3\mathcal{B}_m} \\
\mathbf{q} &= \widehat{\mathbf{m}} - \mathbf{r} \\
\frac{1}{\mathcal{B}_\pi} &= \frac{3(1-n) - \widehat{\mathbf{m}}^T\mathbf{r}}{3\mathcal{B}_m} + \frac{n}{\mathcal{B}_f}
\end{aligned}$$

In fact,

$$\begin{aligned}
\dot{\epsilon}_v^{(\ell)} &= \dot{\epsilon}_v - \dot{\epsilon}_v^{(m)} - \dot{\epsilon}_v^{(n)} = \\
&= \widehat{\mathbf{m}}^T \dot{\epsilon} - \frac{1}{B_m}\left[\frac{\widehat{\mathbf{m}}^T \dot{\sigma}'}{3} - (1-n)\dot{\pi}\right] + n\frac{\dot{\pi}}{B_f} = \\
&= \widehat{\mathbf{m}}^T \dot{\epsilon} - \frac{\widehat{\mathbf{m}}^T}{3B_m}\left[\mathbf{C}(\dot{\epsilon} - \dot{\epsilon}^{(\beta)}) + \mathbf{r}\dot{\pi}\right] + \left[\frac{(1-n)}{B_m} + \frac{n}{B_f}\right]\dot{\pi} = \\
&= \widehat{\mathbf{m}}^T \dot{\epsilon} - \frac{\widehat{\mathbf{m}}^T}{3B_m}\mathbf{C}(\dot{\epsilon} - \dot{\epsilon}^{(\beta)}) + \left[-\frac{\widehat{\mathbf{m}}^T \mathbf{r}}{3B_m} + \frac{(1-n)}{B_m} + \frac{n}{B_f}\right]\dot{\pi} = \\
&= \left[\widehat{\mathbf{m}}^T - \frac{\widehat{\mathbf{m}}^T \mathbf{C}}{3B_m}\right]\dot{\epsilon} + \frac{\widehat{\mathbf{m}}^T \mathbf{C}}{3B_m}\dot{\epsilon}^{(\beta)} + \frac{\dot{\pi}}{B_\pi}
\end{aligned}$$

15.7 Steady Seepage Flow

Let $\mathcal{M}$ be a porous saturated medium of volume Ω bounded by a surface Γ, Fig. 15.3. Conditions of *steady flow* of the fluid phase in $\mathcal{M}$ are met

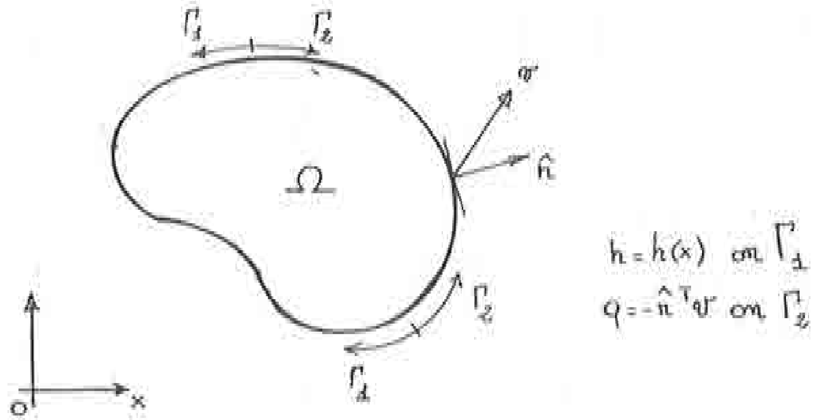


Figure 15.3: Steady seepage flow in a porous solid medium

when, for each elementary volume in $\mathcal{M}$, the fluid phase does not present any acceleration component and the amount of fluid inflowing equals that outflowing, unless local sources or sinks. Mathematically these conditions are expressed by

$$\mathbf{a}^{(f)} = \mathbf{o} \quad \text{and} \quad \dot{\epsilon}^{(\ell)} = g \tag{15.50}$$

where g represents the amount of fluid due to local sinks or sources.

Under the above hypothesis, the flow continuity relation in Eq. 15.49 takes the form

$$\operatorname{div} \mathcal{P} \operatorname{grad} h_o = g \tag{15.51}$$

where

$$h_o = -\frac{\mathbf{b}^T\mathbf{x}}{g} + \frac{\pi_o}{\gamma_f}$$

Usually, in geotechnical engineering, h_o and π_o are called *initial total head* and *initial pore pressure*, respectively. This because they usually represent the initial conditions from which on it starts the stress-strain analysis due to the application of an additional set of external load.

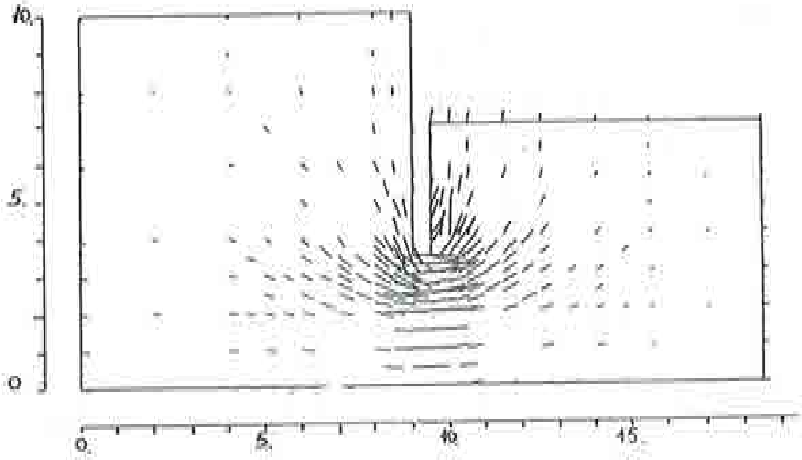
The biarmonic equation in Eq. 15.51, known also as the *Laplace equation*, represents then the governing field equation for a *steady seepage flow*. Provided that all boundary conditions on Γ are assigned, the Laplace equation identifies a unique solution h_o. These boundary conditions can be assigned either in terms of imposed total heads pressure h_o or in terms of imposed flow $q = -\hat{\mathbf{n}}^T\overline{\mathbf{v}}^{(r)}$, Fig. 15.3. Incidentally, the condition of impervious boundary is assigned by imposing $q = 0$.

In geotechnical engineering, seepage problems are subdivided into:

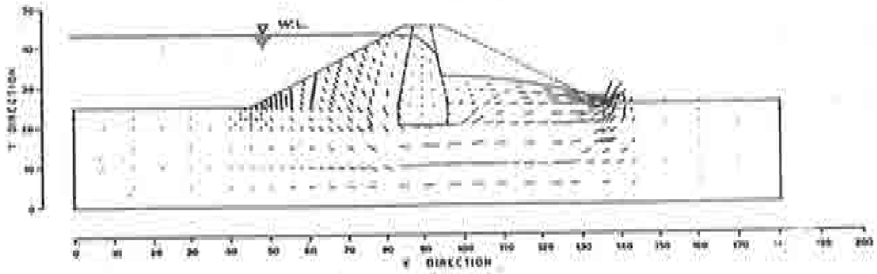
- Confined seepage problems in which all the flow region is known a priori, Fig. 15.4a.

- Unconfined seepage problems in which the flow involves a free surface whose location is an additional unknown of the problem, Fig. 15.4b.

In general, the available exact analytical solutions concern only homogeneous isotropic medium with well defined and simple boundary conditions. For unconfined flow, the geotechnical engineer has traditionally relied on graphical methods, [15, 16]. In the analysis of wells in unconfined flow, groundwater hydrologists have often based their theory on Dupuit's assumptions. Exact analytical methods handling such type of problems have been developed, but they are often difficult to apply. Extensive treatments of these methods can be found in [2, 51, 75]. For the general solution of both confined and



a) Confined flow



b) Unconfined flow

Figure 15.4: Confined and unconfined seepage flow in a porous solid medium

unconfined flow problems in nonhomogeneous porous medium, it is necessary to relay upon numerical solution techniques such as the Finite Element Method, [40].

15.8 The Physics in a Coupled Soild-Fluid Problem

In general, in a porous saturated medium, the equilibrium to a set of external loads is provided by the total stress defined in Eq. 15.30. In particular, due to the action of an additional set of external forces, the solid skeleton of a porous saturated medium deforms trying, therefore, to reduce also the volume of its pores. Consequently, the fluid phase stored in these pores tries to react to this volume reduction increasing its initial pore pressure value π_o. The difference between the current pore pressure value π and the initial one π_o is called *excess pore pressure*, namely

$$\pi_e = \pi - \pi_o \tag{15.52}$$

Due to this increase of pore pressure, the total head value raises its initial value h_o to

$$h = h_o + \frac{\pi_e}{\gamma_f} \tag{15.53}$$

This increment of total head establishes a new flow regime which, eventually, allows the excess quantity of fluid to escape from the pore. Clearly, as the fluid escapes from the pores, the pore pressure value decreases toward its hydrostatic value and the fluid eventually regains its initial flow regime. On the other hand, the value of the effective stress $\boldsymbol{\sigma}'$ increases in order to compensate the reduction of π. This induces additional deformations to the solid skeleton generating, therefore, to the so-called *consolidation process*.

It is of most interest to note that during a consolidation process, the flow continuity equation in Eq. 15.49 takes the form

$$\operatorname{div} \mathcal{P} \left(\frac{\varrho_f \mathbf{a}^{(f)}}{\gamma_f} + \operatorname{grad} \frac{\pi_e}{\gamma_f} \right) = \mathbf{q}^T \dot{\epsilon} + \mathbf{r}^T \dot{\epsilon}^{(\beta)} + \frac{\dot{\pi}_e}{B_\pi} - g \tag{15.54}$$

where g represents the amount of fluid due to local sinks or sources. In fact, according to the generalized Darcy law in Eq. 15.29 and the seepage flow equation in Eq. 15.51

$$-\operatorname{div} \overline{\mathbf{v}}^{(r)} = \operatorname{div} \mathcal{P} \left[\frac{\varrho_f \mathbf{a}^{(f)}}{\gamma_f} + \operatorname{grad} h \right] =$$

$$\begin{aligned} &= \operatorname{div} \mathcal{P}\left[\frac{\varrho_f \mathbf{a}^{(f)}}{\gamma_f} + \operatorname{grad}\left(h_o + \frac{\pi_e}{\gamma_f}\right)\right] = \\ &= \operatorname{div} \mathcal{P} \operatorname{grad} h_o + \operatorname{div} \mathcal{P}\left[\frac{\varrho_f \mathbf{a}^{(f)}}{\gamma_f} + \operatorname{grad} \frac{\pi_e}{\gamma_f}\right] = \\ &= g + \operatorname{div} \mathcal{P}\left[\frac{\varrho_f \mathbf{a}^{(f)}}{\gamma_f} + \operatorname{grad} \frac{\pi_e}{\gamma_f}\right] \end{aligned}$$

15.9 Undrained Conditions

In a coupled solid-fluid problem, *undrained conditions* denote that particular condition in a deformation process in which the fluid phase is not allowed to squeeze out from the pores. For example, such a condition is met at the instant of the application of a set of external loads when the fluid does not have jet the time of escaping. Also, the deformation process that takes place during a short period of dynamic loading conditions due to earthquake may be considered to be in undrained conditions.

Hence, by definition, in undrained conditions

$$\dot{\epsilon}_v^{(\ell)} = \mathbf{q}^T \dot{\epsilon} + \mathbf{r}^T \dot{\epsilon}^{(\beta)} + \frac{\dot{\pi}}{B_\pi} = 0 \tag{15.55}$$

and, consequently, the variation of the pore pressure can be evaluated as

$$\dot{\pi} = -B_\pi \left[\mathbf{q}^T \dot{\epsilon} + \mathbf{r}^T \dot{\epsilon}^{(\beta)}\right] \tag{15.56}$$

The substitution of this expression of $\dot{\pi}$ into Eq. 15.35 yields to the following *total stress constitutive equation*

$$\dot{\sigma} = \mathbf{C}^{(u)}\left(\dot{\epsilon} - \dot{\epsilon}^{(\beta)}\right) + \mathbf{q} B_\pi \dot{\epsilon}_v^{(\beta)} \tag{15.57}$$

where

$$\mathbf{C}^{(u)} = \mathbf{C} + B_\pi \mathbf{q}\mathbf{q}^T$$

can be called the *undrained constitutive matrix* and

$$\dot{\epsilon}_v^{(\beta)} = \widehat{\mathbf{m}}^T \dot{\epsilon}^{(\beta)}$$

In fact,

$$\dot{\sigma} = \mathbf{C}\left(\dot{\epsilon} - \dot{\epsilon}^{(\beta)}\right) - \mathbf{q}\dot{\pi} =$$

$$
\begin{aligned}
&= \mathbf{C}\left(\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}^{(\beta)}\right) + \mathbf{q}B_\pi\left[\mathbf{q}^T\dot{\boldsymbol{\epsilon}} + \mathbf{r}^T\dot{\boldsymbol{\epsilon}}^{(\beta)}\right] = \\
&= \left[\mathbf{C} + B_\pi\mathbf{q}\mathbf{q}^T\right]\dot{\boldsymbol{\epsilon}} - \left[\mathbf{C} - B_\pi\mathbf{q}\mathbf{r}^T\right]\dot{\boldsymbol{\epsilon}}^{(\beta)} = \\
&= \mathbf{C}^{(u)}\dot{\boldsymbol{\epsilon}} - \left[\mathbf{C} - B_\pi\mathbf{q}(\widehat{\mathbf{m}}^T - \mathbf{q}^T)\right]\dot{\boldsymbol{\epsilon}}^{(\beta)} = \\
&= \mathbf{C}^{(u)}\dot{\boldsymbol{\epsilon}} - \left[\mathbf{C} + B_\pi\mathbf{q}\mathbf{q}^T\right]\dot{\boldsymbol{\epsilon}}^{(\beta)} + B_\pi\mathbf{q}\widehat{\mathbf{m}}^T\dot{\boldsymbol{\epsilon}}^{(\beta)} = \\
&= \mathbf{C}^{(u)}\dot{\boldsymbol{\epsilon}} - \mathbf{C}^{(u)}\dot{\boldsymbol{\epsilon}}^{(\beta)} + B_\pi\mathbf{q}\dot{\epsilon}_v^{(\beta)}
\end{aligned}
$$

Note that, if $\mathbf{C}$ is a symmetric matrix, then also $\mathbf{C}^{(u)}$ is a symmetric matrix. Moreove, if tangent stiffness matrix of the solid skeleton behaves as an incrementally isotropic material with the tangent stiffness matrix defined as in Eq. 15.36, then $\mathbf{C}^{(u)}$ may be written as

$$
\mathbf{C}^{(u)} = \frac{E_u}{(1+\nu_u)(1-2\nu_u)}
\begin{bmatrix}
(1-\nu_u) & \nu_u & \nu_u & 0 & . & 0 \\
\nu_u & (1-\nu_u) & \nu_u & 0 & . & 0 \\
\nu_u & \nu_u & (1-\nu_u) & 0 & . & 0 \\
0 & 0 & 0 & (1-2\nu_u) & . & 0 \\
. & . & . & . & . & . \\
0 & 0 & 0 & 0 & . & (1-2\nu_u)
\end{bmatrix}
\tag{15.58}
$$

where

$$
\begin{aligned}
E_u &= E\frac{1+\nu_u}{1+\nu} \\
\nu_u &= \frac{1-u}{2-u} \\
u &= \frac{k}{(1-\nu)k+q^2} \\
k &= \frac{B}{B_\pi} \\
r &= \frac{B}{B_m} \\
q &= 1-r \\
B &= \frac{E}{3(1-2\nu)}
\end{aligned}
$$

For most soils, as already mentioned, $B_m \gg B$ and also $B_\pi \gg B$. In this case, therefore, $k \approx 0$, $r \approx 0$ and $q \approx 1$. Consequently, $u \approx 0$ and

$$
\begin{aligned}
E_u &\approx \frac{2}{3}\frac{E}{1+\nu} \\
\nu_u &\approx \frac{1}{2} \\
B &\approx \infty
\end{aligned}
$$

These results should appear very familiar to any geotechnical engineer since they coincide with the conventional assumption on the behavior of saturated soils for undrained conditions.

However, the important point is that in undrained conditions, coupled solid-fluid geotechnical problems can be solved as any other engineering problem involving only one-phase solid materials. In fact, neglecting $\epsilon^{(\beta)}$, static problems can be solved in terms of total stress with a set of equations by all means analogous to that one used for the solution of one-phase solid problems, Section 14.2.

For dynamic loading conditions, the inertial term, which in a one-phase solid problem is defined as

$$\mathbf{i} = \varrho \dot{\mathbf{v}}$$

in a coupled solid fluid problem has to be defined as

$$\mathbf{i} = (1-n)\varrho_m \dot{\mathbf{v}} + n\varrho_f \dot{\mathbf{v}}^{(f)} \tag{15.59}$$

since $\dot{\mathbf{v}}$ acts on the solid skeleton while $\dot{\mathbf{v}}^{(f)}$ acts on the fluid phase. Taking into account that, Eqs. 15.27 and 15.28,

$$\dot{\mathbf{v}}^{(f)} = \dot{\mathbf{v}} + \dot{\mathbf{v}}^{(r)} = \dot{\mathbf{v}} + \frac{\dot{\overline{\mathbf{v}}}^{(r)}}{n}$$

it follows that Eq. 15.59 can be rewritten as

$$\mathbf{i} = \varrho \dot{\mathbf{v}} + \varrho_f \dot{\overline{\mathbf{v}}}^{(r)} \tag{15.60}$$

where, in this case,

$$\varrho = (1-n)\varrho_m + n\varrho_f$$

represents the mass density of the solid-fluid mixture. Fortunately, the term $\varrho_f \dot{\overline{\mathbf{v}}}^{(r)}$ may be generally neglected since its influence seems to be generally small and it becomes important only at very high frequencies, [96]. Thus, also dynamic problems can be solved in terms of total stress with a set of equations by all means analogous to that one used for the solution of one-phase solid problems, Section 14.2.

Once this system is solved, we can calculate the increase of the pore pressure as in Eq. 15.56 and, then, the variation of the effective stress according to the Principle of Effective Stress in Eq. 15.30.

15.10 The Coupled Theory of Consolidation

In Section 10.9 we have shown that in the special case of undrained conditions, we can solve explicitly the flow continuity equation in terms of excess pore pressure π_e and obtain a constitutive relationship relating total stress $\boldsymbol{\sigma}$ with the strain $\dot{\epsilon}$ of the solid skeleton. This allows to a coupled undrained problem to be solved in terms of total stress through a set of equations formally identical to that one reported in Section 14.2 for the solution of a one-phase solid problem.

In general, however, the flow continuity equation cannot be solved alone and, therefore, the excess pore pressure represents the additional unknown of a coupled solid-fluid problem. Thus, for a complete solution of a coupled solid-fluid problem it is necessary to solve contemporaneously the following field equations:

- The 3 Cauchy Equation of Motions

$$\sigma_{ij,i} + \varrho b_i = \varrho \ddot{u}_i + \varrho_f \ddot{\bar{u}}_i \tag{15.61}$$

  where σ_{ij} are the Cauchy total stress tensor components.

- The 6 Constitutive Relationships, Eq. 15.35,

$$\delta\sigma_{ij} = C_{ijkl}(\delta\epsilon_{kl} - \delta\epsilon_{kl}^{(\beta)}) - Q_{ij}\delta\pi_e \tag{15.62}$$

  where $[C_{ijkl}]$ is the tangential stiffness matrix while $[Q_{ij}]$ is the matrix form of the vector $\mathbf{q}$.

- The 6 Geometric Equations

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{15.63}$$

- The Flow Continuity Equation, Eq. 15.54,

$$\operatorname{div} \mathcal{P}\left(\frac{\varrho_f \mathbf{a}^{(f)}}{\gamma_f} + \operatorname{grad}\frac{\pi_e}{\gamma_f}\right) = \mathbf{q}^T\dot{\epsilon} + \mathbf{r}^T\dot{\epsilon}^{(\beta)} + \frac{\dot{\pi}_e}{B_\pi} - g \tag{15.64}$$

which, neglecting $\varrho_f \mathbf{a}^{(f)}$, represents a system of 16 equations for the 16 unknowns: 6 σ_{ij} stresses, 6 ϵ_{ij} strains, 3 u_i displacements and 1 π_e excess pore pressure.

Although the mathematical problem is completely defined, the solution of the above system of equations still remains a formidable task. As a result of that, analytical solutions are rather limited and generally confined to flexible footings on homogeneous, isotropic and linear elastic semi-infinite half space.

Some of these solutions have been published by Biot, [5, 6], Josselin de Joung [54], Mc Namee and Gibson, [65, 66, 42]. Solutions for consolidation problems involving finite depth have been proposed by Gibson et al. [43] and Booker [12]. A discussion at length on the validity of this theory has been presented by Shiffman et. al., [85],

The important feature of these solutions is the analytical confirmation of the so-called *Mandel-Cryer effect* on the excess pore pressure by which, [63, 20], on the contrary of what one might expect, the excess pore pressure not necessarily monotonically decreases its value with time, Fig. 15.5. Since

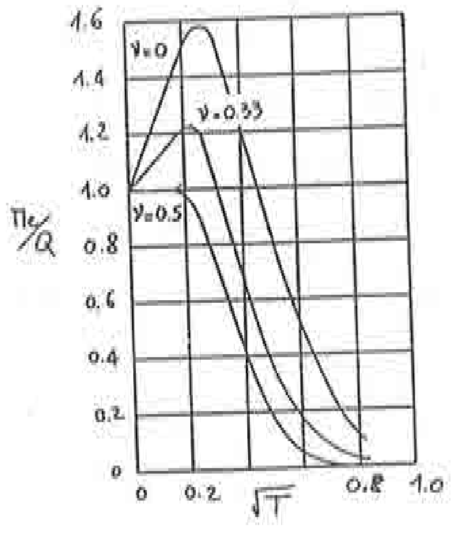


Figure 15.5: Excess pore pressure vs. time factor in the center of a spherical sample loaded uniformly by Q

the excess pore pressure is affected by changes in the mean total stress, it may, at some places within the soil mass, continue to increase for some time after the application of load. The existence of this Mandel-Cryer effect is now supported by a vast experimental evidence. Interesting, for instance, is the experimental study performed by Aboshi, [1].

15.11 The Oedometer Problem

Consider the 1D consolidation problem shown in Fig. 15.6, known as the *oedometer problem*, and assume that:

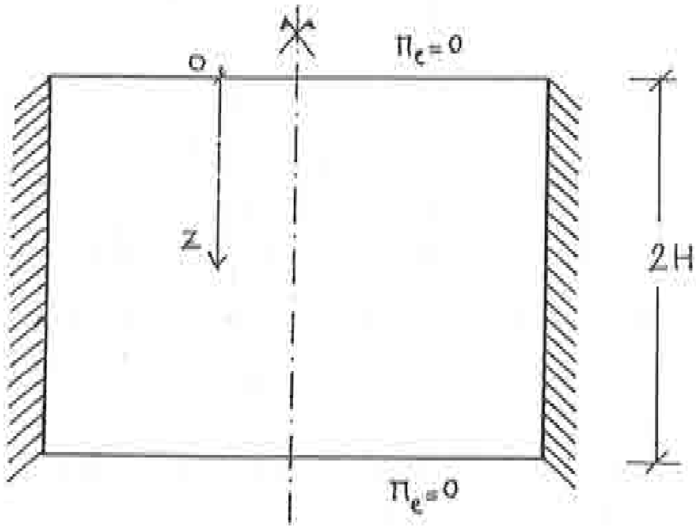


Figure 15.6: 1D consolidation problem

- the porous medium is an isotropic and homogeneous soil in both mechanical and hydraulical sense;
- the solid skeleton behaves as an incrementally isotropic material;
- static loading conditions;
- negligible acceleration in the fluid phase;
- negligible autogeneous strain;
- no sources nor sinks.

Under the above conditions, the Flow Continuity Equation in Eq. 15.54 can be reduced into

$$c_v \frac{\partial^2 \pi_e}{\partial z^2} = \frac{\partial \pi_e}{\partial t} \tag{15.65}$$

where π_e is the excess pore pressure,

$$\begin{aligned} c_v &= \frac{\mathcal{P} B_1}{\gamma_f} \\ B_1 &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \end{aligned}$$

and

$$\begin{aligned} z &= \text{vertical axis;} \\ \mathcal{P} &= \text{permeability coefficient;} \\ \gamma_f &= \text{specific weight of the fluid phase.} \end{aligned}$$

The initial condition for the differential equation in Eq. 15.65 is given by

$$\pi_e(z, t=0) = Q \tag{15.66}$$

for all $z \in [0, 2H]$, where Q is the applied load. While, for all $t > 0$, the boundary conditions are given by

$$\pi_e(z=0, t) = 0 \tag{15.67}$$

$$\pi_e(z=2H, t) = 0 \tag{15.68}$$

15.11.1 The mathematical derivation

In soils, as already remarked in Section 15.5, we can set $\mathbf{r} \simeq \mathbf{o}$ and $\mathbf{q} \simeq \widehat{\mathbf{m}}$. Thus, neglecting $\mathbf{a}^{(f)}$ and $\dot{\epsilon}^{(\beta)}$, the Flow Continuity Equation in Eq. 15.54 reduces to

$$\operatorname{div} \mathcal{P} \operatorname{grad} \frac{\pi_e}{\gamma_f} = \dot{\epsilon}_v + \frac{\dot{\pi}_e}{B_\pi} \tag{15.69}$$

Moreover, since in an incrementally isotropic soil,

$$\dot{\epsilon}_v = \frac{\dot{p}'}{B}$$

it results that

$$\dot{\epsilon}_v^{(\ell)} = \dot{\epsilon}_v + \frac{\dot{\pi}_e}{B_\pi} = \frac{\dot{p}'}{B} + \frac{\dot{\pi}_e}{B_\pi} = \frac{1}{B}\left[\dot{p}' + \frac{B}{B_\pi}\dot{\pi}_e\right] =$$
$$\approx \frac{\dot{p}'}{B} = \dot{\epsilon}_v$$

being in soils $B_\pi \gg B$. Hence, Eq. 15.69, can be reduced to

$$\operatorname{div} \mathcal{P} \operatorname{grad} \frac{\pi_e}{\gamma_f} = \dot{\epsilon}_v \tag{15.70}$$

and, in a 1D isotropic problem, it can be also expressed as

$$\frac{\mathcal{P}}{\gamma_f}\frac{\partial^2 \pi_e}{\partial z^2} = \dot{\epsilon}_v \tag{15.71}$$

For an incrementally isotropic soil, the effective stress constitutive equation has the form, Section 11.7,

$$\dot{\sigma}' = \mathbf{C}\dot{\epsilon} \tag{15.72}$$

where the tangent stiffness matrix $\mathbf{C}$ is of the type of that reported in Eq. 15.36.

In the 1D problem shown in Fig. 15.6, the displacement function $\mathbf{u}(\mathbf{x}, t)$ is of the type

$$\begin{cases} u_1 = u_2 = 0 \\ u_3 = u_3(z, t) \end{cases}$$

This leads to the conclusion that the vertical strain $\dot{\epsilon}_z$ is the only strain component with non zero value. For such type of strain conditions, the constitutive relationship in Eq. 15.72 identifies that the vertical and horizontal effective stresses are related as

$$\dot{\sigma}'_x = \dot{\sigma}'_y = A\dot{\sigma}'_z \tag{15.73}$$

where

$$A = \frac{\nu}{1-\nu}$$

while all the other stress components have null value. Then, according to the Principle of Effective stress in Eq. 15.30, the vertical and horizontal total stresses are related as

$$\dot{\sigma}_x = \dot{\sigma}_y = A\dot{\sigma}_z + (A-1)\dot{\pi}_e \tag{15.74}$$

Under the above stress conditions, the Cauchy Equations of Motion reduce to

$$\begin{cases} \dot{\sigma}_{x,x} = 0 \\ \dot{\sigma}_{y,y} = 0 \\ \dot{\sigma}_{z,z} = 0 \end{cases}$$

and the relative solution is of the type

$$\begin{cases} \sigma_x = \sigma_y = F(z,t) \\ \sigma_z = B(t) \end{cases} \tag{15.75}$$

By imposing the boundary condition

$$\dot{\sigma}_z(z=0,t) = -\dot{Q}(t)$$

we find

$$B(t) = -\dot{Q}(t)$$

Hence, from Eqs. 15.74 and 15.75, it results that

$$\begin{cases} \dot{\sigma}_x = \dot{\sigma}_y = -A\dot{Q} + (A-1)\dot{\pi}_e \\ \dot{\sigma}_z = -\dot{Q} \end{cases} \tag{15.76}$$

which, in terms of effective stress, takes the form

$$\begin{cases} \dot{\sigma}'_x = \dot{\sigma}'_y = A(\dot{\pi}_e - \dot{Q}) \\ \dot{\sigma}'_z = \dot{\pi}_e - \dot{Q} \end{cases} \tag{15.77}$$

Accordingly, the mean effective stress results to be equal to

$$\dot{p}' = \frac{\dot{\sigma}'_z + 2\dot{\sigma}'_x}{3} = \frac{(1+2A)}{3}(\dot{\pi}_e - \dot{Q}) \tag{15.78}$$

and the volumetric strain can be calculated as

$$\dot{\epsilon}_v = \frac{\dot{p}'}{B} = \frac{(1+2A)}{3B}(\dot{\pi}_e + \dot{Q}) = \frac{\dot{\pi}_e - \dot{Q}}{B_1} \tag{15.79}$$

Thus, the flow continuity equation in Eq. 15.71 can be written as

$$\frac{\mathcal{P}}{\gamma_f}\frac{\partial^2 \pi_e}{\partial z^2} = \frac{\dot{\pi}_e - \dot{Q}}{B_1} \tag{15.80}$$

Then, let us consider the case of a constant load function, that is $Q(t) = Q$ for all $t \geq 0$. In this case:

- At time $t = 0$, we can assume undrained conditions, that is

$$\dot{\epsilon}_v^{(\ell)} = \dot{\epsilon}_v = \frac{\dot{\pi}_e - \dot{Q}}{B_1} = 0$$

  Accordingly, we can establish that $\dot{\pi} = \dot{Q}$ confirming therefore the initial condition stated in Eq. 15.66.

- At time $t > 0$, being the load constant, we have $\dot{Q} = 0$. Accordingly, Eq. 15.80 can be reduced into the differential equation in Eq. 15.65.

It is interesting to note that, as remarked in Section 15.9, in undrained conditions we can set $\nu = \nu_u = 1/2$. Consequently, $A = 1$ and, from Eqs. 15.76 and 15.77, being $\dot{\pi}_e = \dot{Q}$,

$$\begin{array}{c} \dot{\sigma}'_x = \dot{\sigma}'_y = \dot{\sigma}'_z = 0 \\ \dot{\sigma}_x = \dot{\sigma}_y = \dot{\sigma}_z = \dot{Q} \end{array} \tag{15.81}$$

from which we can establish that

$$\epsilon_x = \epsilon_y = \epsilon_z = 0 \tag{15.82}$$

While, during the consolidation process, being $\dot{Q} = 0$,

$$\left\{ \begin{array}{l} \dot{\sigma}'_x = \dot{\sigma}'_y = A\dot{\pi}_e \\ \dot{\sigma}'_z = \dot{\pi}_e \end{array} \right. \tag{15.83}$$

and

$$\left\{ \begin{array}{l} \dot{\sigma}_x = \dot{\sigma}_y = (A-1)\dot{\pi}_e \\ \dot{\sigma}_z = 0 \end{array} \right. \tag{15.84}$$

from which we can establish that

$$\epsilon_x = \epsilon_y = 0 \tag{15.85}$$

$$\epsilon_z = \frac{\dot{\pi}_e}{B_1} \tag{15.86}$$

15.11.2 Analytical solution

Let assume the soil to be linear elastic and isotropic. In this case, $c_v = const$ and the 1D Flow Continuity Equation in Eq. 15.65 may be written in the following adimensional form

$$\frac{\partial^2 \pi_e}{\partial Z^2} = \frac{\partial \pi_e}{\partial T_v} \tag{15.87}$$

which can be obtained by setting

$$\begin{aligned} t &= \frac{H^2 T_v}{c_v} \\ z &= ZH \end{aligned}$$

The relative initial condition is given by

$$\pi_e(Z, T_v = 0) = Q \tag{15.88}$$

for all $Z \in [0, 2]$, while for all $t > 0$

$$\begin{aligned} \pi_e(Z = 0, T_v) &= 0 \\ \pi_e(Z = 2, T_v) &= 0 \end{aligned}$$

It is possible to prove that the relative analytical solution is given by, [88] see also [40] Chapter 13,

$$\pi_e(Z, T_v) = \sum_{k=1}^{\infty} \frac{2Q}{N_k} \sin(N_k Z)\, e^{\left(-N_k^2 T_v\right)} \tag{15.89}$$

where

$$N_k = \frac{\pi(2k-1)}{2}$$

Under this hypothesis of linear elasticity, the modulus B_1 in Eq. 15.86 results to be constant and, consequently, the vertical strain mobilized during a consolidation process can be calculated as

$$\Delta\epsilon_z(Z, T_v) = \frac{\Delta\pi_e(Z, T_v)}{B_1} \tag{15.90}$$

where

$$\begin{aligned} \Delta\epsilon_z(Z, T_v) &= \epsilon_z(Z, T_v) - \epsilon_z(Z, T_v = 0) \\ \Delta\pi_e(Z, T_v) &= \pi_e(Z, T_v) - \pi_e(Z, T_v = 0) = \\ &= \pi_e(Z, T_v) - Q \end{aligned}$$

Accordingly, the vertical displacement $\varrho(Z, T_v)$ at depth Z and time T_v can be calculated as

$$\begin{aligned} \varrho(Z, T_v) &= \int_0^Z \Delta\epsilon_z(Z, T_v) dZ = \int_0^Z \frac{\Delta\pi(Z, T_v)}{B_1} dZ = \\ &= \frac{1}{B_1}\left[\int_0^Z \pi_e(Z, T_v) dZ - QZ\right] \end{aligned} \tag{15.91}$$

It is defined as the *average degree of consolidation* $U(T_v)$ at time T_v the displacement ratio

$$U(T_v) = \frac{\varrho(Z = 2, T_v)}{\varrho(Z = 2, T_v \rightarrow \infty)} \tag{15.92}$$

According to Eq. 15.91, this ratio can be expressed in term of excess pore pressure as

$$U(T_v) = 1 - \frac{1}{2Q}\int_0^2 \pi_e(Z, T_v)dZ \tag{15.93}$$

and, finally, substituting to $\pi_e(Z, T_v)$ the analytical expression in Eq. 15.89, we obtain

$$\begin{aligned} U(T_v) &= 1 - \sum_{k=1}^{\infty}\left[\frac{1-\cos(2N_k)}{N_k^2}\right]e^{(-N_k^2 T_v)} = \\ &= 1 - \sum_{k=1}^{\infty}\frac{2}{N_k^2}e^{(-N_k^2 T_v)} \end{aligned} \tag{15.94}$$

15.12 The Uncoupled Theory of Consolidation

Let consider the following consolidation problem in soils:

- solid skeleton behavior representable by a variable moduli model;
- static loading conditions;
- negligible acceleration in the fluid phase;
- negligible autogeneous strains;
- no sources nor sinks.

We have proved in Section 15.11.1 that under the above hypothesis, the Flow Continuity Equation in Eq. 15.54 can be reduced to, Eq. 15.70

$$\operatorname{div}\mathcal{P}\operatorname{grad}\frac{\pi}{\gamma_f} = \dot{\epsilon}_v \tag{15.95}$$

where, depending on the type of problem to be solved, the volumetric strain can be calculated as follows

- 1D problems,

$$\dot{\epsilon}_v = \frac{\dot{p}'}{B} = \frac{\dot{\sigma}'_z}{B_1} \tag{15.96}$$

where

$$B_1 = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$$

since it is possible to prove that in this case, Eq. 15.83,

$$\begin{cases} \dot{\sigma}'_x = \dot{\sigma}'_y = A\dot{\pi}_e \\ \dot{\sigma}'_z = \dot{\pi}_e \end{cases}$$

where

$$A = \frac{\nu}{1-\nu}$$

- 2D plane strain problems,

$$\dot{\epsilon}_v = \frac{\dot{p}'}{B} = \frac{\dot{\sigma}'_x + \dot{\sigma}'_y}{2B_2} \tag{15.97}$$

where

$$B_2 = \frac{E}{2(1+\nu)(1-2\nu)}$$

since it is possible to prove that in this case

$$\dot{\sigma}'_z = \nu(\dot{\sigma}'_x + \dot{\sigma}'_y)$$

- 3D problems,

$$\dot{\epsilon}_v = \frac{\dot{p}'}{B} \tag{15.98}$$

where, Section 11.7,

$$B = \frac{E}{3(1-2\nu)}$$

The *Uncoupled Theory of Consolidation* assumes that, analogously to the 1D problem:

- In a 2D plane strain problem,

$$\frac{\dot{\sigma}'_x + \dot{\sigma}'_y}{2} = \dot{\pi}_e \tag{15.99}$$

- In a 3D problem,

$$\dot{p}' = \dot{\pi}_e \tag{15.100}$$

Accordingly, the Fluid Continuity Equation in Eq. 15.95 can be reduced into

$$\operatorname{div} \mathcal{P} \operatorname{grad} \frac{\pi}{\gamma_f} = \frac{\dot{\pi}_e}{\overline{B}} \tag{15.101}$$

where

$$\overline{B} = \begin{cases} B_1; & \text{1D case, Eq. 15.96.} \\ B_2; & \text{2D plane strain case, Eq. 15.97.} \\ B; & \text{3D case, Eq. 15.98.} \end{cases}$$

The solution of this differential equation may describe the whole dissipation process of a given initial excess pore pressure distribution. The initial value of π_e due to an additional set of external forces can be determined by solving the Cauchy Equations of Equilibrium in undrained conditions, Section 15.9. Knowing at any time the excess pore pressure value, we can then calculate in sequence the effective stress field, the strain field and, eventually, the displacement, [40] Sect. 14.11.

However, unless for the 1D case, the hypothesis of the Uncoupled Theory on the effective stress variation during consolidation are not supported by physical reasons. One of the most evident wrong conclusion of the uncoupled theory is to predict always a monotonically decrease of the excess pore pressure. This can be easily understood by looking the mathematical nature of Eq. 15.96. As a consequence, we may obtain a false stress path σ' which for a nonlinear constitutive equation is a very delicate point. In fact, different stress paths, which even arrive to the same final value, may produce completely different displacement field.

In spite of these theoretical limitations, the Uncoupled Theory still represents the most common approach in the engineering practice. The main reason is because its use is by far more simple and economical than the alternative Coupled Theory. Moreover, in the common engineering problems, the Uncoupled approach produces sufficiently accurate results.

Chapter 16

A Constitutive Equation for Soils

16.1 Introduction

In this last Chapter[1] we present a constitutive model for soils, called SUOLO Standard V1 [24, 25, 26], which satisfies the following assumptions on the mechanical behavior:

1. The (effective) stress σ and the specific volume v are the *State Variables* of the soil, that is: the mechanical soil response $\delta\epsilon$ for a given $\delta\sigma$, and viceversa, is *uniquely* determined only by the current values of (σ, v), regardless the previous stress history.

2. Isotropic mechanical behavior.

3. Soils, subjected to cycles of loading and unloading, present accumulation of irreversible deformations.

4. There exists a single-value function

$$v_{max} = v_{max}(p, q, \theta) \tag{16.1}$$

which bounds always the value of the specific volume v of a soil at a stress state (p, q, θ), that is

$$v \leq v_{max}$$

[1]Chapter written in collaboration with A. De Crescenzo

For any given θ, Eq. 16.1 is the equation of a surface in the (p, q, v) space, called *Limit State Boundary Surface*, LSS.

5. There exists a *Critical State condition* where indeterminate shear strain ϵ_s occurs with no change in the stress σ and in the specific volume v. In the σ space, the locus of all the stress levels corresponding to critical state conditions is called *Critical State Surface*, CSS. For isotropic soils, this CSS is represented by a function of the type

$$q = q_c(p, \theta) \tag{16.2}$$

At this critical state condition:

- the value of v is the maximum compatible with the stress state, that is
$$v = v_{max}(p, q, \theta)$$
- for any given θ, the locus in the (p, q, v) space of the soil states under critical state conditions is a line known as the *Critical State Line*, CSL.

In this Chapter we will always indicate σ as the stress acting on the solid skeleton; thus, in a saturated soil σ will indicate the effective stress.

16.2 Reference Soil Tests

The SUOLO Standard V1 model is based on some basic assumptions of the soil response under triaxial test conditions which we will illustrate in this Section. It is to be noted that the simple triaxial tests presented in this Section are all the required tests to set-up the soil parameters needed by the model.

16.2.1 Isotropic drained tests

Under cycles of isotropic loading and unloading drained tests there is no distorsional strain; the material behavior is assumed to be of the type shown in Fig. 16.1. In this Figure we identify the following 3 parallel straight lines:

- A Normally Consolidated Line, NCL, of equation

$$v = v_{NCL}(p) \tag{16.3}$$

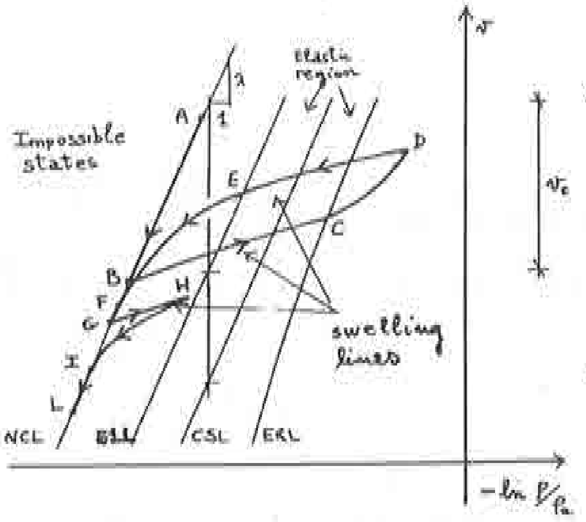


Figure 16.1: Material response in a isotropic loading drained test; p_a is the atmospheric pressure.

where

$$\begin{aligned} p &< 0 \\ v_{NCL}(p) &= v_\lambda - \lambda \ln \frac{p}{p_\lambda} \end{aligned}$$

and λ is a material constant. The pair (p_λ, v_λ) identifies an arbitrary point on NCL and must respect the conditions

$$\begin{aligned} p_\lambda &< 0 \\ v_\lambda &> 0 \end{aligned}$$

- A line CSL$^{(pv)}$, projection in the v vs p plane of the Critical State Line CSL, of equation

$$v = v_{NCL}(p) - v_c \tag{16.4}$$

where the material parameter v_c must respect the inequality

$$v_c \geq \lambda \ln 2$$

or, alternatively,

$$\begin{aligned} p_o &= 0 \\ v_c &\geq (\lambda - \chi)\ln 2 \\ p &< 0 \end{aligned}$$

The above inequalities are assumed for mathematical reasons which will be clear in Sections 16.10 and 16.12.

- An Elastic Left-hand-side Line, ELL, of equation

$$v = v_{NCL}(p) - v_e \tag{16.5}$$

where

$$v_e \in [0, v_c]$$

is a material parameter.

It is also assumed the existence of a family of Swelling Lines, SWL, of equation

$$v = v_{SWL}(p) \tag{16.6}$$

where

- for $p < 0$,

$$v_{SWL}(p) = v^{(1)}_{SWL}(p) = v^{(1)}_{\chi} - \chi \ln \frac{p + p_o}{p^{(1)}_{\chi} + p_o}$$

and

$$\begin{aligned} p_o &< 0 \\ \chi &\in]0, \lambda[\end{aligned} \tag{16.7}$$

are material constants. The pair $(p^{(1)}_{\chi}, v^{(1)}_{\chi})$ identifies an arbitrary point on a SWL and must respect the conditions

$$\begin{aligned} p^{(1)}_{\chi} &< 0 \\ v^{(1)}_{\chi} &> 0 \end{aligned}$$

- for $p \geq 0$,

$$v_{SWL}(p) = v_{SWL}^{(2)}(p) = v_X^{(2)} - \chi \frac{p - p_X^{(2)}}{p_o};$$

The pair $(p_X^{(2)}, v_X^{(2)})$ identifies an arbitrary point on a SWL and must respect the conditions

$$\begin{aligned} p_X^{(2)} &\geq 0 \\ v_X^{(2)} &> 0 \end{aligned}$$

Material states (p, v) on the left-hand-side of the NCL are impossible. If the material state lies on the NCL, then the soil is said to be *Normally Consolidated*, NCS. While, if the material state lies on the right-hand-side of the NCL, then the soil is said to be *Over Consolidated*, OCS. Then:

- The paths AB, FG and IL on the NCL represent the isotropic loading (compression) of a NCS;
- The paths BCD and GH represent the isotropic unloading of a OCS;
- The paths DEF and HI represent the isotropic reloading of a OCS.

The lines ELL and ERL, the Elastic Right-hand-side Line, are assumed to bound the region for purely elastic response of soils subjected to isotropic loading or unloading conditions. The unloading paths BC and GH and the reloading path DE are assumed to be on SWL of equation of the type in Eq. 16.6.

16.2.2 Undrained triaxial tests

A set of identical samples of NCS are isotropically loaded (compression) under drained conditions up to a confining pressure $p^{(o)}$, point P_o in Fig. 16.2a. Then, they are isotropically unloaded, always in drained conditions, up to different confining pressures $p_1 < p_2 < \ldots < p_5$; points P_i, for $i = 1, 2, \ldots, 5$, in Fig. 16.2a.

These samples of OCS, subjected to triaxial (compression) undrained conditions, are assumed to respond as shown in Fig. 16.2. We note that:

- All the material states (p, v) terminate on the $CSL^{(pv)}$, Eq. 16.4.

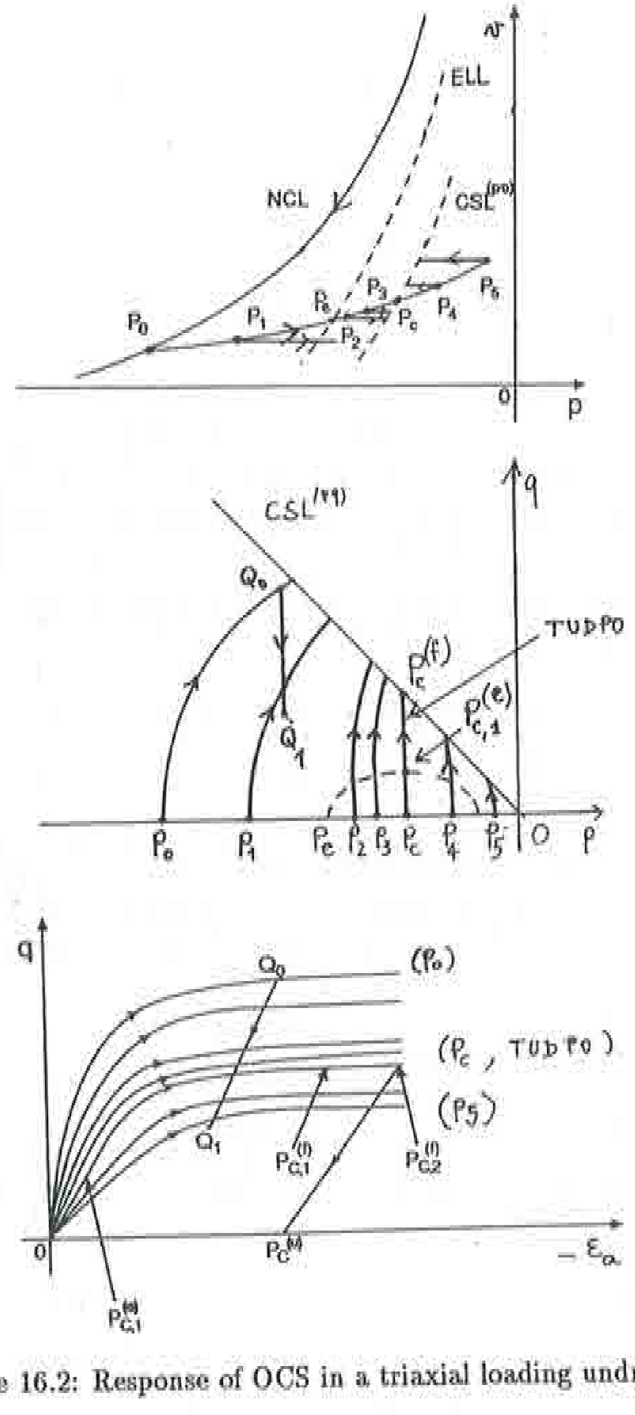


Figure 16.2: Response of OCS in a triaxial loading undrained test.

- All the material states (p,q) terminate on the $\mathrm{CSL}^{(pq)}$, projection in the q vs p plane of the CSL. By definition the $\mathrm{CSL}^{(pq)}$, may be seen also as the intersection of the CSS with the q vs p plane, for $\theta = \pi/6$. We assume that the CSS is given by the Modified Mohr-Coulomb Surface, Eq. 11.15,

$$q = q_c(p,\theta) \tag{16.8}$$

where

$$q_c = -M(\theta)p$$

- The stress paths (p,q) of the OCS whose initial confining pressure p_i lye within the elastic region first rise vertically and then bend toward $\mathrm{CSL}^{(pq)}$.

- The points where the stress path (p,q) of these OCS start to bend, define an ellipse of center P_e in the p vs. q plane.

- The stress path (p,q) of the OCS whose preconsolidation pressure coincides with the center of this ellipse rises vertically toward the $\mathrm{CSL}^{(pq)}$. We will refer to this particular test as the TUDP0 test which stands for Triaxial Undrained $\delta p = 0$ test.

16.3 The Mathematical Framework

The SUOLO Standard V1 model is based on the general theoretical framework presented in Chapter 13. In this case, the 10 main hypotheses listed in Section 13.2 are specialized as follows:

1. There exists a *Bounding Surface*

$$\overline{F} = \overline{F}(p,q,\theta,\overline{p}_y) \tag{16.9}$$

which, as the name indicates, bounds always the location of the current stress point. Accordingly, if $(p,q,\theta,\overline{p}_y)$ represents the current material state, the only admissible alternative conditions are

$$\begin{cases} \overline{F}(p,q,\theta,\overline{p}_y) = 0; & \text{i.e. } \boldsymbol{\sigma} \text{ lies on } \overline{F}. \\ \overline{F}(p,q,\theta,\overline{p}_y) < 0; & \text{i.e } \boldsymbol{\sigma} \text{ lies inside } \overline{F} \end{cases}$$

while

$$\overline{F}(p,q,\theta,\overline{p}_y) > 0$$

is not admissible. Moreover, it is required that $\overline{F}$ is a convex simply connected surface in the principal stress space and that it is a continuous monotonically increasing function of $\overline{p}_y$, namely

$$\frac{\partial \overline{F}}{\partial \overline{p}_y} > 0 \tag{16.10}$$

2. The hardening parameter $\overline{p}_y$ is a continuous monotonically increasing function of the plastic specific volume $v^{(p)}$, that is

$$\overline{p}_y = \overline{p}_y(v^{(p)}) \tag{16.11}$$

with

$$\frac{d\overline{p}_y}{dv^{(p)}} > 0$$

Accordingly, its incremental variation is given by

$$\delta\overline{p}_y = \frac{d\overline{p}_y}{dv^{(p)}}\delta v^{(p)} \tag{16.12}$$

where, Eq. 16.80,

$$\delta v^{(p)} = v\delta\epsilon_v^{(p)} = v\delta\lambda\frac{\partial G}{\partial p} = v\delta\lambda\frac{\partial F}{\partial p} \tag{16.13}$$

where the function G and the scalar $\delta\lambda$ are defined as reported in items 8 and 9, respectively.

3. There exists a *Yield Surface*

$$F = F(p, q, \theta, \overline{p}_y, p_y) \tag{16.14}$$

which always follows the location of the current stress point. Accordingly, the only admissible material condition is

$$F(p, q, \theta, \overline{p}_y, p_y) = 0 \tag{16.15}$$

while

$$F(p, q, \theta, \overline{p}_y, p_y) \neq 0$$

is not admissible.

4. The functional relationships for the hardening parameter p_y are given by:

(a) There exists a scalar function, derived from Eq. 16.15, of the type

$$p_y = p_y(p, q, \theta, \overline{p}_y) \tag{16.16}$$

so that for any given material state $(p, q, \theta, \overline{p}_y)$ the current location of the yield surface $F(p, q, \theta, \overline{p}_y, p_y)$ can be uniquely determined.

(b) If plasticity occurs, then the incremental variation of the hardening parameter p_y is given by

$$\delta p_y = \frac{\partial p_y}{\partial v^{(p)}} \delta v^{(p)} + \frac{\partial p_y}{\partial \epsilon_s^{(p)}} \delta \epsilon_s^{(p)} \tag{16.17}$$

where

$$\epsilon_s^{(p)} = \int \delta \epsilon_s^{(p)} \tag{16.18}$$

is the total plastic shear strain and, Eq. 16.77,

$$\delta \epsilon_s^{(p)} = \delta \lambda \sqrt{\frac{2}{3}} \|\nabla_s G\| = \delta \lambda \left[\left(\frac{\partial F}{\partial q} \right)^2 + \frac{1}{q^2} \left(\frac{\partial F}{\partial \theta} \right)^2 \right] \tag{16.19}$$

Moreover,

$$\frac{\partial p_y}{\partial v^{(p)}} = \frac{\partial p_y}{\partial v^{(p)}} (p_y, \overline{p}_y) \tag{16.20}$$

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} = \frac{\partial p_y}{\partial \epsilon_s^{(p)}} (p_y, \overline{p}_y) \tag{16.21}$$

and

$$\frac{\partial p_y}{\partial v^{(p)}} \quad \begin{cases} \geq \dfrac{d\overline{p}_y}{dv^{(p)}} & \text{for } \delta v^{(p)} \leq 0 \\ \leq \dfrac{d\overline{p}_y}{dv^{(p)}} & \text{for } \delta v^{(p)} > 0 \end{cases} \tag{16.22}$$

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} \quad \leq 0 \tag{16.23}$$

It is immediate to prove that, being $\delta \epsilon_s^{(p)} \geq 0$, the properties in Eqs. 16.22 and 16.23 assure that, when plastic deformations take place, the *distance* between F and $\overline{F}$ always decreases, that is

$$\delta \left(p_y - \overline{p}_y \right) = \delta p_y - \delta \overline{p}_y = \left(\frac{\partial p_y}{\partial v^{(p)}} - \frac{d\overline{p}_y}{dv^{(p)}} \right) \delta v^{(p)} + \frac{\partial p_y}{\partial \epsilon_s^{(p)}} \delta \epsilon_s^{(p)} \leq 0$$

5. The space region bounded by F is a subspace of $\overline{F}$, that is
$$F(p,q,\theta,\overline{p}_y,p_y) \subseteq \overline{F}(p,q,\theta,\overline{p}_y) \tag{16.24}$$
If the current stress point σ lies on $\overline{F}$, then F and $\overline{F}$ must coincide, that is
$$F(p,q,\theta,\overline{p}_y,p_y) \equiv \overline{F}(p,q,\theta,\overline{p}_y) \tag{16.25}$$
and
$$\begin{aligned} p_y &= \overline{p}_y \\ \frac{\partial p_y}{\partial v^{(p)}} &= \frac{d\overline{p}_y}{dv^{(p)}} \\ \frac{\partial p_y}{\partial \epsilon_s^{(p)}} &= 0 \end{aligned}$$

6. There exists an *Elastic Surface*
$$\widehat{F} = \widehat{F}(p,q,\theta,\overline{p}_y) \tag{16.26}$$
defined as
$$\widehat{F}(p,q,\theta,\overline{p}_y) \equiv F(p,q,\theta,\overline{p}_y,p_y = \widehat{p}_y)$$
where
$$\widehat{p}_y = \widehat{p}_y(\overline{p}_y) \tag{16.27}$$
Note that the above definition implies that, if the current stress point σ lies on $\widehat{F}$, then F coincides with $\widehat{F}$.

7. In general, for $p_y \to \widehat{p}_y$,
$$\begin{aligned} \frac{\partial p_y}{\partial v^{(p)}} &\to \infty \\ \frac{\partial p_y}{\partial \epsilon_s^{(p)}} &\to \infty \\ A &\to +\infty \end{aligned}$$
The definition of the *Plastic Modulus* A is postponed to item 9. If $\widehat{F}$ and $\overline{F}$ always coincide, the above assumptions do not apply.

8. There exists a *Potential Function* G for plastic deformations which is assumed to coincide with the yield surface (associative flow rule), that is
$$G = F(p,q,\theta,\overline{p}_y,p_y) \tag{16.28}$$

9. The infinitesimal strain increment $\delta\epsilon$ is given by

$$\delta\epsilon = \delta\epsilon^{(e)} + \delta\epsilon^{(p)} \qquad (16.29)$$

where:

- $\delta\epsilon^{(e)}$ represents the elastic (fully recoverable) component which may be calculated according to the generalized Hooke's law,

$$\delta\epsilon^{(e)} = \left(\mathbf{C}^{(e)}\right)^{-1} \delta\sigma \qquad (16.30)$$

where

$$\mathbf{C}^{(e)} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & . & 0 \\ \nu & (1-\nu) & \nu & 0 & . & 0 \\ \nu & \nu & (1-\nu) & 0 & . & 0 \\ 0 & 0 & 0 & (1-2\nu) & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & (1-2\nu) \end{bmatrix}_{(9\times 9)}$$

is the tangential elastic symmetric stiffness matrix. The *Young Modulus* E and the *Poisson Coefficient* ν may be, in general, function of the state variables (p, q, θ, v), but the relationships of E and ν must be of certain form so that the elastic bulk modulus, defined as

$$B^{(e)} = \frac{E}{3(1-2\nu)}$$

may be eventually expressed as

$$B^{(e)} = vB^{(e)}(p) \qquad (16.31)$$

where $B^{(e)}(p)$ is a continuous function of p only.

- $\delta\epsilon^{(p)}$ is the plastic (irreversible) component defined as

$$\delta\epsilon^{(p)} = \delta\lambda\mathbf{b} \qquad (16.32)$$

where

$$\delta\lambda \begin{cases} \geq 0, & \text{if elasto-plastic response occurs.} \\ = 0, & \text{if elastic response occurs.} \end{cases}$$

and, because of the associative flow rule $G \equiv F$,

$$\mathbf{b} \equiv \mathbf{a} = \frac{\partial F}{\partial \sigma} \qquad (16.33)$$

According to Eqs. 13.19-13.21 specialized for the associative flow rule assumption $G \equiv F$, the value of the plastic multiplier $\delta\lambda$ can be calculated as:

- If $\delta\boldsymbol{\sigma}$ is assigned

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T\delta\boldsymbol{\sigma}}{A}, & \text{if } A \neq 0. \\ \text{indeterminate}, & \text{if } A = 0. \end{cases} \tag{16.34}$$

- if $\delta\boldsymbol{\epsilon}$ is assigned

$$\delta\lambda = \begin{cases} \dfrac{\mathbf{a}^T\delta\boldsymbol{\sigma}^{(e)}}{A + \mathbf{a}^T\mathbf{c}^{(F)}}, & \text{if } A \neq -\mathbf{a}^T\mathbf{c}^{(F)}. \\ \text{indeterminate}, & \text{if } A = -\mathbf{a}^T\mathbf{c}^{(F)}. \end{cases} \tag{16.35}$$

where

$$\begin{aligned} \delta\boldsymbol{\sigma}^{(e)} &= \mathbf{C}^{(e)}\delta\boldsymbol{\epsilon} \\ \mathbf{c}^{(F)} &= \mathbf{C}^{(e)}\mathbf{a} \end{aligned}$$

The *Plastic Modulus* A, which according to Eq. 13.21, is defined as

$$A = -\frac{1}{\delta\lambda}\left[\frac{\partial F}{\partial \overline{p}_y}\delta\overline{p}_y + \frac{\partial F}{\partial p_y}\delta p_y\right]$$

because of the relationships in Eqs. 16.12 and 16.17, results to be given by

$$A = -\left[v\left(\frac{\partial F}{\partial \overline{p}_y}\frac{d\overline{p}_y}{dv^{(p)}} + \frac{\partial F}{\partial p_y}\frac{\partial p_y}{\partial v^{(p)}}\right)\frac{\partial F}{\partial p} + \frac{\partial F}{\partial p_y}\frac{\partial p_y}{\partial \epsilon_s^{(p)}}\overline{c}_2\right] \tag{16.36}$$

where

$$\overline{c}_2 = \left[\left(\frac{\partial F}{\partial q}\right)^2 + \frac{1}{q^2}\left(\frac{\partial F}{\partial \theta}\right)^2\right]$$

According to the above definition of elasto-plastic deformation, the stress increment $\delta\boldsymbol{\sigma}$ resulting from strain increment $\delta\boldsymbol{\epsilon}$, starting from a material state (p, q, θ, v), can be calculated as, Eq. 13.22,

$$\delta\boldsymbol{\sigma} = \mathbf{C}\delta\boldsymbol{\epsilon} \tag{16.37}$$

where **C**, the tangent elasto-plastic constitutive matrix, is a symmetric matrix defined as

$$\mathbf{C} = \begin{cases} \mathbf{C}^{(e)}; & \text{if a purely elastic response occurs.} \\ \mathbf{C}^{(e)} - \mathbf{C}^{(p)}; & \text{if plasticity develops.} \end{cases}$$

and

$$\mathbf{C}^{(p)} = \begin{cases} \dfrac{\mathbf{c}^{(F)}\mathbf{c}^{(F)T}}{A + \mathbf{a}^T\mathbf{c}^{(F)}}; & \text{for } A \neq -\mathbf{a}^T\mathbf{c}^{(F)}; \\ \text{indeterminate} & \text{for } A = -\mathbf{a}^T\mathbf{c}^{(F)}; \end{cases}$$

The type of mechanical response is established according to the following criterion.

10. By definition, if

$$\hat{F}(p, q, \theta, \overline{p}_y) < 0 \tag{16.38}$$

the material response is always elastic, regardless the applied stress increment $\delta\boldsymbol{\sigma}$ or strain increment $\delta\boldsymbol{\epsilon}$. Instead, if

$$\hat{F}(p, q, \theta, \overline{p}_y) \geq 0 \tag{16.39}$$

the type of material response is established as follows:

- *Stress Based Criterion.* Let $\delta\boldsymbol{\sigma}$ be a stress increment applied on any material state $(p, q, \theta, \overline{p}_y)$, then:

  (a) Elasto-plastic response occurs if

  $$\begin{array}{rcl} A > 0 & ; & \mathbf{a}^T\delta\boldsymbol{\sigma} \geq 0 \\ A = 0 & ; & \mathbf{a}^T\delta\boldsymbol{\sigma} = 0 \\ A < 0 & ; & \mathbf{a}^T\delta\boldsymbol{\sigma} = 0 \end{array}$$

  (b) Elastic response occurs if

  $$A \geq 0 \;\; ; \;\; \mathbf{a}^T\delta\boldsymbol{\sigma} < 0$$

  (c) Either elastic or elasto-plastic response may occur if

  $$A < 0 \;\; ; \;\; \mathbf{a}^T\delta\boldsymbol{\sigma} < 0$$

  This is the only ambiguous situation which this model does not solve by itself.

(d) Stress increments by which

$$A \leq 0 \quad ; \quad \mathbf{a}^T \delta\sigma > 0$$

are not admissible; that is, according to the theory, the material cannot sustain such type of stress increment.

- *Strain Based Criterion.* Let $\delta\epsilon$ be a strain increment applied on any material state $(p, q, \theta, \overline{p}_y)$, then:

(a) Elasto-plastic response occurs if

$$\begin{array}{lcl} A > -\mathbf{a}^T\mathbf{c}^{(F)} & ; & \mathbf{a}^T\delta\sigma^{(e)} \geq 0 \\ A = -\mathbf{a}^T\mathbf{c}^{(F)} & ; & \mathbf{a}^T\delta\sigma^{(e)} = 0 \\ A < -\mathbf{a}^T\mathbf{c}^{(F)} & ; & \mathbf{a}^T\delta\sigma^{(e)} = 0 \end{array}$$

(b) Elastic response occurs if

$$A \geq -\mathbf{a}^T\mathbf{c}^{(F)} \quad ; \quad \mathbf{a}^T\delta\sigma^{(e)} < 0$$

(c) Either elastic or elasto-plastic response may occur if

$$A < -\mathbf{a}^T\mathbf{c}^{(F)} \quad ; \quad \mathbf{a}^T\delta\sigma^{(e)} < 0$$

This is the only ambiguous situation which this model does not solve by itself.

(d) Strain increments by which

$$A \leq -\mathbf{a}^T\mathbf{c}^{(F)} \quad ; \quad \mathbf{a}^T\delta\sigma^{(e)} > 0$$

are not admissible; that is, according to the theory, the material cannot sustain such type of strain increment.

Moreover, SUOLO/Standard V1 assumes that the functional relationships for $\overline{F}$, G and A eventually satisfy the following conditions:

1. The system of equations

$$\left\{ \begin{array}{lcl} \overline{F}(p, q, \theta, \overline{p}_y) & = & 0 \\ \dfrac{\partial G}{\partial p}(p, q, \theta, \overline{p}_y, p_y) & = & 0 \end{array} \right. \tag{16.40}$$

admits a solution (p, q, θ). For any given θ this solution is unique and satisfies the critical state surface equation, defined in item 5 of Section 16.1,

$$q = q_c(p, \theta)$$

2. For any soil state where

$$\begin{cases} \overline{F}(p,q,\theta,\overline{p}_y) & < \quad 0 \\ \dfrac{\partial G}{\partial p}(p,q,\theta,\overline{p}_y,p_y) & = \quad 0 \end{cases} \qquad (16.41)$$

the plastic modulus A, defined in Eq. 16.36, can not be equal to zero, namely

$$A \neq 0 \qquad (16.42)$$

It will be shown that the above 10 plus 2 mathematical conditions are sufficient for assuring the respect of the physical hypothesis listed in Section 16.1.

16.4 The Bounding, the Yield and the Elastic Surfaces

It is assumed that in the q vs. p plane the bounding, the yield and the elastic surfaces in Eqs. 16.9, 16.14 and 16.26 are represented by the three homothetic ellipses shown in Fig. 16.3. Their general mathematical description is given by

$$\overline{F}(p,q,\theta,\overline{p}_y) = f(p,q,\theta,p_m=\overline{p}_y,\overline{p}_c,\overline{p}_y) \qquad (16.43)$$

$$F(p,q,\theta,p_y,\overline{p}_y) = f(p,q,\theta,p_m=p_y,\overline{p}_c,\overline{p}_y) \qquad (16.44)$$

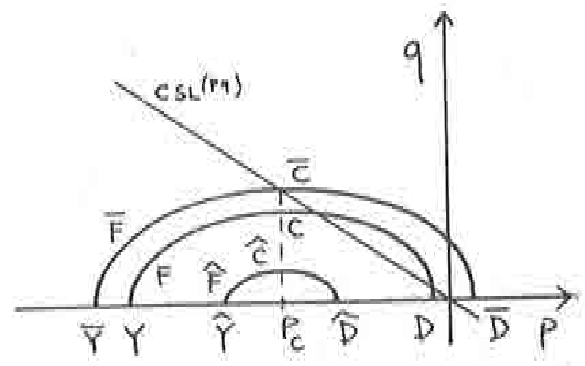
$$\hat{F}(p,q,\theta,\overline{p}_y) = f(p,q,\theta,p_m=\hat{p}_y,\overline{p}_c,\overline{p}_y) \qquad (16.45)$$

where

$$f(p,q,\theta,p_m,\overline{p}_c,\overline{p}_y) = (p-\overline{p}_c)^2 + \frac{q^2}{M(\theta)^2}\left(\frac{\overline{p}_y-\overline{p}_c}{\overline{p}_c}\right)^2 - (p_m-\overline{p}_c)^2 \quad (16.46)$$

and

- $M(\theta)$ is given in Eq. 11.15;
- Geometrically, $\overline{p}_y$ represents the intersection of smaller value of $\overline{F}$ with the p-axis, Fig. 16.3, and the hypothesis in item 2 in Section 16.3 requires that $\overline{p}_y = \overline{p}_y(v^{(p)})$. In Section 16.8 it will be proved that, according to the material behavior reported in Section 16.2, the value



Points Coordinates:

$\overline{Y}(\overline{p}_y, 0)$; $Y(p_y, 0)$; $\widehat{Y}(\widehat{p}_y, 0)$

$P_c(\overline{p}_c, 0)$; $\widehat{C}(\overline{p}_c, \widehat{q}_c)$; $C(\overline{p}_c, q_c)$; $\overline{C}(\overline{p}_c, \overline{q}_c)$

$\widehat{D}(\widehat{p}_d, 0)$; $D(p_d, 0)$; $\overline{D}(\overline{p}_d, 0)$

Figure 16.3: The bounding, the yield and the elastic surfaces in the q vs. p plane

of $\overline{p}_y$ may be calculated as function of $v^{(p)}$ by inverting the functional relationship

$$v^{(p)} = \overline{v}_y^{(p)}(\overline{p}_y) = v_\lambda - \lambda \ln \frac{\overline{p}_y}{p_\lambda} + \chi \ln \frac{\overline{p}_y + p_o}{p_\lambda + p_o} \tag{16.47}$$

Alternatively, the value of $\overline{p}_y$ may be calculated as function of the current values of (p, v) by solving the implicit equation, Section 16.9,

$$v - c(p) = v_\lambda - \lambda \ln \frac{\overline{p}_y}{p_\lambda} + \chi \ln \frac{\overline{p}_y + p_o}{p_\lambda} \tag{16.48}$$

where

$$c(p) = \begin{cases} -\chi \ln \frac{p + p_o}{p_\lambda}; & \text{for } p \leq 0 \\ -\chi \left[\ln \frac{p_o}{p_\lambda} + \frac{p}{p_o}\right]; & \text{for } p > 0 \end{cases}$$

- Geometrically, $\overline{p}_c$ represents the centre of the homothetic surfaces $\overline{F}$, F and $\widehat{F}$, Fig. 16.3. In Section 16.10 it will be proved that, according to the material behavior reported in Section 16.2, the value of $\overline{p}_c$ may be calculated as function of $\overline{p}_y$ by solving the implicit equation

$$-\lambda \ln \frac{\overline{p}_c}{\overline{p}_y} + \chi \ln \frac{\overline{p}_c + p_o}{\overline{p}_y + p_o} - v_e = 0 \qquad (16.49)$$

Alternatively, the value of $\overline{p}_c$ may be calculated as function of the current values of (p, v) by solving the implicit equation

$$v - c(p) = v_\lambda - \lambda \ln \frac{\overline{p}_c}{p_\lambda} + \chi \ln \frac{\overline{p}_c + p_o}{p_\lambda} - v_e \qquad (16.50)$$

where $c(p)$ is given in Eq. 16.48.

- Geometrically, p_y represents the intersection of smaller value of F with the p-axis, Fig. 16.3, and the hypothesis in item 4a in Section 16.3 requires that $p_y = p_y(p, q, \theta, \overline{p}_y)$. It is easy to verify that from Eq. 16.44 it results that the value of p_y may be calculated as function of the current values of $(p, q, \theta, \overline{p}_y)$ as

$$p_y = \overline{p}_c - \left[(p - \overline{p}_c)^2 + \frac{q^2}{M(\theta)^2} \left(\frac{\overline{p}_y - \overline{p}_c}{\overline{p}_c} \right)^2 \right]^{\frac{1}{2}} \qquad (16.51)$$

where, according to Eq. 16.49, $\overline{p}_c = \overline{p}_c(\overline{p}_y)$. Note that since $\overline{p}_y$ may be calculated as function of (p, v) we can indicate Eq. 16.51 as

$$p_y = p_y(p, q, \theta, v) \qquad (16.52)$$

- Geometrically, $\widehat{p}_y$ represents the intersection of smaller value of $\widehat{F}$ with the p-axis, Fig. 16.3, and the hypothesis in Eq. 16.27 requires that $\widehat{p}_y = \widehat{p}_y(\overline{p}_y)$. In Section 16.11 it will be proved that, according to the material behavior reported in Section 16.2, the value of $\widehat{p}_y$ may be calculated as function of $\overline{p}_y$ by solving the implicit equation

$$-\lambda \ln \frac{\widehat{p}_y}{\overline{p}_y} + \chi \ln \frac{\widehat{p}_y + p_o}{\overline{p}_y + p_o} - v_e = 0 \qquad (16.53)$$

Alternatively, the value of $\widehat{p}_y$ may be calculated as function of the current values of (p, v) by solving the implicit equation

$$v - c(p) = v_\lambda - \lambda \ln \frac{\widehat{p}_y}{p_\lambda} + \chi \ln \frac{\widehat{p}_y + p_o}{p_\lambda} - v_e \qquad (16.54)$$

where $c(p)$ is given in Eq. 16.48.

The surface $\hat{F}$ is identified as the dashed line indicated in Fig. 16.2b. In fact, in undrained conditions we have, Eq. 16.78,

$$\delta v = \frac{\delta p}{B^{(e)}(p)} + v\delta\lambda\frac{\partial F}{\partial p} = 0$$

and since along the vertical paths in Fig. 16.2b

$$\delta p = 0$$

it follows that

$$\delta\lambda = 0$$

This can be justified only if the vertical paths are inside the elastic surface $\hat{F}$.

The analytical expressions of the equations of the three surfaces $\overline{F}, F$ and $\hat{F}$ are then obtained imposing that:

- $\overline{F}, F$ and $\hat{F}$ are three homothetic surfaces, i.e. three ellipses with the same center point P_c and the same axis-ratio, Fig. 16.3.
- The point of intersection of $\overline{F}$ with $CSL^{(pq)}$ lies on the vertical line crossing the homothetic center.

It is worth mentioning that the elliptic form of these surfaces is not the essential point of the proposed model. Other type of geometrical forms can be employed; however, they must respect the following essential features:

- The three surfaces $\overline{F}, F$ and $\hat{F}$ must be convex and homothetic with respect to a point P_c lying on the p-axis.
- The point of intersection of $\overline{F}$ with $CSL^{(pq)}$ lies on the vertical line crossing the homothetic center.
- At the point of intersection of F with the p-axis, the gradient ∇F must have horizontal direction.
- At the point of intersection of F with the vertical line drawn from the homothetic center P_c, the gradient ∇F has vertical direction.

This assures that:

- The requirements in Eqs. 16.24, 16.25 and 16.26 are respected.

- At the point of intersection of F with the p-axis, the gradient ∇F has horizontal direction. Consequently, in accordance with the experimental evidences, Section 16.2.1, the plastic deformations mobilized by pure isotropic loading conditions do not contain distortional components, i.e.

$$\delta \epsilon_s^{(p)} = 0 \tag{16.55}$$

- At the point of intersection of F with the vertical line drawn from the homothetic center P_c the gradient ∇F has vertical direction. Consequently, in accordance with the experimental results, Fig. 16.2, the undrained triaxial (effective) stress path starting from P_c keeps always a vertical direction. In fact, in undrained conditions $\delta v = 0$ and, consequently, Eq. 16.78,

$$\delta v^{(e)} = -\delta v^{(p)}$$

On the other hand, at the point of intersection of F with the vertical line drawn from the homothetic center P_c the gradient ∇F has vertical direction, i.e.

$$\frac{\partial F}{\partial p} = 0$$

so that, Eq. 16.80,

$$\delta v^{(p)} = v\delta\lambda\frac{\partial F}{\partial p} = 0 \tag{16.56}$$

This implies that $\delta v^{(e)} = 0$, from which it follows, Eq. 16.79,

$$\delta p = 0$$

- It easy to verify that, according to the requirement stated in Eq. 16.40, the system of equations

$$\begin{cases} \overline{F}(p,q,\theta) & = & 0 \\ \dfrac{\partial F}{\partial p} & = & 2\,(p - \overline{p}_c) = 0 \end{cases}$$

admits a unique solution (p,q,θ). For any given θ this solution lies on the $\mathrm{CSL}^{(pq)}$ and satisfies the CSS equation.

16.5 Remarks on the Mathematical Formulation

From the mathematical assumptions reported in the previous Sections 16.3 and 16.4, it is possible to derive easily the following important relationships which are going to be useful for what it follows;

1. From the definitions of the NCL and of the Bounding surface $\overline{F}$, it follows that on the NCL

$$\overline{p}_y \equiv p \tag{16.57}$$

and, in any case,

$$\overline{p}_y < 0 \tag{16.58}$$

2. According to the elastic strain definition in Eq. 16.30, it follows that the elastic strain, its deviatoric component and invariants result to be defined as, Section 11.7,

$$\begin{aligned} \delta\boldsymbol{\epsilon}^{(e)} &= \frac{\delta p}{3B^{(e)}}\widehat{\mathbf{m}} + \frac{1}{2G^{(e)}}\delta\mathbf{s} & (16.59)\\ \delta\mathbf{e}^{(e)} &= \frac{\delta\mathbf{s}}{2G^{(e)}} & (16.60)\\ \delta\epsilon_v^{(e)} &= \frac{\delta p}{B^{(e)}} & (16.61)\\ \delta\epsilon_s^{(e)} &= \frac{1}{\sqrt{6}G^{(e)}}\left\{\delta\mathbf{s}^T\delta\mathbf{s}\right\}^{1/2} & (16.62) \end{aligned}$$

where

$$\begin{aligned} B^{(e)} &= \frac{E}{3(1-2\nu)}\\ G^{(e)} &= \frac{E}{2(1+\nu)} \end{aligned}$$

3. Since the potential-yield function $F \equiv G$ is a stress invariant function, its gradient in the stress space can be calculated as, Appendix A,

$$\begin{aligned} \mathbf{a} &= \left\{\frac{\partial F}{\partial \sigma_{ij}}\right\}^T = c_1\widehat{\mathbf{m}} + c_2\mathbf{s} + c_3\mathbf{v} & (16.63)\\ \nabla_p F &= \left\{\frac{\partial F}{\partial p}\frac{\partial p}{\partial \sigma_{ij}}\right\}^T = c_1\widehat{\mathbf{m}} & (16.64)\\ \nabla_s F &= \left\{\frac{\partial F}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial \sigma_{ij}}\right\}^T = c_2\mathbf{s} + c_3\mathbf{v} & (16.65) \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{1}{3}\frac{\partial F}{\partial p} \\ c_2 &= \frac{3}{2q}\left(\frac{\partial F}{\partial q} - \frac{\sin 3\theta}{q}\widetilde{\frac{\partial F}{\partial \theta}}\right) \\ c_3 &= -\frac{9}{2q^3\cos 3\theta}\frac{\partial F}{\partial \theta} \end{aligned}$$

and

$$\widetilde{\frac{\partial F}{\partial \theta}} = \frac{1}{\cos 3\theta}\frac{\partial F}{\partial \theta}$$

and the elements of $\mathbf{v}$ are function of $\mathbf{s}$. Moreover,

$$\|\nabla_s F\| = \left\{\frac{3}{2}\left[\left(\frac{\partial F}{\partial q}\right)^2 + \frac{1}{q^2}\left(\frac{\partial F}{\partial \theta}\right)^2\right]\right\}^{1/2} \tag{16.66}$$

4. The scalar product $\mathbf{a}^T\boldsymbol{\sigma}$ can be expressed as in terms of invariant values as

$$\mathbf{a}^T\boldsymbol{\sigma} = \frac{\partial F}{\partial p}\delta p + \frac{\partial F}{\partial q}\delta q + \frac{\partial F}{\partial \theta}\delta\theta \tag{16.67}$$

5. The partial derivatives of $F \equiv G$ with respect to the stress invariants are given by

$$\frac{\partial F}{\partial p} = 2(p - \overline{p}_c) \tag{16.68}$$

$$\frac{\partial F}{\partial q} = \frac{2q}{M^2}\left(\frac{\overline{p}_y - \overline{p}_c}{\overline{p}_c}\right)^2 \tag{16.69}$$

$$\frac{\partial F}{\partial \theta} = -\frac{2q^2}{M^3}\left(\frac{\overline{p}_y - \overline{p}_c}{\overline{p}_c}\right)^2\frac{dM}{d\theta} \tag{16.70}$$

$$\widetilde{\frac{\partial F}{\partial \theta}} = \frac{1}{\cos 3\theta}\frac{\partial F}{\partial \theta} = -\frac{2q^2}{M^3}\left(\frac{\overline{p}_y - \overline{p}_c}{\overline{p}_c}\right)^2\widetilde{\frac{dM}{d\theta}} \tag{16.71}$$

and the expressions of $dM/d\theta$ and $\widetilde{dM}/d\theta$ are reported in Section 11.4.

6. The partial derivatives of F with respect to the hardening parameters $\overline{p}_y$ and p_y can be calculated as

$$\frac{\partial F}{\partial \overline{p}_y} = \left[\frac{\partial f}{\partial \overline{p}_y} + \frac{\partial f}{\partial \overline{p}_c}\frac{d\overline{p}_c}{d\overline{p}_y}\right]_{p_m = p_y} \tag{16.72}$$

$$\frac{\partial F}{\partial p_y} = \left(\frac{\partial f}{\partial p_m}\right)_{p_m = p_y} \tag{16.73}$$

where

$$\begin{aligned}
\frac{\partial f}{\partial \overline{p}_y} &= \frac{2q^2}{M^2}\left(\frac{\overline{p}_y - \overline{p}_c}{\overline{p}_c}\right)\frac{1}{\overline{p}_c} \\
\frac{\partial f}{\partial \overline{p}_c} &= 2\,(p_m - p) - \frac{2q^2}{M^2}\left(\frac{\overline{p}_y - \overline{p}_c}{\overline{p}_c}\right)\frac{\overline{p}_y}{\overline{p}_c^2} \\
\frac{\partial f}{\partial p_m} &= -2\,(p_m - \overline{p}_c)
\end{aligned}$$

and the expression of $d\overline{p}_c/d\overline{p}_y$ is reported in Eq. 16.127.

7. According to the $\mathbf{a} \equiv \mathbf{b}$ expression in Eq. 16.63, it follows that the plastic strain, its deviatoric components and invariants result to be defined as, Section 12.10,

$$\delta\boldsymbol{\epsilon}^{(p)} = \delta\lambda(c_1\widehat{\mathbf{m}} + c_2\mathbf{s} + c_3\mathbf{v}) \tag{16.74}$$

$$\delta\mathbf{e}^{(p)} = \delta\lambda(c_2\mathbf{s} + c_3\mathbf{v}) \tag{16.75}$$

$$\delta\epsilon_v^{(p)} = \delta\lambda\frac{\partial F}{\partial p} \tag{16.76}$$

$$\begin{aligned}
\delta\epsilon_s^{(p)} &= \delta\lambda\sqrt{\frac{2}{3}}\|\nabla_s F\| = \\
&= \delta\lambda\left[\left(\frac{\partial F}{\partial q}\right)^2 + \frac{1}{q^2}\left(\frac{\partial F}{\partial \theta}\right)^2\right]^{1/2}
\end{aligned} \tag{16.77}$$

8. The incremental variation of the specific volume is given by

$$\delta v = \delta v^{(e)} + \delta v^{(p)} \tag{16.78}$$

where

$$\delta v = v\delta\epsilon_v$$

$$\delta v^{(e)} = v\delta\epsilon_v^{(e)} = \frac{\delta p}{B^{(e)}(p)} \tag{16.79}$$

$$\delta v^{(p)} = v\delta\epsilon_v^{(p)} = v\delta\lambda\frac{\partial G}{\partial p} = v\delta\lambda\frac{\partial F}{\partial p} \tag{16.80}$$

and, from Eq. 16.90,

$$\mathcal{B}^{(e)} = \frac{B^{(e)}}{v} = \begin{cases} -\frac{p+p_o}{\chi}; & \text{for } p < 0 \\ -\frac{p_o}{\chi}; & \text{for } p \geq 0 \end{cases} \tag{16.81}$$

Moreover, taking into account that $p_o \leq 0$ and $\chi > 0$, Eq. 16.7,

$$\mathcal{B}^{(e)} > 0 \tag{16.82}$$

9. The requirement on $\overline{F}$ in Eq. 16.10 has two important consequences:

   - Being $\overline{F}$ a continuous monotonically increasing function of $\overline{p}_y$, the equation

     $$\overline{F}(p, q, \theta, \overline{p}_y) = 0$$

     admits only one solution $\overline{p}_y$, for any admissible stress state (p, q, θ).
   - The value of the current $\overline{p}_y$ of a soil at an admissible stress state (p, q, θ) is bounded as

     $$\overline{p}_y \leq \overline{p}_{ymax}(p, q, \theta) \tag{16.83}$$

     where $\overline{p}_{ymax}$ is the unique solution of the equation

     $$\overline{F}(p, q, \theta, \overline{p}_y) = 0$$

     in terms of $\overline{p}_y$. In fact, by definition,

     $$\overline{F}(p, q, \theta, \overline{p}_{ymax}) = \overline{F}^{(1)} = 0$$

     while, according to the requirement in item 1 in Section 16.3,

     $$\overline{F}(p, q, \theta, \overline{p}_y) = \overline{F}^{(2)} \leq 0$$

     Thus, being $\overline{F}$ a continuous monotonically increasing function of $\overline{p}_y$ and $\overline{F}^{(2)} \leq \overline{F}^{(1)}$, it results

     $$\overline{p}_y \leq \overline{p}_{ymax}$$

16.6 The Kinematics of the Bounding Surface $\overline{F}$

According to the requirement in item 1 in Section 16.3, the bounding surface equation

$$\overline{F} = \overline{F}(p, q, \theta, \overline{p}_y) \tag{16.84}$$

must satisfy the condition

$$\frac{\partial \overline{F}}{\partial \overline{p}_y} > 0 \tag{16.85}$$

With refence to Eq. 16.43, the partial derivative of $\overline{F}$ with respect to its hardening parameter $\overline{p}_y$ can be calculated as

$$\frac{\partial \overline{F}}{\partial \overline{p}_y} = \left[\frac{\partial f}{\partial \overline{p}_y} + \frac{\partial f}{\partial \overline{p}_c}\frac{d\overline{p}_c}{d\overline{p}_y} + \frac{\partial f}{\partial p_m}\right]_{p_m=\overline{p}_y} \tag{16.86}$$

where the expressions for the partial derivatives of f are given in item 6 in Section 16.5 while the expression for $d\overline{p}_c/d\overline{p}_y$ is given in Eq. 16.127. It has been verified numerically that, being $\overline{p}_d > 0$, Section 16.12, the condition in Eq. 16.85 is satisfied. This mathematical condition implies that the bounding surface expands or contracts itself as $\overline{p}_y$ decreases or increases, Fig. 16.4. In

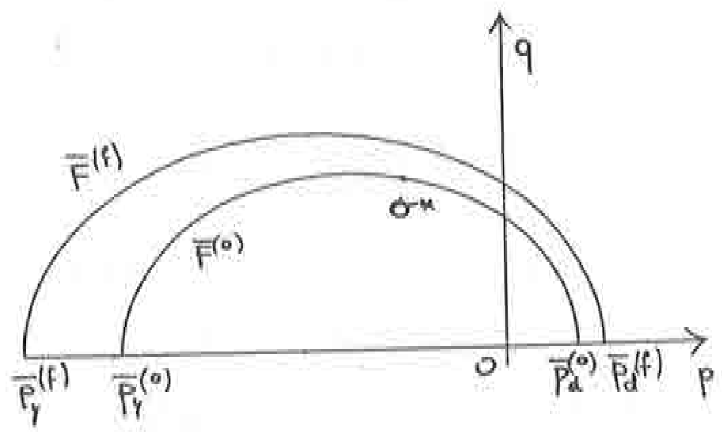


Figure 16.4: Expansion of $\overline{F}$ in the q vs. p plane

fact, consider an initial location of the bounding surface

$$\overline{F}^{(o)}(\sigma) = \overline{F}(\sigma, \overline{p}_y = \overline{p}_y^{(o)})$$

whose intersections with the p axis are equal to

$$p = \begin{cases} \overline{p}_y = \overline{p}_y^{(o)} \\ \overline{p}_d = \overline{p}_d^{(o)} \end{cases}$$

Suppose that the hardening parameter is varied by an infinitesimal amount $d\overline{p}_y$, so that the new location of the bounding surface has equation

$$\overline{F}^{(f)}(\sigma) = \overline{F}(\sigma, \overline{p}_y = \overline{p}_y^{(f)} = \overline{p}_y^{(o)} + d\overline{p}_y)$$

and, consequently, the intersections with the p axis result to be equal to

$$p = \begin{cases} \overline{p}_y = \overline{p}_y^{(f)} = \overline{p}_y^{(o)} + d\overline{p}_y \\ \overline{p}_d = \overline{p}_d^{(f)} = \overline{p}_d^{(o)} + d\overline{p}_d \end{cases}$$

The value that $\overline{F}^{(f)}$ takes for any stress point σ^* lying on $\overline{F}^{(o)}$, for which

$$\overline{F}^{(o)}(\sigma^*) = \overline{F}(\sigma = \sigma^*, \overline{p}_y = \overline{p}_y^{(o)}) = 0$$

may be *exactly* evaluated as

$$\overline{F}^{(f)}(\sigma^*) = \left[\overline{F}^{(o)} + \frac{\partial \overline{F}^{(o)}}{\partial \overline{p}_y} d\overline{p}_y\right]_{\sigma=\sigma^*} = \left(\frac{\partial \overline{F}^{(o)}}{\partial \overline{p}_y}\right)_{\sigma=\sigma^*} d\overline{p}_y \qquad (16.87)$$

From Eqs. 16.87 and 16.85, it results that, in general,

$$\overline{F}^{(f)}(\sigma^*) = \begin{cases} < 0; & \text{for } d\overline{p}_y < 0. \\ > 0; & \text{for } d\overline{p}_y > 0. \end{cases} \qquad (16.88)$$

This implies that:

- From the 1st inequality in Eq. 16.88, it follows that if $d\overline{p}_y < 0$ the stress point σ^* results to be internal to the new location $\overline{F}^{(f)}$, so that

  $$\overline{F}(p, q, \theta, \overline{p}_y^{(f)}) \subset \overline{F}^{(o)}(p, q, \theta, \overline{p}_y^{(o)})$$

  and

  $$\overline{p}_d^{(o)} < \overline{p}_d^{(f)} = \overline{p}_d^{(o)} + d\overline{p}_d$$

  from which

  $$d\overline{p}_d > 0$$

- From the 2nd inequality in Eq. 16.88, it follows that if $d\overline{p}_y > 0$ the stress point σ^* results to be external to the new location $\overline{F}^{(f)}$, so that

$$\overline{F}^{(o)}(p,q,\theta,\overline{p}_y^{(o)}) \subset \overline{F}(p,q,\theta,\overline{p}_y^{(f)})$$

and

$$\overline{p}_d^{(o)} > \overline{p}_d^{(f)} = \overline{p}_d^{(o)} + d\overline{p}_d$$

from which

$$d\overline{p}_d < 0$$

The above results show that $\overline{F}$ expands or contracts as $\overline{p}_y$ decreases or increases and also that

$$\frac{d\overline{p}_d}{d\overline{p}_y} < 0 \tag{16.89}$$

16.7 The Elastic Parameters

According to the elastic material response remarked in Section 16.2.1, it results that the elastic parameters E and ν in the elastic constitutive matrix $\mathbf{C}^{(e)}$ in Eq. 16.30 are given by

$$E = E(v,p) = E_o v e(p) \tag{16.90}$$

$$\nu \in \,]-1, 0.5[\tag{16.91}$$

where

$$E_o = -\frac{3(1-2\nu)}{\chi}$$

$$e(p) = \begin{cases} (p+p_o); & \text{for } p \leq 0 \\ p_o; & \text{for } p > 0 \end{cases}$$

and χ and p_o are the constant material parameters in Eq. 16.6. In fact, we recall that according to the elastic constitutive equation in Eq. 16.30, the elastic volumetric strain can be calculated as, Eq. 16.78,

$$\delta v^{(e)} = v\frac{\delta p}{B^{(e)}} = \frac{\delta p}{\mathcal{B}^{(e)}} \tag{16.92}$$

On the other hand, we have observed in Section 16.2.1 that the elastic material response under drained unloading isotropic stress path is governed by an equation of the type, Eq. 16.6,

$$v^{(e)} = \begin{cases} v_X^{(1)} - \chi \ln \dfrac{p + p_o}{p_X^{(1)} + p_o}; & \text{for } p < 0. \\ v_X^{(2)} - \chi \dfrac{p - p_X^{(2)}}{p_o}; & \text{for } p \geq 0. \end{cases}$$

from which we can establish that

$$\delta v^{(e)} = \begin{cases} -\dfrac{\chi}{p + p_o} \delta p; & \text{for } p < 0. \\ -\dfrac{\chi}{p_o} \delta p; & \text{for } p \geq 0. \end{cases} \tag{16.93}$$

Equalizing Eqs. 16.92 and 16.93, we identify

$$B^{(e)} = \begin{cases} -\dfrac{v(p + p_o)}{\chi}; & \text{for } p < 0. \\ -\dfrac{v p_o}{\chi}; & \text{for } p \geq 0. \end{cases} \tag{16.94}$$

from which

$$E = 3(1 - 2\nu) B^{(e)} = \begin{cases} -\dfrac{3(1 - 2\nu)}{\chi} v(p + p_o); & \text{for } p < 0. \\ -\dfrac{3(1 - 2\nu)}{\chi} v p_o; & \text{for } p \geq 0. \end{cases}$$

proving therefore the expression of E in Eq. 16.90.

16.8 Functional Relationship for $\overline{p}_y(v^{(p)})$

Geometrically, $\overline{p}_y$ represents the intersection of smaller value of $\overline{F}$ with the p-axis, Fig. 16.3. Also, it represents the p-value at the intersection of the NCL with the SWL line crossing the current value of p, Fig. 16.5, see Section 16.9.

By definition, Eq. 16.11, $\overline{p}_y$ is a monotonically increasing function of the plastic volumetric strain $v^{(p)}$. This implies that it can be inverted to obtain $v^{(p)}$ as a monotonically increasing function of $\overline{p}_y$, that is

$$v^{(p)} = \overline{v}_y^{(p)}(\overline{p}_y) \tag{16.95}$$

$$\frac{dv^{(p)}}{d\overline{p}_y} > 0 \tag{16.96}$$

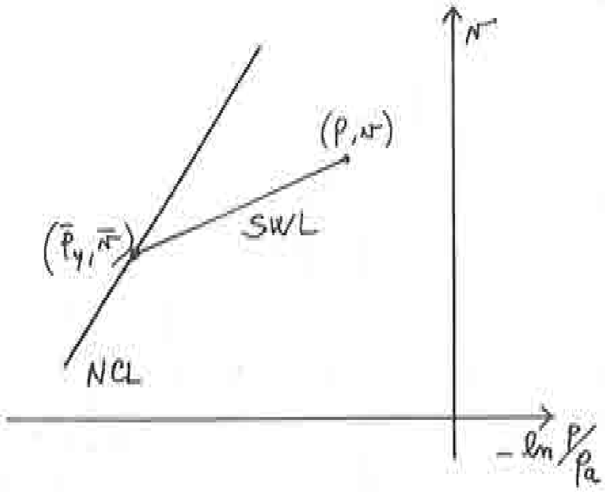


Figure 16.5: Determination of $\overline{p}_y$

From the material behavior observed in Section 16.2.1, it results that

$$v^{(p)} = \overline{v}_y^{(p)}(\overline{p}_y) = v_\lambda - \lambda \ln \frac{\overline{p}_y}{p_\lambda} + \chi \ln \frac{\overline{p}_y + p_o}{p_\lambda + p_o} \tag{16.97}$$

and, accordingly,

$$\frac{d\overline{v}_y^{(p)}}{d\overline{p}_y} = -\frac{\lambda}{\overline{p}_y} + \frac{\chi}{\overline{p}_y + p_o} > 0 \tag{16.98}$$

where the positive character can be proved taking into account the material constant definitions in Eq. 16.6 and the inequality in Eq. 16.58.

Consequently, the function

$$\overline{p}_y = \overline{p}_y(v^{(p)}) \tag{16.99}$$

defined in Eq. 16.11 can be obtained inverting the functional relationship in Eq. 16.97. Accordingly,

$$\frac{d\overline{p}_y}{dv^{(p)}} = \left[\frac{d\overline{v}_y^{(p)}}{d\overline{p}_y}\right]^{-1} = \left[-\frac{\lambda}{\overline{p}_y} + \frac{\chi}{\overline{p}_y + p_o}\right]^{-1} > 0 \tag{16.100}$$

In fact, Eq. 16.78,

$$\delta v^{(p)} = \delta v - \delta v^{(e)}$$

where $\delta v^{(e)}$ can be calculated as in Eq. 16.79 and, according to Eq. 16.95, $\delta v^{(p)}$ is only function of $\delta \overline{p}_y$. Since $\overline{p}_y$ does not depend on the type of stress path, we may consider the particular stress path on the NCL in which, Eq. 16.58,

$$\overline{p}_y \equiv p$$

In this case,

$$\delta v^{(p)}(\overline{p}_y) = \delta v_{NCL}(\overline{p}_y) - \frac{\delta \overline{p}_y}{B^{(e)}(\overline{p}_y)} \tag{16.101}$$

The integration of this differential equation between $(v_o^{(p)}, \overline{p}_{yo})$ and the current soil state $(v^{(p)}, \overline{p}_y)$, yields to

$$v^{(p)} = \overline{v}_y^{(p)}(\overline{p}_y) \tag{16.102}$$

where

$$\overline{v}_y^{(p)}(\overline{p}_y) = v_o^{(p)} + v_{NCL}(\overline{p}_y) - v_{NCL}(\overline{p}_{yo}) - \int_{\overline{p}_{yo}}^{\overline{p}_y} \frac{1}{B^{(e)}(p)} dp$$

Accordingly,

$$\frac{d\overline{v}_y^{(p)}}{d\overline{p}_y} = \frac{dv_{NCL}(\overline{p}_y)}{d\overline{p}_y} - \frac{1}{B^{(e)}(\overline{p}_y)} \tag{16.103}$$

Then:

- expressing v_{NCL} and $B^{(e)}$ as in Eqs. 16.3 and 16.81, respectively,
- setting the arbitrary constants $(v_o^{(p)}, \overline{p}_{yo})$ equal to (v_λ, p_λ),

we obtain the functional relationships in Eqs. 16.97 and 16.98. It can be verified that the values of $(v_o^{(p)}, \overline{p}_{yo})$ do not enter in the constitutive equation.

16.9 Functional Relationship for $\overline{p}_y(p, v)$

The main hypotheses of the model listed in Section 16.3 lead to the conclusion that ther exists a single-value and invertible function

$$v = v(p, \overline{p}_y) \tag{16.104}$$

where

$$v(p,\overline{p}_y) = v_\lambda + \int_{p_\lambda}^{p} \frac{1}{B^{(e)}(p)} dp + \overline{v}_y^{(p)}(\overline{p}_y) - \overline{v}_y^{(p)}(p_\lambda)$$

From the functional relationships presented in the Sections 16.7 and 16.8, it results that this single-value function is given by

$$v = v(p,\overline{p}_y) = v_\lambda + c(p) - \lambda \ln \frac{\overline{p}_y}{p_\lambda} + \chi \ln \frac{\overline{p}_y + p_o}{p_\lambda} \tag{16.105}$$

where

$$c(p) = \begin{cases} -\chi \ln \frac{p+p_o}{p_\lambda}; & \text{for } p \leq 0. \\ -\chi \left[\ln \frac{p_o}{p_\lambda} + \frac{p}{p_o} \right]; & \text{for } p > 0. \end{cases}$$

This implies that, regardless the type of stress path, we can calculate the current value of v as function of the current values of $(p,\overline{p}_y)$. Moreover, being v a continuous function of p and $\overline{p}_y$ and

$$\frac{\partial v}{\partial p} = f_p(p) = \frac{1}{B^{(e)}(p)} > 0 \tag{16.106}$$

$$\frac{\partial v}{\partial \overline{p}_y} = f_{\overline{p}}(\overline{p}_y) = \frac{d\overline{v}_y^{(p)}(\overline{p}_y)}{d\overline{p}_y} > 0 \tag{16.107}$$

where the inequalities are established in Eqs. 16.82 and 16.98, the functional relationship in Eq. 16.104 can be partially inverted to obtain

$$p = p(\overline{p}_y, v) \tag{16.108}$$

$$\overline{p}_y = \overline{p}_y(p, v) \tag{16.109}$$

In order to derive Eq. 16.104, we recall that, Eq. 16.78,

$$\delta v = \delta v^{(e)} + \delta v^{(p)}$$

where, Eqs. 16.79 and 16.95,

$$\delta v^{(e)} = \frac{1}{B^{(e)}(p)} \delta p$$

$$\delta v^{(p)} = \frac{d\overline{v}_y^{(p)}(\overline{p}_y)}{d\overline{p}_y} \delta \overline{p}_y$$

Consequently, δv has the general form

$$\delta v = f_p(p)\delta p + f_{\overline{p}}(\overline{p}_y)\delta \overline{p}_y \tag{16.110}$$

and, since

$$\frac{\partial f_p(p)}{\partial \overline{p}_y} = \frac{\partial f_{\overline{p}}(\overline{p}_y)}{\partial p}$$

being both equal to zero, according to a well-known theorem of Differential Calculus, we recognize that Eq. 16.110 is an *exact differential form*. This implies that there exists a scalar function $v = v(p, \overline{p}_y)$ whose expression can be obtained by the integration of the differential form in Eq. 16.110, so that

$$\begin{aligned} v &= v_{yo} + \int_{\overline{p}_{yo}}^{p} \frac{1}{B^{(e)}(p)} dp + \int_{\overline{p}_{yo}}^{\overline{p}_y} \frac{d\overline{v}_y^{(p)}(\overline{p}_y)}{d\overline{p}_y} d\overline{p}_y = \\ &= v_{yo} + \int_{\overline{p}_{yo}}^{p} \frac{1}{B^{(e)}(p)} dp + \overline{v}_y^{(p)}(\overline{p}_y) - \overline{v}_y^{(p)}(\overline{p}_{yo}) \end{aligned} \tag{16.111}$$

where $(\overline{p}_{yo}, v_{yo})$ represents an admissible soil state. Then, setting $(\overline{p}_{yo}, v_{yo})$ equal to (p_λ, v_λ), which represents an admissible soil state on the NCL, we obtain the general functional relationship in Eq. 16.104. According to the expressions of $B^{(e)}$ in Eq. 16.81, and taking into account that $p_\lambda < 0$, it results:

- for $p < 0$

$$\begin{aligned} \int_{p_\lambda}^{p} \frac{1}{B^{(e)}(p)} dp &= -\int_{p_\lambda}^{p} \frac{\chi}{p + p_o} dp = -\chi \ln \frac{p + p_o}{p_\lambda + p_o} = \\ &= c(p) - \chi \ln \frac{p_\lambda}{p_\lambda + p_o} \end{aligned}$$

- for $p \geq 0$

$$\begin{aligned} \int_{p_\lambda}^{p} \frac{1}{B^{(e)}(p)} dp &= -\int_{p_\lambda}^{o} \frac{\chi}{p + p_o} dp - \int_{o}^{p} \frac{\chi}{p_o} dp = \\ &= -\chi \ln \frac{p_o}{p_\lambda + p_o} - \chi \frac{p}{p_o} = \\ &= c(p) - \chi \ln \frac{p_\lambda}{p_\lambda + p_o} \end{aligned}$$

Then, substituting the above expressions in Eq. 16.104 and expressing $\overline{v}_y^{(p)}(\overline{p}_y)$ as reported in Eq. 16.97, we obtain the extended form in Eq. 16.105.

It is interesting to note that Eq. 16.105 can be alternatively derived equalizing the NCL equation, Eq. 16.3, and the SWL equation relative to the current material state (p, v), Eq. 16.6, that is, for $p < 0$,

$$\begin{cases} \overline{v} = v_\lambda - \lambda \ln \dfrac{\overline{p}_y}{p_\lambda} \\ \\ \overline{v} = v - \chi \ln \dfrac{\overline{p}_y + p_o}{p + p_o} \end{cases} \tag{16.112}$$

This implies that $\overline{p}_y$ represents geometrically the point of intersection of the above two lines, Fig. 16.5.

16.10 Functional Relationships for $\overline{p}_c$

The parameter $\overline{p}_c$ is the value of p at P_c, the center of the homothetic relationship among F, $\hat{F}$ and $\overline{F}$, Fig. 16.3. From the material behavior observed in Section 16.2.1, it results that the value of $\overline{p}_c$ is a monotonically increasing function of $\overline{p}_y$, that is

$$\overline{p}_c = \overline{p}_c(\overline{p}_y) \tag{16.113}$$

$$\frac{d\overline{p}_c}{d\overline{p}_y} > 0 \tag{16.114}$$

where the functional relationship in Eq. 16.113 can be obtained by solving the implicit equation

$$-\lambda \ln \frac{\overline{p}_c}{\overline{p}_y} + \chi \ln \frac{\overline{p}_c + p_o}{\overline{p}_y + p_o} - v_c = 0 \tag{16.115}$$

Thus, being $\overline{p}_y$ a continuous monotonically increasing function of $v^{(p)}$, Section 16.8, it follows that

$$\overline{p}_c = \overline{p}_c(v^{(p)}) \tag{16.116}$$

$$\frac{d\overline{p}_c}{dv^{(p)}} = \frac{d\overline{p}_c}{d\overline{p}_y}\frac{d\overline{p}_y}{dv^{(p)}} > 0 \tag{16.117}$$

where the functional relationship in Eq. 16.116 can be obtained by solving the implicit equation

$$v^{(p)} = \overline{v}_c^{(p)}(\overline{p}_c) \tag{16.118}$$

where

$$\overline{v}_c^{(p)}(\overline{p}_c) = v_\lambda - \lambda \ln \frac{\overline{p}_c}{p_\lambda} + \chi \ln \frac{\overline{p}_c + p_o}{p_\lambda + p_o} - v_c$$

Finally, equalizing Eq. 16.115 to Eq. 16.105 we obtain an implicit equation for $\overline{p}_c$ in terms of (p, v) of the form

$$v - c(p) = v_\lambda - \lambda \ln \frac{\overline{p}_c}{p_\lambda} + \chi \ln \frac{\overline{p}_c + p_o}{p_\lambda} - v_c \tag{16.119}$$

In fact, we preliminary observe that $\overline{p}_c$ is also the value of p at the point $\overline{C}$ in Fig. 16.3, i.e. the point of intersection of $\overline{F}$ with the CSL$^{(pq)}$. According to Eq. 16.4, the specific volume of a soil whose state lies on the CSL$^{(pq)}$ may be calculated as

$$v = v_{NCL}(p = \overline{p}_c) - v_c$$

On the other hand, according to Eq. 16.111, the specific volume of a soil, whose state is characterized by $(p = \overline{p}_c, \overline{p}_y)$, may be calculated as

$$v = v_{yo} + \int_{\overline{p}_{yo}}^{\overline{p}_c} \frac{1}{\mathcal{B}^{(e)}(p)} dp + \overline{v}_y^{(p)}(\overline{p}_y) - \overline{v}_y^{(p)}(\overline{p}_{yo})$$

where, being $(\overline{p}_{yo}, v_{yo})$ an arbitrary material state, we may choose a state on the NCL given by

$$\begin{aligned} \overline{p}_{yo} &= \overline{p}_c \\ v_{yo} &= v_{NCL}(\overline{p}_c) \end{aligned}$$

Hence, equalizing the above two alternative expressions for v, we obtain

$$\overline{v}_y^{(p)}(\overline{p}_c) - \overline{v}_y^{(p)}(\overline{p}_y) - v_c = 0 \tag{16.120}$$

which, substituting the expression for $\overline{v}_y^{(p)}(\overline{p}_c)$ and $\overline{v}_y^{(p)}(\overline{p}_y)$ in Eq. 16.97, yields to the functional relationship in Eq. 16.115. Moreover,

$$v^{(p)} = \overline{v}_c^{(p)}(\overline{p}_c) = \overline{v}_y^{(p)}(\overline{p}_c) - v_c \tag{16.121}$$

which yields to the functional relationship in Eq. 16.118. It will be useful to observe that, according to Eq. 16.115,

- If

$$\begin{aligned} p_o &< 0 \\ v_c &\geq \lambda \ln 2 \end{aligned}$$

as required in Section 16.2.1, the quantity $\overline{p}_c/\overline{p}_y$ increases as the positive quantity $p_o/\overline{p}_y$ increases and it is bounded by

$$e^{-\frac{v_c}{\lambda-\chi}} < \frac{\overline{p}_c}{\overline{p}_y} < e^{-\frac{v_c}{\lambda}} \tag{16.122}$$

Consequently,

$$\overline{p}_y < 2\overline{p}_c \tag{16.123}$$

- If

$$p_o = 0 \tag{16.124}$$
$$v_c > (\lambda-\chi)\ln 2 \tag{16.125}$$
$$p < 0 \tag{16.126}$$

as required in Section 16.2.1, it results

$$\frac{\overline{p}_y}{\overline{p}_c} = \text{ constant } = e^{\frac{v_c}{\lambda-\chi}}$$

from which, substituting v_c as in Eq. 16.125, we obtain the inequality in Eq. 16.123. The additional condition in Eq. 16.126 assures that the elastic matrix $\mathbf{C}^{(e)}$ is positive definite. In fact, Eq. 16.90,

$$E = \begin{cases} -v\dfrac{3(1-2\nu)}{\chi}p & \text{for } p < 0 \\ 0 & \text{for } p \geq 0 \end{cases}$$

- From Eqs. 16.117, 16.118 and 16.100, it follows that

$$\begin{aligned} \frac{d\overline{p}_c}{d\overline{p}_y} &= \left(\frac{dv^{(p)}}{d\overline{p}_c}\right)^{-1}\left(\frac{d\overline{p}_y}{dv^{(p)}}\right)^{-1} = \\ &= \left[-\frac{\lambda}{\overline{p}_c}+\frac{\chi}{\overline{p}_c+p_o}\right]^{-1}\left[-\frac{\lambda}{\overline{p}_y}+\frac{\chi}{\overline{p}_y+p_o}\right] > 0 \end{aligned} \tag{16.127}$$

16.11 Functional Relationships for $\hat{p}_y$

Geometrically, $\hat{p}_y$ represents the intersection of smaller value of $\hat{F}$ with the p-axis, Fig. 16.3, and the hypothesis in Eq. 16.27 requires that $\hat{p}_y$ is a function of $\overline{p}_y$ ony. From the material behavior observed in Section 16.2.1, it results that the value of $\widehat{p}_y$ is a monotonically increasing function of $\overline{p}_y$, that is

$$\widehat{p}_y = \widehat{p}_y(\overline{p}_y) \tag{16.128}$$

$$\frac{d\widehat{p}_y}{d\overline{p}_y} > 0 \tag{16.129}$$

where the functional relationship in Eq. 16.128 can be obtained by solving the implicit equation

$$-\lambda \ln \frac{\widehat{p}_y}{\overline{p}_y} + \chi \ln \frac{\widehat{p}_y + p_o}{\overline{p}_y + p_o} - v_e = 0 \tag{16.130}$$

Thus, being $\overline{p}_y$ a continuous monotonically increasing function of $v^{(p)}$, Section 16.8, it follows that

$$\widehat{p}_y = \widehat{p}_y(v^{(p)}) \tag{16.131}$$

$$\frac{d\widehat{p}_y}{dv^{(p)}} = \frac{d\widehat{p}_y}{d\overline{p}_y}\frac{d\overline{p}_y}{dv^{(p)}} > 0 \tag{16.132}$$

where the functional relationship in Eq. 16.131 can be obtained by solving the implicit equation

$$v^{(p)} = \widehat{v}_y^{(p)}(\widehat{p}_y) \tag{16.133}$$

where

$$\widehat{v}_y^{(p)}(\widehat{p}_y) = v_\lambda - \lambda \ln \frac{\widehat{p}_y}{p_\lambda} + \chi \ln \frac{\widehat{p}_y + p_o}{p_\lambda + p_o} - v_e$$

Finally, equalizing Eq. 16.130 to Eq. 16.105 we obtain an implicit equation for $\widehat{p}_y$ in terms of (p, v) of the form

$$v - e(p) = v_\lambda - \lambda \ln \frac{\widehat{p}_y}{p_\lambda} + \chi \ln \frac{\widehat{p}_y + p_o}{p_\lambda} - v_e \tag{16.134}$$

The proof is analogous to that in the previous Section 16.10 and it is based on the fact that $\widehat{p}_y$ moves along the ELL. According to Eq. 16.5, the specific volume of a soil whose state lies on the ELL may be calculated as

$$v = v_{NCL}(p = \widehat{p}_y) - v_e$$

16.12 Functional Relationship for $\overline{p}_d$

Geometrically, $\overline{p}_d$ represents the intersection of larger value of $\overline{F}$ with the p-axis, Fig. 16.3, and its value can then be calculated as

$$\overline{p}_d = \overline{p}_d(\overline{p}_y) \tag{16.135}$$

where

$$\overline{p}_d(\overline{p}_y) = 2\overline{p}_c - \overline{p}_y$$

and $\overline{p}_c$ may be calculated as function of $\overline{p}_y$ as reported in Section 16.10. Taking into account the inequality in Eq. 16.123 it is possible to state that

$$\overline{p}_d > 0 \tag{16.136}$$

Moreover, according to Eq. 16.89, $\overline{p}_d$ is a decreasing function of $\overline{p}_y$:

$$\frac{d\overline{p}_d}{d\overline{p}_y} = 2\frac{d\overline{p}_c}{d\overline{p}_y} - 1 < 0 \tag{16.137}$$

where $d\overline{p}_c/d\overline{p}_y$ is given in Section 16.18. Thus, being $\overline{p}_y$ a continuous monotonic increasing function of $v^{(p)}$, Section 16.8, it follows that

$$\overline{p}_d = \overline{p}_d(v^{(p)}) \tag{16.138}$$

$$\frac{d\overline{p}_d}{dv^{(p)}} = \frac{d\overline{p}_d}{d\overline{p}_y}\frac{d\overline{p}_y}{dv^{(p)}} < 0 \tag{16.139}$$

16.13 Functional Relationship for $\hat{p}_d$

Geometrically, $\hat{p}_d$ represents the intersection of larger value of $\hat{F}$ with the p-axis, Fig. 16.3. Because of the elliptical shape of $\hat{F}$, it results that its value can be calculated as

$$\hat{p}_d = \hat{p}_d(\overline{p}_y) \tag{16.140}$$

where

$$\hat{p}_d(\overline{p}_y) = 2\overline{p}_c - \hat{p}_y$$

and $\overline{p}_c$ and $\hat{p}_y$ are functionally related to $\overline{p}_y$ as reported in Sections 16.10 and 16.11, respectively. Thus, being $\overline{p}_y$ a continuous function of $v^{(p)}$, Section 16.8, it follows that

$$\hat{p}_d = \hat{p}_d(v^{(p)}) \tag{16.141}$$

In general, the sign of $d\hat{p}_d/dv^{(p)}$ cannot be determined.

16.14 The $v^{(p)}$ vs. p Plane

In a $v^{(p)}$ vs. p plane we can draw the following five lines, Fig. 16.6:

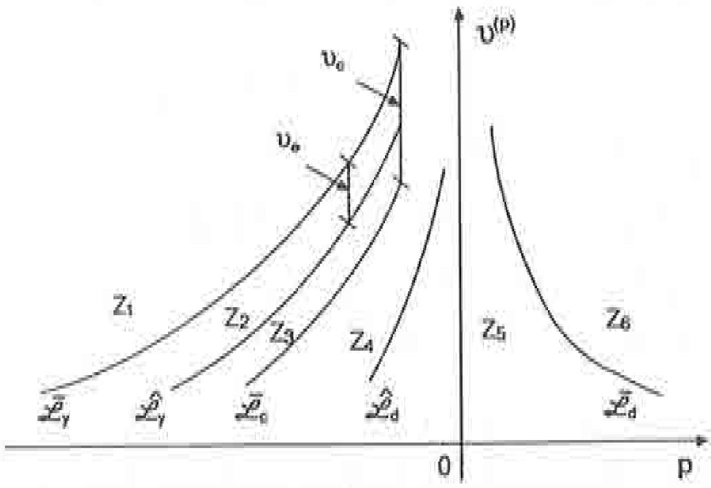


Figure 16.6: Plane $v^{(p)}$ vs. p.

- A line $\overline{\mathcal{L}}_y$ of equation, Section 16.8,

$$v^{(p)} = \overline{v}_y^{(p)}(p) \text{ with } \frac{dv^{(p)}}{dp} > 0$$

  on which $p = \overline{p}_y$;

- A line $\hat{\mathcal{L}}_y$ of equation, Section 16.11,

$$v^{(p)} = \hat{v}_y^{(p)}(p) \text{ with } \frac{dv^{(p)}}{dp} > 0$$

  on which $p = \hat{p}_y$;

- A line $\overline{\mathcal{L}}_c$ of equation, Section 16.10,

$$v^{(p)} = \overline{v}_c^{(p)}(p) \text{ with } \frac{dv^{(p)}}{dp} > 0$$

  on which $p = \overline{p}_c$;

- A line $\widehat{\mathcal{L}}_d$ of equation, Section 16.13,

$$p = \widehat{p}_d(v^{(p)})$$

on which $p = \widehat{p}_d$;

- A line $\overline{\mathcal{L}}_d$ of equation, the inverse of Eq. 16.139,

$$v^{(p)} = \overline{v}_d^{(p)}(p) \text{ with } \frac{dv^{(p)}}{dp} \leq 0$$

on which $p = \overline{p}_d$;

In this graph, the zones denoted by Z indicate:

- Z_1, Z_6 - impossible states zones;
- Z_2, Z_5 - elasto-plastic zones;
- Z_3, Z_4 - elastic zones.

16.15 Functional Relationships for p_y

Geometrically, p_y represents the intersection of smaller value of F with the p-axis, Fig. 16.3. In accordance with the hypothesis in item 4(a) in Section 16.3, the value of p_y can be calculated as

$$p_y = p_y(p, q, \theta, \overline{p}_y) \tag{16.142}$$

where, from Eq. 16.44,

$$p_y(p, q, \theta, \overline{p}_y) = \overline{p}_c - \left[(p - \overline{p}_c)^2 + \frac{q^2}{M(\theta)^2}\left(\frac{\overline{p}_y - \overline{p}_c}{\overline{p}_c}\right)^2\right]^{\frac{1}{2}}$$

and, according to Eq. 16.113, $\overline{p}_c = \overline{p}_c(\overline{p}_y)$. We note that since $\overline{p}_y = \overline{p}_y(p, v)$, Eq. 16.109, the functional relationship in Eq. 16.142 can be also expressed as

$$p_y = p_y(p, q, \theta, v) \tag{16.143}$$

Moreover, according to the requirement in item 5 in Section 16.3, if $F \equiv \overline{F}$ then it results

$$p_y = \overline{p}_y$$

as it can be easily verified solving Eq. 16.43 in terms of $p_m = \overline{p}_y$. The hypothesis in item 4 (b) in Section 16.3 states that, if plasticity occurs, then the incremental variation of the p_y is given by

$$\delta p_y = \frac{\partial p_y}{\partial v^{(p)}} \delta v^{(p)} + \frac{\partial p_y}{\partial \epsilon_s^{(p)}} \delta \epsilon_s^{(p)} \tag{16.144}$$

where the partial derivatives are function of $(p_y, \overline{p}_y)$ only. Consequently, the functional relationships for the partial derivatives can be found from any experimental stress path. In particular:

- the functional relationship for $\partial p_y / \partial v^{(p)}$ can be conveniently determined refering to experimental stress paths in which

$$\delta \epsilon_s^{(p)} = 0$$

  since in this case, Eq. 16.144,

$$\frac{\partial p_y}{\partial v^{(p)}} = \frac{\delta p_y}{\delta v^{(p)}}$$

  Hence, we can refer to a drained isotropic loading test of the type described in Section 16.2.1, in which the above condition is satisfied, Eq. 16.55.

- the functional relationship for $\partial p_y / \partial \epsilon_s^{(p)}$ can be conveniently determined refering to experimental stress paths in which

$$\delta v^{(p)} = 0$$

  since in this case, Eq. 16.144,

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} = \frac{\delta p_y}{\delta \epsilon_s^{(p)}}$$

  Hence, we can refer to an TUDP0 test, triaxial undrained $\delta p = 0$, of the type described in Section 16.2.2, stress path $P_c P_c^{(f)}$ in Fig. 16.2b. In this type of stress path, the above condition is satisfied, Eq. 16.56.

16.16 Functional Relationship for $\partial p_y/\partial v^{(p)}$

We have pointed out in Section 16.15 that the law of the variation of p_y with respect to $v^{(p)}$ only, can be experimentally determined from the response of an OCS during an isotropic drained test. The analysis of the soil response shown in Fig. 16.1 leads to the conclusion that the variation of p_y with respect to $v^{(p)}$ can be mathematically expressed as, Section 16.16.2,

$$\frac{\partial p_y}{\partial v^{(p)}} = \begin{cases} f_1(p_y, \overline{p}_y); & \text{for } p \leq \overline{p}_c. \\ f_2(p_y, \overline{p}_y); & \text{for } p > \overline{p}_c. \end{cases} \tag{16.145}$$

A particular expression for f_1 and f_2 is reported in Section 16.16.3.

16.16.1 v^p vs. p in an isotropic test

Figure 16.7 reports the qualitative diagram of the plastic component of the specific volume which, according with the proposed model, is mobilized during the isotropic test shown in Fig. 16.1. The values of $v^{(p)}$ reported in Fig. 16.7 are calculated from the value v reported in Fig. 16.1 taking into account that, Eq. 16.78,

$$\delta v^{(p)} = \delta v - \frac{\delta p}{\mathcal{B}^{(e)}(p)}$$

Figure 16.8 reports an extrapolation of the experimental response in Fig. 16.7 which results in accordance with the proposed model, [25]. The extrapolation consists in extending the unloading stress path CD until the $\overline{\mathcal{L}}_d$ line is met. From there, the soil is again loaded until the NCL is reached.

16.16.2 v^p vs. p_y in an isotropic test

From the experimental results reported in Fig. 16.8 it is possible to establish that in an isotropic test, $v^{(p)}$ is related to p_y as shown in Fig. 16.9. In fact, we remark in Fig. 16.3 that the F surface crosses the p-axis at two points of coordinate $q = 0$ and

$$p = \begin{cases} p_y; & \text{for } p \leq \overline{p}_c. \\ p_d = 2\overline{p}_c - p_y; & \text{for } p > \overline{p}_c \end{cases}$$

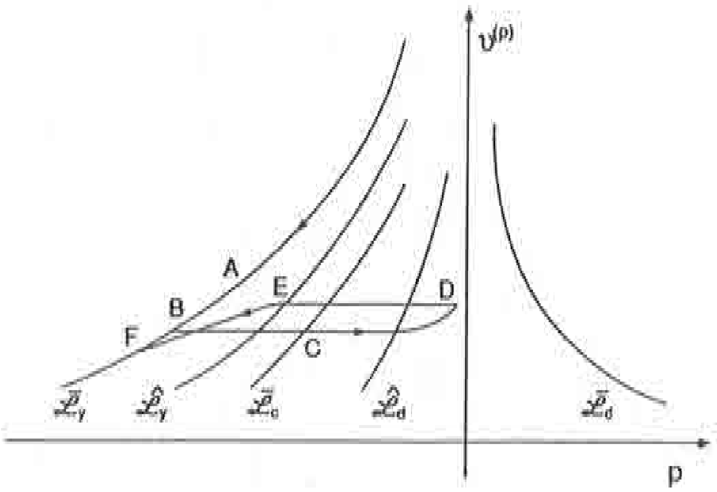


Figure 16.7: Experimental graph $v^{(p)}$ vs. p of an isotropic test

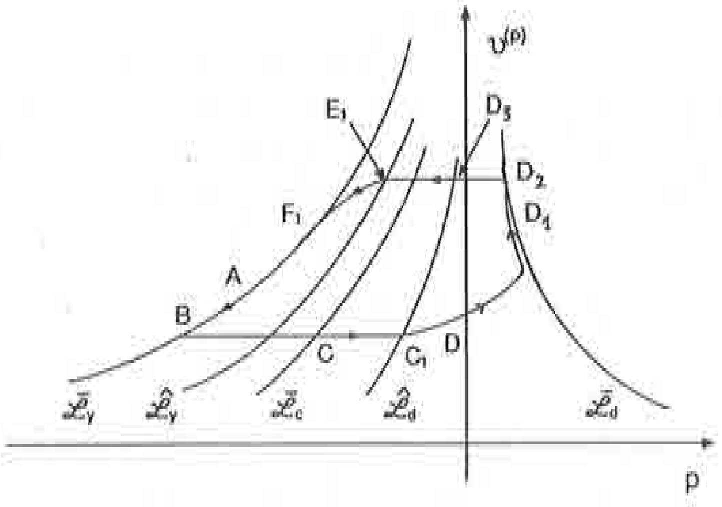


Figure 16.8: Extrapolated graph $v^{(p)}$ vs. p, of an isotropic test

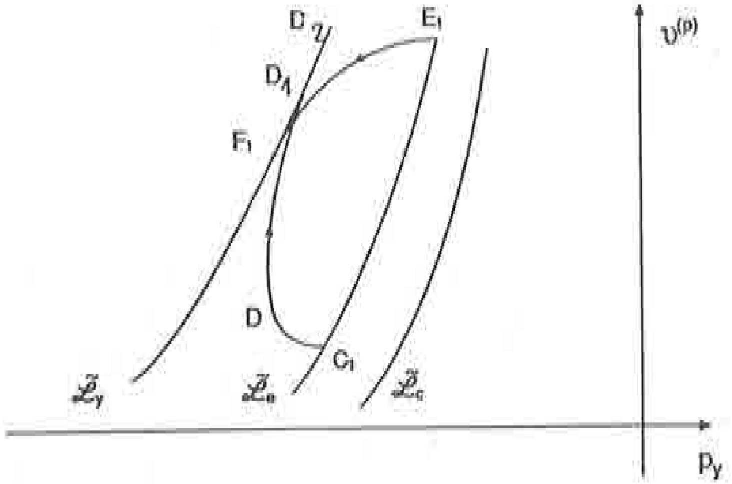


Figure 16.9: Extrapolated graph $v^{(p)}$ vs. p_y of an isotropic test

It follows that in an isotropic test p_y may be calculated as

$$p_y = \begin{cases} p; & \text{for } p \leq \overline{p}_c. \\ 2\overline{p}_c - p; & \text{for } p > \overline{p}_c. \end{cases} \tag{16.146}$$

where, Section 16.10, $\overline{p}_c$ is function of $v^{(p)}$ only. Moreover, the graph in Fig. 16.8 shows that plastic deformations take place on the stress path AB, $C_1DD_1D_2$ and E_1F_1. Thus, for each of these piecewise stress paths we may express

$$p = p(v^{(p)})$$

and the relationship in Eq. 16.146 takes the form

$$p_y = p_y(v^{(p)})$$

This relationship allows to map the experimental values $(p, v^{(p)})$ in Fig. 16.8 in the $v^{(p)}$ vs. p_y diagram in Fig. 16.9. This diagram shows also three lines $\tilde{\mathcal{L}}_y$, $\tilde{\mathcal{L}}_e$ and $\tilde{\mathcal{L}}_c$ which have been obtained as follows:

- The line $\tilde{\mathcal{L}}_y$, which results to be of equation

$$v^{(p)} = \tilde{v}_y^{(p)}(p_y) \tag{16.147}$$

is the mapping of the lines $\overline{\mathcal{L}}_y$ and $\overline{\mathcal{L}}_d$, shown in Fig. 16.8.

- The line $\tilde{\mathcal{L}}_e$, which results to be of equation

$$v^{(p)} = \hat{v}_y^{(p)}(p_y) = \overline{v}_y^{(p)}(p_y) - v_e \tag{16.148}$$

  is the mapping of the lines $\tilde{\mathcal{L}}_y$ and $\tilde{\mathcal{L}}_d$ shown in Fig. 16.8.

- The line $\tilde{\mathcal{L}}_c$, which results to be of equation

$$v^{(p)} = \overline{v}_c^{(p)}(p_y) = \overline{v}_y^{(p)}(p_y) - v_c \tag{16.149}$$

  is the mapping of the line $\overline{\mathcal{L}}_c$ shown in Fig. 16.8.

where the function $\overline{v}_y^{(p)}$ is defined in Eq. 16.97.

The experimental results in Fig. 16.9 suggest the existence of two families of curves $\tilde{\mathcal{R}}_1$ and $\tilde{\mathcal{R}}_2$ for isotropic drained test, of the type shown in Fig. 16.10. The functional relationships of these curves are of the type

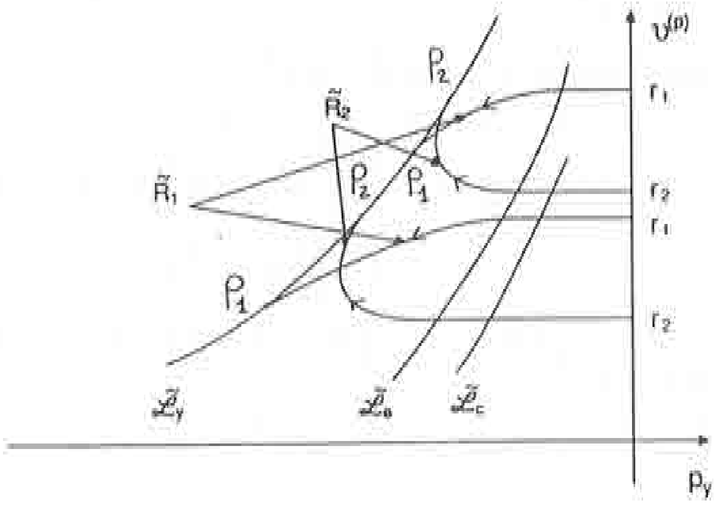


Figure 16.10: Families of curves $v^{(p)}$ vs. p_y for isotropic drained test.

$$p_y = \begin{cases} p_y^{(1)}\left(v^{(p)}, r_1\right); & \text{on } \tilde{\mathcal{R}}_1, \text{ for } p \le \overline{p}_c. \\ p_y^{(2)}\left(v^{(p)}, r_2\right); & \text{on } \tilde{\mathcal{R}}_2, \text{ for } p > \overline{p}_c. \end{cases} \tag{16.150}$$

or, equivalently, taking into account Eq. 16.95,

$$p_y = \begin{cases} p_y^{(1)}\left(\overline{p}_y, r_1\right); & \text{on } \tilde{\overline{\mathcal{R}}}_1, \text{ for } p \leq \overline{p}_c. \\ p_y^{(2)}\left(\overline{p}_y, r_2\right); & \text{on } \tilde{\overline{\mathcal{R}}}_2, \text{ for } p > \overline{p}_c. \end{cases} \tag{16.151}$$

where r_1 and r_2 are the parameters of the families of curves. The main geometrical properties of these curves are:

- The curves of the family $\tilde{\overline{\mathcal{R}}}_1$ (and $\tilde{\overline{\mathcal{R}}}_2$) do not intersect each others. This implies that, for any pair of values $(\overline{p}_y, p_y)$, we can determine a unique curve $\tilde{\overline{\mathcal{R}}}_1$ (or $\tilde{\overline{\mathcal{R}}}_2$); consequently, there exists a unique correspondence between the values $(\overline{p}_y, p_y)$ and the value r_1 (or r_2). Hence, it results

$$\frac{\partial p_y}{\partial v^{(p)}} = \begin{cases} f_1(p_y, \overline{p}_y); & \text{on } \tilde{\overline{\mathcal{R}}}_1 \\ f_2(p_y, \overline{p}_y); & \text{on } \tilde{\overline{\mathcal{R}}}_2 \end{cases}$$

  in accordance with the requirement in Eq. 16.20 of item 4(b), Section 16.3.

- The curves of the family $\tilde{\overline{\mathcal{R}}}_1$, on which

$$\delta v^{(p)} \leq 0$$

  keep always a positive bending and respect the property

$$0 < \frac{d\overline{p}_y}{dv^{(p)}} \leq \frac{\partial p_y}{\partial v^{(p)}} \tag{16.152}$$

  in accordance with the requirement in Eq. 16.22 in item 4(b), Section 16.3.

- The curves of the family $\tilde{\overline{\mathcal{R}}}_2$, on which

$$\delta v^{(p)} \geq 0$$

  change the sign of their bending and respect the property

$$\frac{d\overline{p}_y}{dv^{(p)}} \geq \frac{\partial p_y}{\partial v^{(p)}} \tag{16.153}$$

  in accordance with the requirement in Eq. 16.22 in item 4(b), Section 16.3.

- The curves of the family $\tilde{\mathcal{R}}_1$ (and $\tilde{\mathcal{R}}_2$) eventually overlay the line $\bar{\mathcal{L}}_y$, at the points P_1 and P_2. Consequently, for $p_y = \overline{p}_y$,

$$\frac{\partial p_y}{\partial v^{(p)}} = \frac{d\overline{p}_y}{dv^{(p)}}$$

in accordance with the requirement in item 5 in Section 16.3.

- The curves of the family $\tilde{\mathcal{R}}_1$ (and $\tilde{\mathcal{R}}_2$) terminate on one end horizontally, namely, for $p_y \to \hat{p}_y$,

$$\begin{aligned} \frac{\partial p_y}{\partial v^{(p)}} &\to +\infty; \quad \text{on } \tilde{\mathcal{R}}_1 \\ \frac{\partial p_y}{\partial v^{(p)}} &\to -\infty \quad \text{on } \tilde{\mathcal{R}}_2 \end{aligned}$$

in accordance with the requirement in item 7 in Section 16.3.

16.16.3 A mathematical expression for $\partial p_y/\partial v^{(p)}$

Based on the functional relationships in Eq. 16.150 representing the families of curves $\tilde{\mathcal{R}}_1$ and $\tilde{\mathcal{R}}_2$ in Fig. 16.10, the following mathematical expression for the partial derivative of p_y with respect to $v^{(p)}$ is proposed, [25]:

$$\frac{\partial p_y}{\partial v^{(p)}} = \left[-\frac{\lambda}{p_y} + \frac{\chi}{p_y + p_o}\right]^{-1} * \begin{cases} [Y_1(X)]^{-1}; & \text{for } p \leq \overline{p}_c. \\ [Y_2(X)]^{-1}; & \text{for } p > \overline{p}_c. \end{cases} \tag{16.154}$$

where

$$\begin{aligned} Y_1(X) &= 1 - \left[\sum_{k=1}^{n} A_k X^k\right]^{\frac{1}{n+1}} \\ Y_2(X) &= \left\{1 - \left[\frac{\sum_{k=1}^{m_1} B_k X^k}{\sum_{k=1}^{m_2} B_{m_1+k}(v_c - X)^k}\right]^{\frac{1}{m_1+1}}\right\}^{-1} \end{aligned}$$

The independent variable X is defined as

$$X = X(p_y, \overline{p}_y) = -\lambda \ln \frac{p_y}{\overline{p}_y} + \chi \ln \frac{p_y + p_o}{\overline{p}_y + p_o}$$

or, alternatively, being $\overline{p}_y = \overline{p}_y(p, v)$, Section 16.9,

$$X = X(p, v, p_y) = -\lambda \ln \frac{p_y}{p_\lambda} + \chi \ln \frac{p_y + p_o}{p_\lambda} + c(p) + v_\lambda - v \qquad (16.155)$$

The material parameters controlling the $Y_1(X)$ function can be determined as follows, [25]:

- $(n-1)$, with $n > 0$, is the total number of experimental points $(v, p, dv/dp)$ measured on elasto-plastic isotropic reloading paths of the type EF and HI in Fig. 16.1.

- A_k, for $k = 1, 2, \ldots, n$, are n constants which can be determined from the solution of the linear system of n equations

$$\left\{ \begin{array}{l} \sum_{k=1}^{n} A_k v_e^k = 1 \\ \sum_{k=1}^{n} A_k X_i^k = [1 - Y_1(X_i)]^{n+1} \end{array} \right. \qquad (16.156)$$

  where $i = 1, \ldots, n-1$.

- The values of $Y_1(X_i)$ and X_i are given by, Eqs. 16.154 and 16.155,

$$\begin{aligned} Y_1(X_i) &= \left[\left(-\frac{\lambda}{(p_y)_i} + \frac{\chi}{(p_y)_i + p_o} \right) \left(\frac{\partial p_y}{\partial v^{(p)}} \right)_i \right]^{-1} \\ X_i &= -\lambda \ln \frac{(p_y)_i}{p_\lambda} + \chi \ln \frac{(p_y)_i + p_o}{p_i + p_o} + v_\lambda - v_i \end{aligned}$$

  where, Eq. 16.146,

$$(p_y)_i = p_i$$

  and it can be proved that, [25],

$$\left(\frac{\partial p_y}{\partial v^{(p)}} \right)_i = \left(\frac{dv}{dp} + \frac{\chi}{p + p_o} \right)_i^{-1}$$

- To make sure that the experimental values do not belong to the elastic region bounded by ELL and ERL, it is necessary to verify that

$$0 < X_i < v_e$$

However, once the system of equation in Eq. 16.156 has been solved for A_k, it is necessary to verify that, for all $0 < X < v_e$, it results

$$0 < \sum_{k=1}^{n} A_k X^k < 1$$

If this condition is not satisfied, then the experimental points cannot be exactly interpolated by the proposed functional relationship in Eq. 16.154. If one wants still to use this functional relationship, he must adjust the experimental data.

The material parameters controlling the $Y_2(X)$ function can be determined as follows, [25].

- $(m_1 + m_2 - 1)$, with $m_1, m_2 > 0$, is the total number of experimental points $(v, p, dv/dp)$, with $p < 0$, measured on elasto-plastic isotropic unloading paths of the type CD in Fig. 16.1.
- B_k, for $k = 1, 2, \ldots, (m_1 + m_2)$, are $(m_1 + m_2)$ constants which may be determined from the solution of the linear system of $(m_1 + m_2)$ equations

$$\begin{cases} B_{m_1+1} = 1 \\ \displaystyle\sum_{k=1}^{m_1} B_k X_i^k = \left\{1 - [Y_2(X_i)]^{-1}\right\}^{m_1+1} * \sum_{k=1}^{m_2} B_{m_1+k} (v_e - X_i)^k \end{cases} \tag{16.157}$$

where $i = 1, \ldots, (m_1 + m_2) - 1$.

- The values of $Y_2(X_i)$ and X_i are given by Eqs. 16.154 and 16.155,

$$\begin{aligned} Y_2(X_i) &= \left[\left(-\frac{\lambda}{(p_y)_i} + \frac{\chi}{(p_y)_i + p_o}\right)\left(\frac{\partial p_y}{\partial v^{(p)}}\right)_i\right]^{-1} \\ X_i &= -\lambda \ln \frac{(p_y)_i}{p_\lambda} + \chi \ln \frac{(p_y)_i + p_o}{p_i + p_o} + v_\lambda - v_i \end{aligned}$$

where, Eq. 16.146,

$$(p_y)_i = (2\overline{p}_c - p)_i$$

and it can be proved that, [25],

$$\left(\frac{\partial p_y}{\partial v^{(p)}}\right)_i = 2\left(-\frac{\lambda}{\overline{p}_c} + \frac{\chi}{\overline{p}_c + p_o}\right)_i^{-1} - \left(\frac{dv}{dp} + \chi \frac{1}{p + p_o}\right)_i^{-1}$$

where, according to Eq. 16.119, the value of $(\overline{p}_c)_i$ can be calculated as function of $(p, v)_i$.

- To make sure that the experimental values do not belong to the elastic region bounded by ELL and ERL, it is necessary to verify that

$$0 < X_i < v_e$$

However, once the system of equation in Eq. 16.157 has been solved for B_k, it is necessary to verify that, for all $0 < X < v_e$ it results that

$$\frac{\sum_{k=1}^{m_1} B_k X^k}{\sum_{k=1}^{m_2} B_{m_1+k} (v_e - X)^k} > 0$$
$$\sum_{k=1}^{m_2} B_{m_1+k} (v_e - X)^k \neq 0 \tag{16.158}$$

and in any case

$$\sum_{k=1}^{m_1} B_k v_e^k \neq 0 \tag{16.159}$$

If these conditions are not satisfied, then the experimental points cannot be exactly interpolated by the proposed functional relationship in Eq. 16.154. If one wants still to use this functional relationship, he must adjust the experimental data.

16.17 Functional Relationship for $\partial p_y / \partial \epsilon_s^{(p)}$

We have pointed out in Section 16.15 that the law of the variation of p_y with respect to $\epsilon_s^{(p)}$ only, can be experimentally determined from the response of an OCS in a TUDP0 test. The analysis of the soil response shown in Fig. 16.2, stress path $P_cP^{(f)}$, leads to the conclusion that the variation of p_y with respect to $\epsilon_s^{(p)}$ can be mathematically expressed as, Section 16.17.2,

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} = f_3(p_y, \overline{p}_y) \tag{16.160}$$

A particular expression for f_3 is reported in Section 16.17.3.

16.17.1 $\epsilon_s^{(p)}$ vs. q in a triaxial test

Figure Fig. 16.11 reports the qualitative diagram of the total plastic shear strain $\epsilon_s^{(p)}$, which, according to the model, is mobilized during the undrained (compression) triaxial test $P_cP_c^{(f)}$ in Fig. 16.2. The values of $(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$ re-

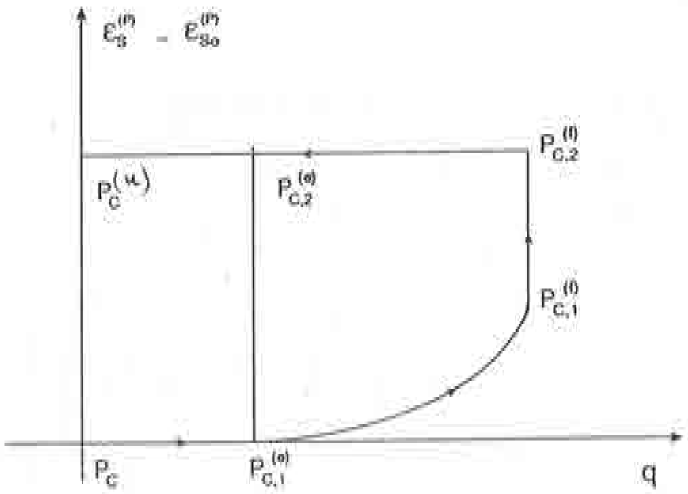


Figure 16.11: Experimental graph $(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$ vs. q, in the undrained triaxial test TUDP0

ported in Fig. 16.11 are calculated from the values of ϵ_a reported in Fig. 16.2 taking into account that, by definition,

$$\delta\epsilon_a = \delta\epsilon_a^{(e)} + \delta\epsilon_a^{(p)}$$

where for the particular stress path $P_cP_c^{(f)}$, it is easy to prove that

$$\begin{aligned} \delta\epsilon_a^{(e)} &= -\frac{1}{3G^{(e)}}\delta q \\ \delta\epsilon_s^{(p)} &= |\delta\epsilon_a^{(p)}| \end{aligned} \qquad (16.161)$$

It follows that the plastic component $\epsilon_s^{(p)} - \epsilon_{so}^{(p)}$ shown in Fig. 16.11 may be calculated by integrating

$$\delta\epsilon_s^{(p)} = \left|\delta\epsilon_a + \frac{1}{3G^{(e)}}\delta q\right|$$

where $\delta\epsilon_a$, δq can be found from the experimental values shown in Fig. 16.2c. The quantity $\epsilon_{so}^{(p)}$ represents the initial condition.

It is interesting to note that the stress path $P_{c,1}^{(e)}P_{c,1}^{(f)}$ reaches the bounding surface $\overline{F}$ in $P_{c,1}^{(f)}$, where $A = 0$ and the critical state conditions are met. It is easy to verify that the diagram in Fig. 16.11 is consistent with the proposed model.

16.17.2 $\epsilon_s^{(p)}$ vs. p_y in a triaxial test

From the experimental results of the type in Fig. 16.11 it is possible to establish that, along the TUDP0 stress path $P_cP_c^{(f)}$ in Fig. 16.2, $\epsilon_s^{(p)}$ is related to p_y as shown in Fig. 16.12. In fact, because of the homothetic

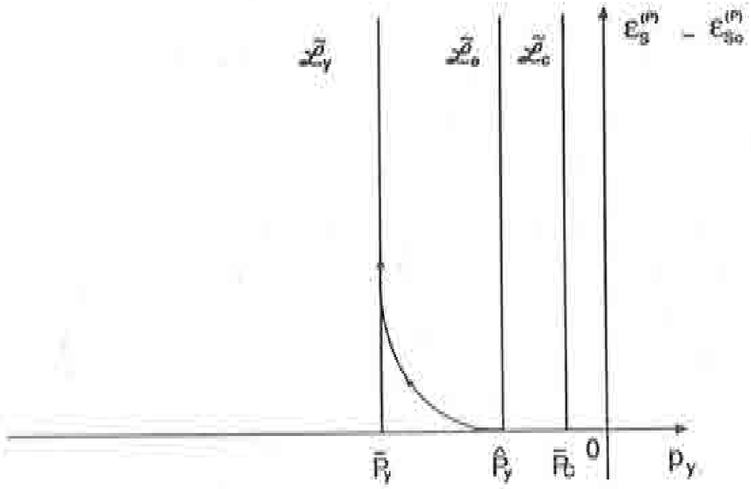


Figure 16.12: Experimental graph $(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$ vs. p_y, in the undrained triaxial test TUDP0

relationship between $\overline{F}$ and F with respect to the point P_c, Fig. 16.3, we may establish that

$$\frac{\overline{q}_c}{\overline{p}_c - \overline{p}_y} = \frac{q}{\overline{p}_c - p_y}$$

where, Eq. 16.8,

$$\overline{q}_c = q_c(p,\theta) = -M\left(\theta = \frac{\pi}{6}\right)\overline{p}_c$$

and the value of θ is established because of the triaxial compression condition. It follows that, in this triaxial test, p_y may be calculated as

$$p_y = \overline{p}_c + \frac{\overline{p}_y - \overline{p}_c}{\overline{q}_c} q \tag{16.162}$$

where $\overline{p}_c$ and $\overline{p}_y$, being only function of $v^{(p)}$, are constant values; consequently, also $\overline{q}_c$ is constant. Moreover, from the graph in Fig. 16.11 we note that plastic deformation takes place on the stress path $P_{c,1}^{(e)} P_{c,1}^{(f)} P_{c,2}^{(f)}$ Thus, for this stress path we may express

$$q = q(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$$

and the relationship in Eq. 16.162 takes the form

$$p_y = p_y(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$$

This relationship in Eq. 16.162 allows to map the experimental values $(\epsilon_s^{(p)}, q)$ in Fig. 16.11 into the $\epsilon_s^{(p)}$ vs. p_y diagram in Fig. 16.12. This diagram shows three lines $\tilde{\mathcal{L}}_y$, $\tilde{\mathcal{L}}_e$ and $\tilde{\mathcal{L}}_c$ which have been obtained as follows:

- The line $\tilde{\mathcal{L}}_y$, which results to be of equation

  $$p_y = \overline{p}_y = \text{ constant} \tag{16.163}$$

  is the mapping of the line $P_{c,1}^{(f)} P_{c,2}^{(f)}$ shown in Fig. 16.11.

- The line $\tilde{\mathcal{L}}_e$, which results to be of equation

  $$p_y = \hat{p}_y = \text{ constant} \tag{16.164}$$

  is the mapping of the line $P_{c,1}^{(e)} P_{c,2}^{(e)}$ shown in Fig. 16.11.

- The line $\tilde{\mathcal{L}}_c$, which results to be of equation

  $$p_y = \overline{p}_c = \text{ constant} \tag{16.165}$$

  is the mapping of the line $P_c P_c^{(u)}$ shown in Fig. 16.11.

The experimental results shown in Fig. 16.12 suggest the existence of the family of curves $\tilde{\mathcal{R}}_3$ drawn in Fig. 16.13. The functional relationship of these curves is of the type

$$p_y = p_y(\epsilon_s^{(p)} - \epsilon_{so}^{(p)}, \overline{p}_y) \tag{16.166}$$

where $\overline{p}_y$ is the parameter of the family of curves. The main geometrical properties of these curves are:

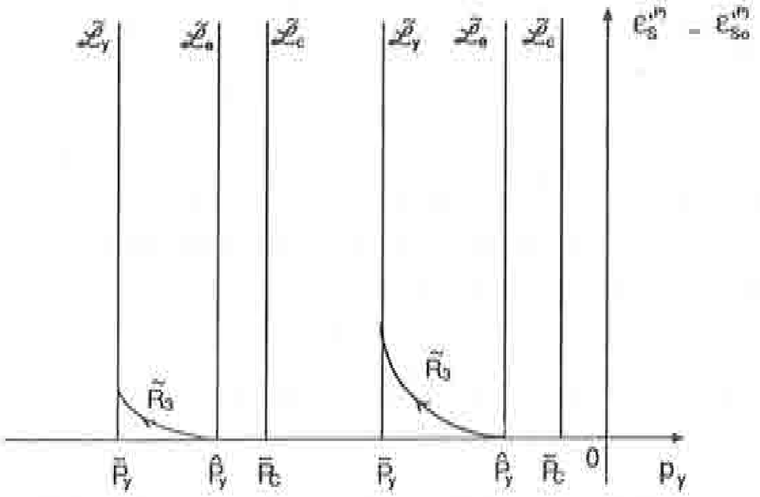


Figure 16.13: Family of curves $(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$ vs. p_y for undrained triaxial test TUDP0.

- The curves of the family $\tilde{\mathcal{R}}_3$ do not intersect each others. This implies that, for any pair of values $(\overline{p}_y, p_y)$, we can determine a unique curve $\tilde{\mathcal{R}}_3$; consequently, there exists a unique correspondence between the values $(\overline{p}_y, p_y)$ and the value $(\epsilon_s^{(p)} - \epsilon_{so}^{(p)})$. Hence, it results

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} = \frac{\partial p_y}{\partial \epsilon_s^{(p)}}(p_y, \overline{p}_y)$$

  in accordance with the requirement in Eq. 16.21 in item 4(b), Section 16.3.

- The curves of the family $\tilde{\mathcal{R}}_3$ keep always a negative slope, namely

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} \leq 0 \tag{16.167}$$

  in accordance with the requirement in Eq. 16.23 in item 4(b), Section 16.3.

- The curves of the family $\tilde{\mathcal{R}}_3$ eventually overlay the line $\tilde{\mathcal{L}}_y$, consequently, for $p_y = \overline{p}_y$,

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} = 0$$

in accordance with the requirement in item 5 in Section 16.3.

- The curves of the family $\tilde{\mathcal{R}}_3$ terminate on one end horizontally, namely, for $p_y \to \bar{p}_y$,

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} \to -\infty$$

in accordance with the requirement in item 7 in Section 16.3.

16.17.3 A mathematical expression for $\partial p_y/\partial \epsilon_s^{(p)}$

Based on the functional relationship $p_y = p_y(\epsilon_s^{(p)} - \epsilon_{so}^{(p)}, \bar{p}_y)$ in Eq. 16.166 representing the family of curves $\tilde{\mathcal{R}}_3$ in Fig. 16.13, the following mathematical expression for the partial derivative of p_y with respect to $\epsilon_s^{(p)}$ is proposed, [25]:

$$\frac{\partial p_y}{\partial \epsilon_s^{(p)}} = \left[-\frac{\lambda}{p_y} + \frac{\chi}{p_y + p_o}\right]^{-1} * [Y_3(X)]^{-1} \tag{16.168}$$

where

$$Y_3(X) = -\left[\frac{\sum_{k=1}^{p_2} C_{p_1+k} X^{k-1}}{\sum_{k=1}^{p_1} C_k X^k}\right]^{\frac{1}{p_1+1}}$$

and the independent variable X is defined as in Eq. 16.155. The material parameters controlling the $Y_3(X)$ function can be determined as follows, [25]:

- $[(p_1 + p_2) - 2]$, with $p_1 > 0$ and $p_2 > 1$, is the total number of experimental points $(q, dq/d\epsilon_a)$ measured on a TUDP0 elasto-plastic loading path of the type $P_c^{(e)} P_c^{(f)}$ in Fig. 16.2c.
- C_k, for $k = 1, 2, \ldots, (p_1 + p_2)$, are $(p_1 + p_2)$ constants which can be determined from the solution of the linear system of $(p_1 + p_2)$ equations

$$\begin{cases} C_{p_1+1} = 1 \\ \sum_{k=1}^{p_2} C_{p_1+k} v_c^{k-1} = 0 \\ \sum_{k=1}^{p_2} C_{p_1+k} X_i^{k-1} = [-Y_3(X_i)]^{p_1+1} \sum_{k=1}^{p_1} C_k X_i^k \end{cases} \tag{16.169}$$

where $i = 1, \ldots, (p_1 + p_2) - 2$.

- The values of X_i and $Y_3(X_i)$ are given by, Eqs. 16.155 and 16.168,

$$\begin{aligned} Y_3(X_i) &= \left[\left(-\frac{\lambda}{(p_y)_i} + \frac{\chi}{(p_y)_i + p_o} \right) \left(\frac{\partial p_y}{\partial \epsilon_s^{(p)}} \right)_i \right]^{-1} \\ X_i &= -\lambda \ln \frac{(p_y)_i}{\overline{p}_y} + \chi \ln \frac{(p_y)_i + p_o}{\overline{p}_y + p_o} \end{aligned}$$

where, Eq. 16.162

$$\begin{aligned} (p_y)_i &= \overline{p}_c + \left(\frac{\overline{p}_c - \overline{p}_y}{\overline{p}_c} \right) \frac{q_i}{\overline{M}} \\ \overline{M} &= M(\theta = \pi/6) \end{aligned}$$

and, it can be proved that, [25],

$$\left(\frac{\partial p_y}{\partial \epsilon_s^{(p)}} \right)_i = \frac{\overline{p}_y - \overline{p}_c}{\overline{M}\,\overline{p}_c} \frac{1}{\left(\dfrac{\delta q}{\delta \epsilon_a} \right)_i^{-1} + \dfrac{2(1+\nu)}{3E_o v\,(\overline{p}_c + p_o)}}$$

Moreover:

  - $(q, \delta q/\delta \epsilon_a)_i$ are the experimental values.
  - $\overline{p}_y$ is the preconsolidation value of p, that is the value of p at the end of the isotropic compression, point P_o in Fig. 16.2a. Note that $\overline{p}_y$ is a constant value during a TUDP0 test.
  - $\overline{p}_c$ can be calculate as function of $\overline{p}_y$ by solving the implicit equation in Eq. 16.115.
  - v is the specific volume measured at the begin of the TUDP0 test. Note that v is a constant value during a TUDP0 test.
  - E_o and p_o are elastic material parameters, Section 16.7.

- In order to be sure that the experimental values $(q, dq/d\epsilon_a)_i$ are measured within the elasto-plastic region of a TUDP0 stress path of the type $P_e^{(e)}P_e^{(f)}$ in Fig. 16.12a, it is necessary to verify that

$$0 < X_i < v_e$$

However, once the system of equation in Eq. 16.169 has been solved for C_k, it is necessary to verify that, for all $0 < X < v_e$ it results that

$$\frac{\sum_{k=1}^{p_2} C_{p_1+k} X^{k-1}}{\sum_{k=1}^{p_1} C_k X^k} > 0$$

$$\sum_{k=1}^{p_1} C_k X^k \neq 0$$

If this condition is not satisfied, then the experimental points cannot be exactly interpolated by the proposed functional relationship in Eq. 16.168. If one wants still to use this functional relationship, he must adjust the experimental data.

16.18 Plastic Modulus

The general expression of the plastic modulus for the constitutive equation presented in this Chapter is of the type, Eq. 16.36,

$$A = -\left[v\left(\frac{\partial F}{\partial \overline{p}_y}\frac{d\overline{p}_y}{dv^{(p)}} + \frac{\partial F}{\partial p_y}\frac{\partial p_y}{\partial v^{(p)}}\right)\frac{\partial F}{\partial p} + \frac{\partial F}{\partial p_y}\frac{\partial p_y}{\partial \epsilon_s^{(p)}}\overline{c}_2\right] \qquad (16.170)$$

where

$$\overline{c}_2 = \left[\left(\frac{\partial F}{\partial q}\right)^2 + \frac{1}{q^2}\left(\frac{\partial F}{\partial \theta}\right)^2\right]^{1/2}$$

and

- The expressions of the partial derivatives of F with respect to p, q, and θ are given in Eqs. 16.68-16.71.
- The expressions of the partial derivatives of F with respect to the parameters $\overline{p}_y$ and p_y are reported in Eqs. 16.72 and 16.73, respectively.
- The expression for $d\overline{p}_y/dv^{(p)}$ is reported in Eq. 16.100.
- In general, the partial derivative of p_y with respect to $v^{(p)}$ are of the type, Sections 16.16 and 16.17,

$$\frac{\partial p_y}{\partial v^{(p)}} = \begin{cases} f_1(\overline{p}_y, p_y); & \text{for } p \le \overline{p}_c, \\ f_2(\overline{p}_y, p_y); & \text{for } p > \overline{p}_c. \end{cases}$$

$$\frac{\partial p_y}{\partial \epsilon_q^{(p)}} = f_3(\overline{p}_y, p_y)$$

Particular expressions for f_1, f_2 and f_3 are reported in Sections 16.16.3 and 16.17.3.

It is possible to prove that the expression of A in Eq. 16.170 satisfies the requirement stated in Eqs. 16.41 and 16.42.

Moreover, looking back to the mathematical relationships presented in the previous Sections, we can verify that the plastic modulus A is eventually function of the variables (p, q, θ, v), that is

$$A = A(p, q, \theta, v) \tag{16.171}$$

It can be proved that if $F \equiv \overline{F}$, then the plastic modulus expression in Eq. 16.170 reduces to, Section 13.4,

$$A = -v \frac{\partial \overline{F}}{\partial \overline{p}_y} \frac{d\overline{p}_y}{dv^{(p)}} \frac{\partial F}{\partial p} \tag{16.172}$$

where

$$\frac{\partial F}{\partial p} = 2(p - \overline{p}_c)$$

and the expression for the partial derivative of F with respect to $\overline{p}_y$ is reported in Eq. 16.86. Since v and $d\overline{p}_y/dv^{(p)}$ in Eq. 16.100 are always positive quantities and Eq. 16.10,

$$\frac{\partial \overline{F}}{\partial \overline{p}_y} > 0$$

it follows that

$$A \begin{cases} > 0; & \text{for } p < \overline{p}_c. \\ = 0; & \text{for } p = \overline{p}_c. \\ < 0; & \text{for } p > \overline{p}_c. \end{cases}$$

16.19 The Limit State Surface

It is simple to verify that the presented model leads to the conclusion that there exists a *Limit State Surface* of equation

$$v_{max} = v_{max}(p, q, \theta) \tag{16.173}$$

which bounds always the value of the specific volume v of a soil at an admissible stress state (p, q, θ), that is

$$v \leq v_{max}(p, q, \theta)$$

This function can be obtained specializing Eq. 16.104 as

$$\begin{aligned} v(p,q,\theta) &= v_\lambda + \int_{p_\lambda}^{p} \frac{1}{B^{(e)}(p)} dp + \overline{v}_y^{(p)}(\overline{p}_{ymax}) - \overline{v}_y^{(p)}(p_\lambda) = \\ &= v_\lambda + c(p) - \lambda \ln \frac{\overline{p}_{ymax}}{p_\lambda} + \chi \ln \frac{\overline{p}_{ymax} + p_o}{p_\lambda} \end{aligned} \qquad (16.174)$$

where $\overline{p}_{ymax}$ is the unique solution of Eq. 16.43 for the given stress state, item 9 in Section 16.5.

In fact, Eq. 16.107 indicates that the value of v monotonically increases with $\overline{p}_y$. Moreover, for any admissible material state $(p, q, \theta, \overline{p}_y)$, it results, item 9 in Section 16.5,

$$\overline{p}_y \leq \overline{p}_{ymax}$$

Consequently, according to Eq. 16.104

$$v = v(p, \overline{p}_y) \leq v\left(p, \overline{p}_{ymax}(p, q, \theta)\right) = v_{max}(p, q, \theta)$$

For a given θ, Eq. 16.173 describes a surface in the (p, q, v) space, Fig. 16.14, called the *Limit State Boundary Surface*, LSS, since it bounds all possible soil states. The intersection of the LSS with the v vs p plane, on which

$$\begin{aligned} q &= 0 \\ \theta &= \text{indeterminate} \end{aligned}$$

identifies the NCL

$$v = v_{NCL}(p) \qquad (16.175)$$

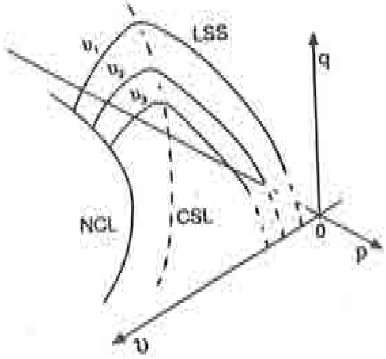
in Eq. 16.3. Hence, extending the definition of Normally and Overconsolidated Soils given in Section 16.2.1, we classify soils as follows:

- soils whose state (p, q, v) is on the LSS are defined as *Normally Consolidated Soils*, NCS,

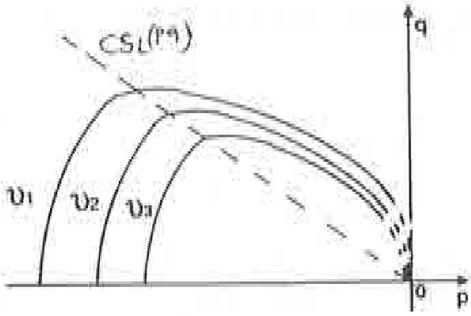
$$v = v_{max}(p, q, \theta)$$

- soils whose state (p, q, v) is inside the LSS are defined as *Overconsolidated Soils*, OCS,

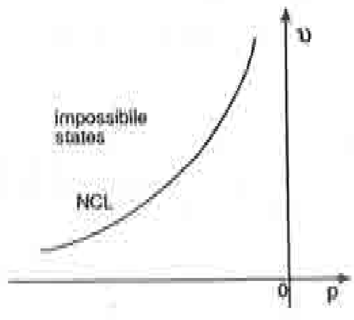
$$v < v_{max}(p, q, \theta)$$



a) LSS in the p, q, v space.



b) LSS projected on the q vs. p plane.



c) LLS intersection with the v vs. p plane.

Figure 16.14: The Limit State Surface, LSS:

16.20 Critical State Conditions

Consider a material state (p, q, θ, v) which satisfies both conditions

$$\begin{cases} q = q_c(p,\theta) \\ v = v_{max}(p,q,\theta) \end{cases} \tag{16.176}$$

that is the stress point lies on the Critical State Surface defined in Eq. 16.8 and the specific volume lies on the Limit State Surface defined in Eq. 16.173. In this case, the presented model predicts the *Critical State Conditions*

$$\begin{cases} \boldsymbol{\sigma} & = & \text{constant} \\ v & = & \text{constant} \\ \delta e_s & = & \text{indeterminate} \end{cases} \tag{16.177}$$

In fact, by definition,

$$\begin{aligned} \delta v &= \delta v^{(e)} + \delta v^{(p)} \\ \delta \mathbf{e} &= \delta \mathbf{e}^{(e)} + \delta \mathbf{e}^{(p)} \end{aligned}$$

If $v = v_{max}$, then the material state lies on $\overline{F}$, Section 16.19; thus, the stress point satisfies both

$$\begin{cases} q = q_c(p,\theta) \\ \overline{F}(p,q,\theta,\overline{p}_y) = 0 \end{cases} \tag{16.178}$$

which then identifies the point $\overline{C}$ in Fig. 16.3. At point $\overline{C}$ we have $p = \overline{p}_c$ and, consequently, Eq. 16.172,

$$\begin{aligned} \frac{\partial F}{\partial p} &= 0 \\ A &= 0 \end{aligned}$$

Being $A = 0$, from Eq. 16.34 it follows

$$\delta\lambda = \text{indeterminate}$$

for any given $\delta\boldsymbol{\sigma}$. Hence, Eqs. 16.80 and 16.75,

$$\begin{cases} \delta v^{(p)} = 0 \\ \delta \mathbf{e}^{(p)} = \text{indeterminate} \end{cases}$$

Consequently, for $\delta\boldsymbol{\sigma} = 0$, it results

$$\begin{aligned} \delta v &= 0 \\ \delta \mathbf{e} &= \text{indeterminate} \end{aligned}$$

being $\delta v^{(e)} = 0$ and $\delta \mathbf{e}^{(e)} = \mathbf{o}$. This proves the critical state conditions stated in Eq. 16.177. The proof for the reverse proposition, that is the critical state conditions occur only at material states satisfying Eq. 16.177, can be found in [24].

16.21 Verification of the Physical Hypotheses

The mathematical developments presented in the previous Sections indicate that SUOLO Standard V1 verifies all the physical hypotheses listed in Section 16.1. In fact:

1. The mechanical behavior of the material is fully controlled by the state variables (σ, v) since:
   - the elastic mechanical behavior depends on $E = E(v, p)$ and ν is assumed to be a constant, Section 16.7;
   - the plastic mechanical behavior, items 9 and 10 in Section 16.3, depends on $A = A(p, q, \theta, v)$ and $\mathbf{a} = \mathbf{a}(\sigma, p_y, \overline{p}_y, \overline{p}_c, v)$, only, Eqs. 16.63 and 16.171, and all the hardening parameters $(p_y, \overline{p}_y, \overline{p}_c)$ eventually depend on the state variables (p, q, θ, v) only, Section 16.4.
2. The isotropicity requirement is satisfied since the material state is given in terms of invariant quantities, so that the relative constitutive equation is invariant under a change of reference frame.
3. The model predicts plastic deformations at any stress level so that, subjected to cycles of loading and unloading, the soil presents accumulation of irreversible deformations.
4. The value of the specific volume v associated with an admissible material state is always bounded by the function value v_{max} defined in Eq. 16.173.
5. Material states satisfying both the conditions

$$\begin{cases} q = q_c(p, \theta) \\ v = v_{max}(p, q, \theta) \end{cases}$$

   are at *Critical State Conditions*, Section 16.20.

16.22 Soil Model Predictions

SUOLO/Standard V1 has been implemented in Ω which is a Finite Element Code able to solve the Coupled-Solid Fluid problem on the basis of the theory developed in [40, 41].

In this Section we present the numerical predictions of SUOLO/Standard V1 calculated by Ω on a sample of kaolin subjected to triaxial loading conditions. The following material parameters have been established to fit at best the available experimental tests reported in Fig. 16.15, [3].

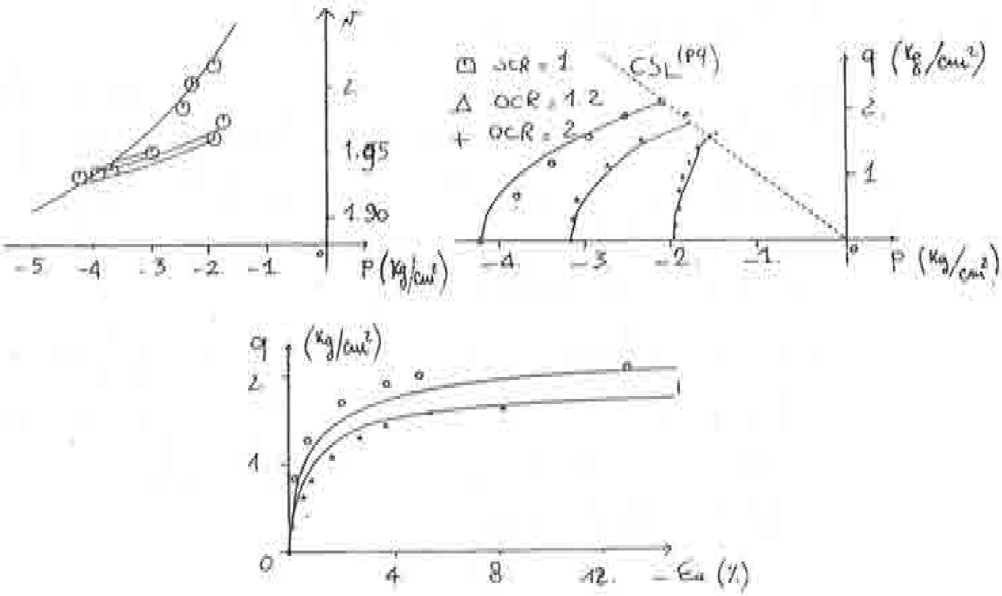


Figure 16.15: Reference experimental tests

$$p_\lambda = -2.11\ Kg/cm^2; \quad v_\lambda = 2.02; \quad \lambda = 0.14$$

$$\chi = 0.05; \quad p_o = -5.E - 10\ Kg/cm^2 \quad \nu = 0.18; \quad v_e = 0.06$$

$$\phi = 26.5^o; \quad v_c = 0.098$$

$$n = 2; \quad A_1 = 7.553; \quad A_2 = 151.9$$

$$p_1 = 1; \quad p_2 = 2; \quad C_1 = 16.67; \quad C_2 = 1.0; \quad C_3 = -16.67$$

Because of the lack of informations, we have then assumed the following material parameter values:

$$\theta_1 = -25^o; \quad \theta_2 = 25^o$$

$$m_1 = 1; \quad m_2 = 1; \quad B_1 = 1.0; \quad B_2 = 1.0$$

The continuous lines in Fig. 16.15 represent the numerical interpolation of the available experimental data, while:

- Figure 16.16 shows the predicted response of the soil under a complete loading/unloading/reloading isotropic drained test, with $p < 0$.

- Figure 16.17 shows the predicted response of three samples of soil subjected to triaxial drained conditions. These samples have been previously subjected to isotropic compression drained stress paths up to the confining pressures $p_1 > p_2 > p_3$, Fig. 16.17a. Then, they have been isotropically unloaded, always under drained conditions, up to the same confining pressure $p^{(o)}$.

- Figure 16.18 shows the predicted response of four samples of OCS soil subjected to triaxial undrained conditions. These samples have been previously subjected to isotropic drained stress paths up to the confining pressures $p_1 > p_2 > p_3 > p_4$, Fig. 16.18a. Then, they have been isotropically unloaded, always under drained conditions, up to the same specific volume value $v^{(o)}$.

  It is interesting to note that in triaxial undrained conditions the sign of the mean stress increment is given by

$$\delta p \begin{cases} \geq 0 & \text{for } p \leq \overline{p}_c \\ < 0 & \text{for } p > \overline{p}_c \end{cases} \tag{16.179}$$

  which can be also analytically verified from the equation $\delta v = 0$, for any undrained test, either triaxial, or not triaxial.

  Moreover, the values of q, p at failure are uniquely determined as,

$$p = \overline{p}_c \tag{16.180}$$

$$q = \overline{q} = -\overline{M}\,\overline{p}_c \tag{16.181}$$

where

$$\overline{M} = M(\theta = \pi/6) \tag{16.182}$$

$$\overline{p}_c = p_\lambda \exp\left(-\frac{v - v_\lambda + v_c}{\lambda}\right) \tag{16.183}$$

and v is the constant specific volume.

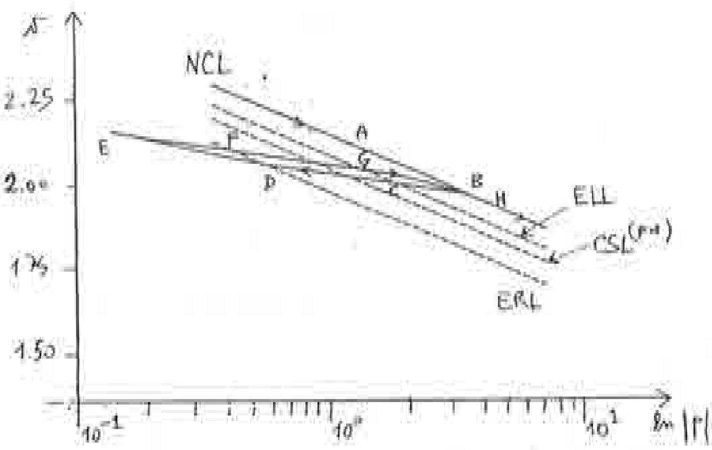


Figure 16.16: Isotropic drained test

16.23 Undrained Triaxial Cyclic Test Simulation

Figures 16.19 and 16.20 show the predicted response of two samples of fully saturated NCS subjected to undrained triaxial cyclic loading. Both samples have been previously isotropically compressed under drained conditions, up to the same confining pressure $p^{(o)}$, Figs. 16.19a and 16.20a. Then, they have been subjected to cycles of axial loads ranging between $q = \left[0, q_M^{(1)}\right]$ and $q = \left[0, q_M^{(2)}\right]$, respectively, where

$$q_M^{(1)} \leq \tilde{q}^{(t)} < q_M^{(2)} \leq \overline{q}^{(t)}$$

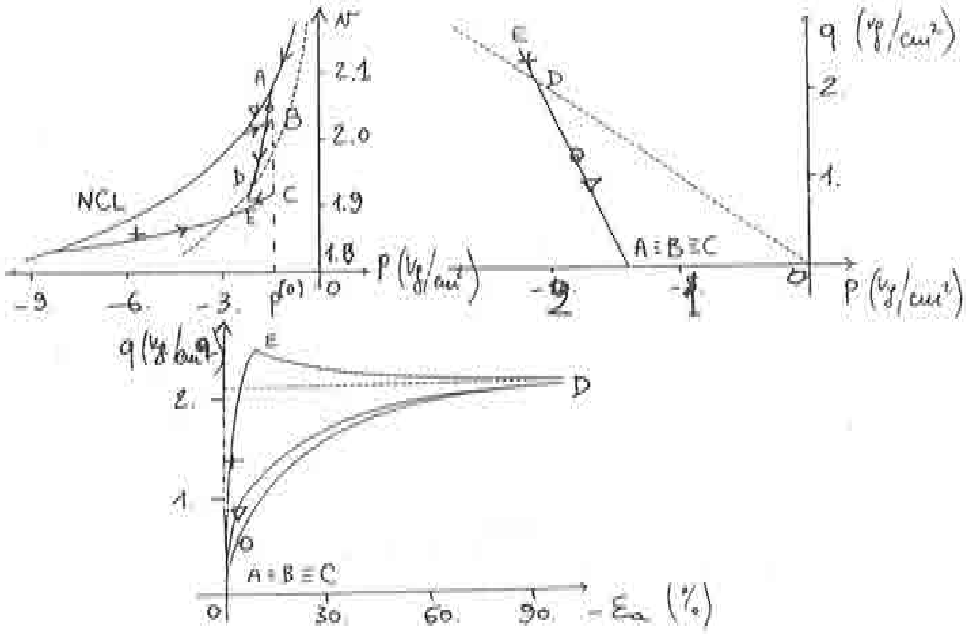


Figure 16.17: Triaxial drained test

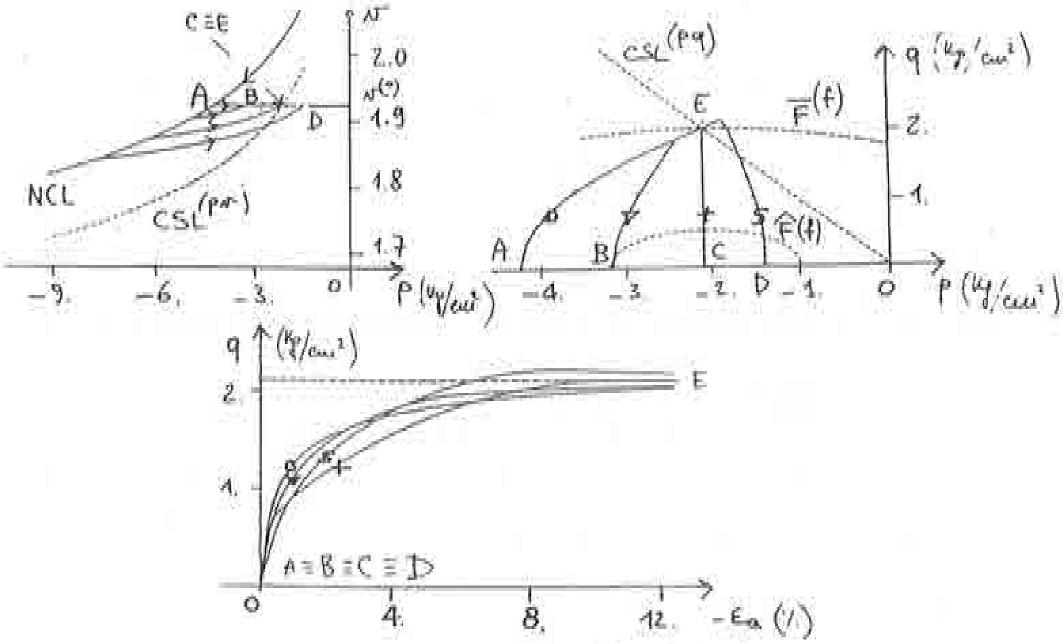


Figure 16.18: Triaxial undrained test

and $\hat{q}^{(t)}$ and $\overline{q}^{(t)}$ are two threshold values calculated as, [25],

$$\hat{q}^{(t)} = -\overline{M}\overline{p}_c^{(t)}\frac{\hat{p}_y^{(t)} - \overline{p}_c^{(t)}}{\overline{p}_y^{(t)} - \overline{p}_c^{(t)}} \quad (16.184)$$

$$\overline{q}^{(t)} = -\overline{M}\overline{p}_c^{(t)} \quad (16.185)$$

where

- $\overline{M}$ and $\overline{p}_c^{(t)}$ are given in Eq. 16.181;
- the values of the parameters $\overline{p}_y^{(t)}$ and $\hat{p}_y^{(t)}$ can be found as function of $(p = \overline{p}_c^{(t)}, v)$ from the solution of the implicit equations in Eqs. 16.48 and 16.54, respectively;
- finally, being v constant during an undrained test, its value is that one at the end of the consolidation stage.

In both numerical tests we observe that the stress path in the q vs. p plane eventually reaches a stable condition along a straight line parallel to the q axis, that is with $\delta p = 0$, Figs. 16.19b and 16.20b. At this stage there is no more permanent accumulation of excess pore pressure, Figs. 16.19c and 16.20c. However:

- in the case $q_M^{(1)} \leq \hat{q}^{(t)}$, the stability is reached at $p = p_s$ where the value p_s is the solution for p of the elastic surface equation, with $q = q_M^{(1)}$, namely
$$(p_s - \overline{p}_c)^2 + \frac{q_M^{(1)^2}}{\overline{M}^2}\left(\frac{\overline{p}_y - \overline{p}_c}{\overline{p}_c}\right)^2 - (\hat{p}_y - \overline{p}_c)^2 = 0$$
where $\overline{p}_y, \overline{p}_c$ and $\hat{p}_y$ are found as function of $(p = p_s, v)$ from the solution of the implicit equations in Eqs. 16.48, 16.50 and 16.54. From this stage, the material response is purely elastic, Fig. 16.19d.
- in the case $q_M^{(2)} > \hat{q}^{(t)}$, the stability condition is reached at $p = \overline{p}_c$. From this stage, the material response continues to be elasto-plastic with
$$\delta v^{(p)} = 0$$
$$\delta \epsilon_s^{(p)} > 0$$
and there is no bound for permanent deformations, Fig. 16.20d.

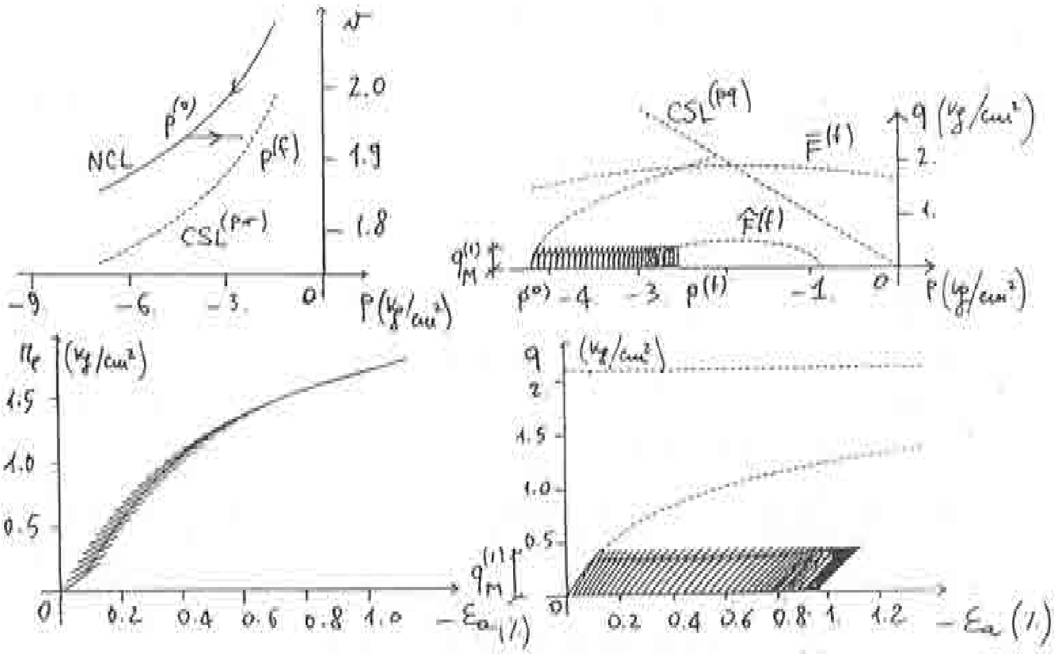


Figure 16.19: Triaxial cyclic undrained test ($q_M^{(1)} \leq \bar{q}^{(t)}$)

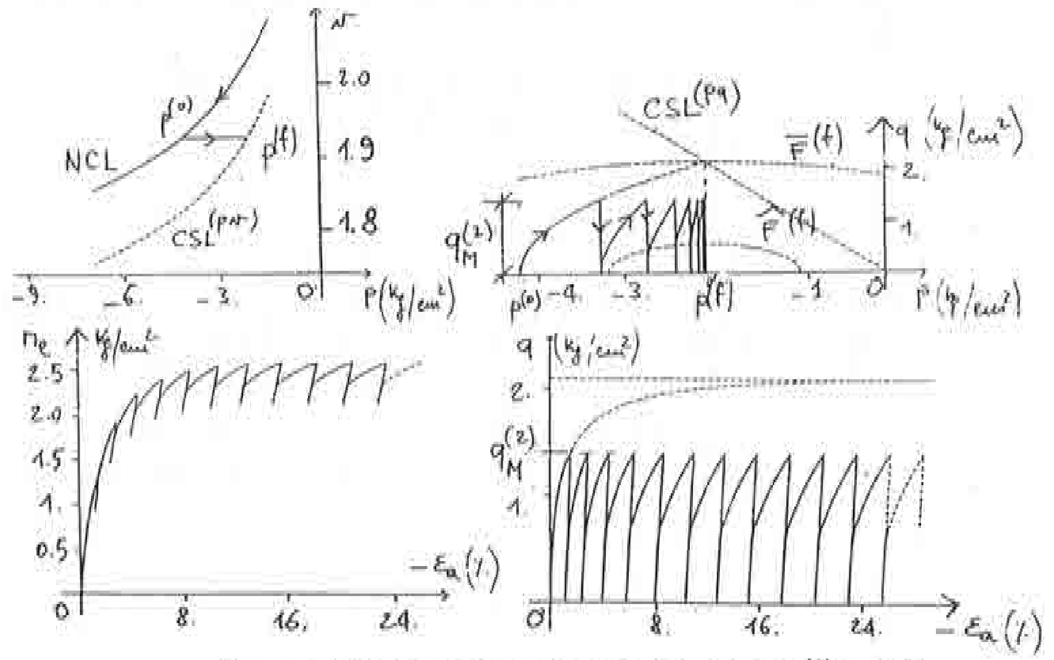


Figure 16.20: Triaxial cyclic undrained test ($q_M^{(2)} > \bar{q}^{(t)}$)

16.24 Undrained Shear Cyclic Test Simulation

In a shear test, the soil sample is subjected to a uniform shear strain field of the type shown in Fig. 16.21. Figures 16.22 and 16.23 show the predicted

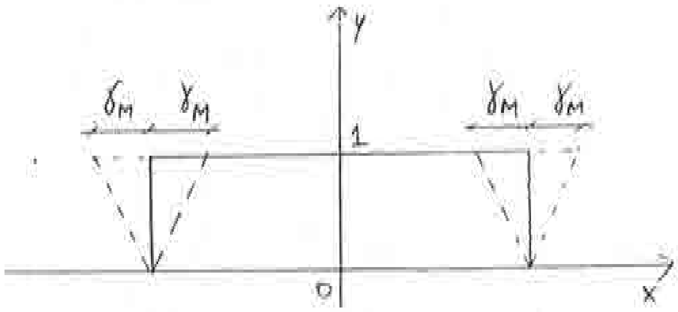


Figure 16.21: Imposed pure shear strain

undrained response of two samples of fully saturated soils subjected to cycles of imposed pure shear strain. Both samples have been previously isotropically compressed under drained conditions, up to the same confining pressure $p^{(o)}$, Figs. 16.22a and 16.23a. Then, they have been subjected to cycles of pure shear strain ranging between $\gamma = \left[-\gamma_M^{(1)}, \gamma_M^{(1)}\right]$ and $\gamma = \left[-\gamma_M^{(2)}, \gamma_M^{(2)}\right]$, respectively, where

$$\gamma_M^{(1)} = 0.1\ \%; \quad \gamma_M^{(2)} = 1.0\ \%$$

In both numerical tests we observe that, [25]:

- The stress path in the p vs. q plane eventually reaches a stable condition along a straight line parallel to the q-axis at the value of $p = \overline{p}_c^{(t)}$, Figs. 16.22b and 16.23b, where $\overline{p}_c^{(t)}$ can be calculated as reported Eq. 16.181.
- The θ invariant keeps changing its value.
- The pore pressure keeps its initial value, that is, there is no accumulation of excess pore pressure.
- In general, the dynamical shear modulus G_d, defined as the *slope* of the *hysteresis* loop in Figs. 16.22c and 16.23c, slightly decreases with the number of cycles.

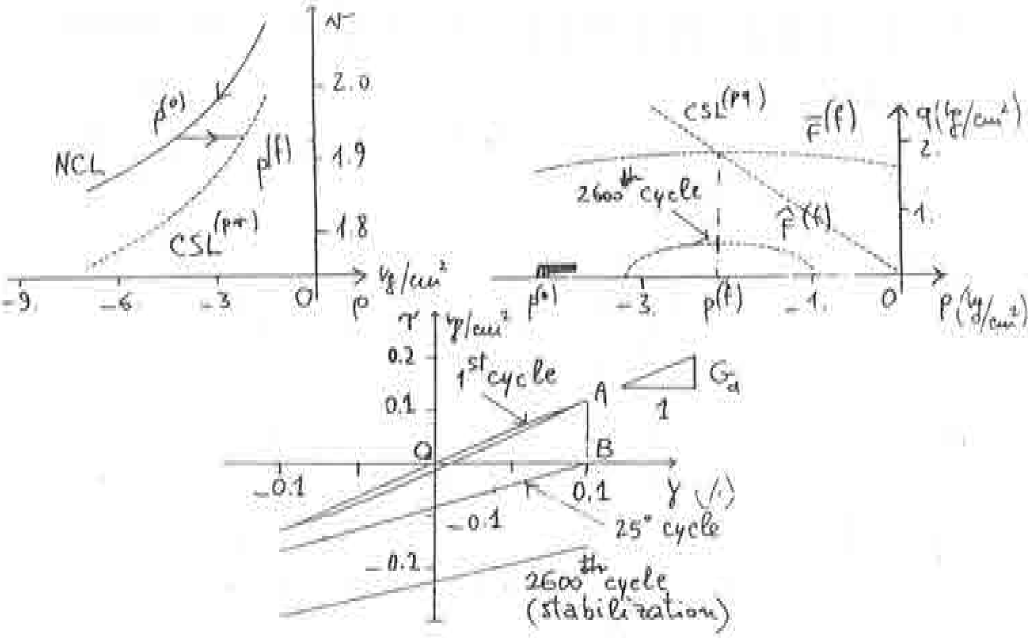


Figure 16.22: Shear cyclic undrained test ($\gamma_{max} = \gamma_M^{(1)}$)

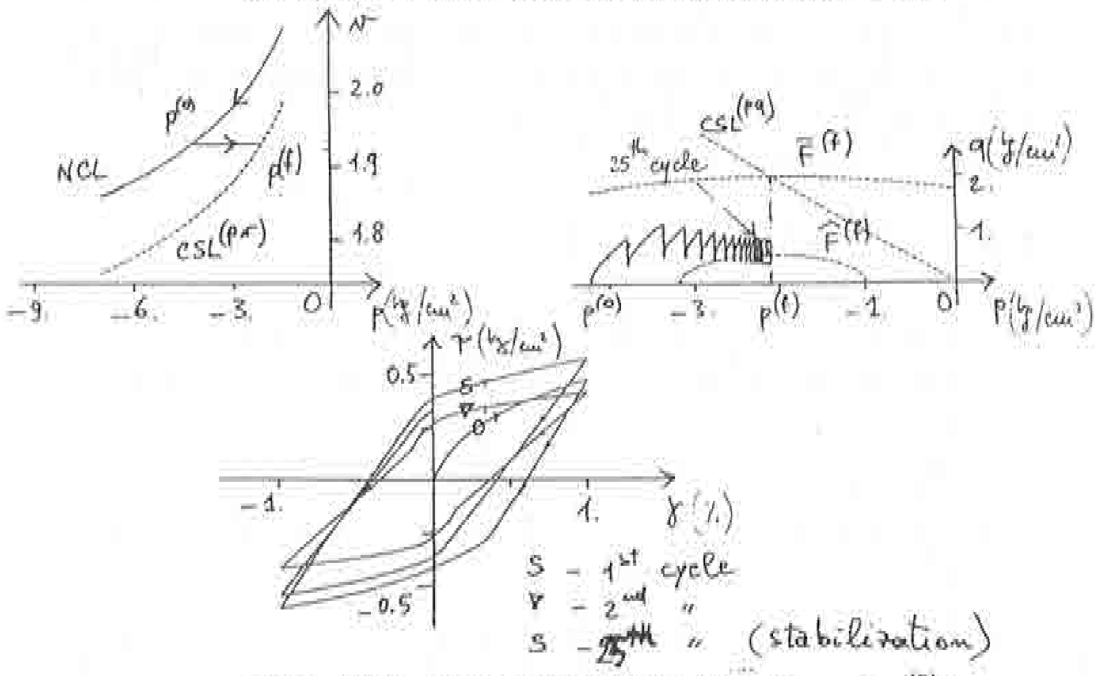


Figure 16.23: Shear cyclic undrained test ($\gamma_{max} = \gamma_M^{(2)}$)

- In general, the area of the hysteresis loop in Figs. 16.22c and 16.23c slightly decreases with the number of cycles.

Finally, Figs. 16.24a and 16.24b report the variation of the dynamic modulus G_d and of the *damping coefficient* λ, with the maximum amplitude of the imposed shear deformation, γ_M. The damping coefficient λ, according to [50], is defined as

$$\lambda = \frac{\text{hysteresis loop area}}{4\pi A_{OAB}}$$

and OAB represents the triangle shown in Fig. 16.22c.

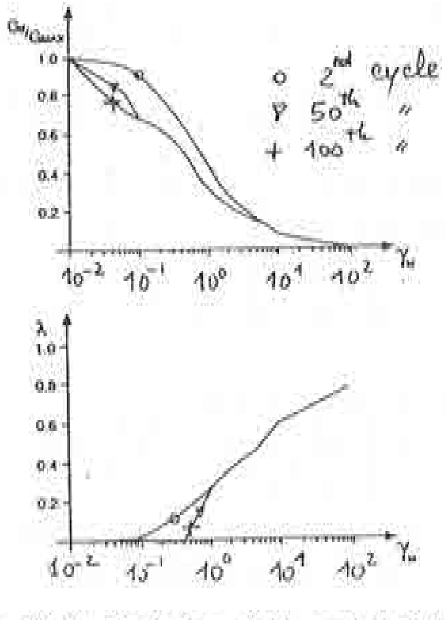


Figure 16.24: Variation of G_d and λ with γ_M.

16.25 Conclusions

The simulation of SUOLO Standard V1 is in good agreement with the available experimental data which, however, regard only monotonic loading conditions, Fig. 16.15.

With regard to cyclic loading, the soil behavior predicted by the model shows the following main characteristics:

- Soils subjected to isotropic loading/unloading/reloading drained tests present an unclosed hysteresis loop and the reloading stress path reaches the NCL for a finite value of p, Fig. 16.16.
- Strongly overconsolidated soils subjected to drained triaxial tests present a peak and then a *softening* in the q vs ϵ_a graph, Fig. 16.17.
- The behavior of a soil subjected to undrained cyclic triaxial tests depends on the maximum value of the imposed axial load, Figs. 16.19 and 16.20:
  - If the value of the applied q is greater than the threshold value $\bar{q}^{(t)}$ given in Eq. 16.184, the soil presents an unbounded progressive accumulation of irreversible (plastic) deformations, with no limit.
  - On the other hand, if the value of the applied q is smaller than $\bar{q}^{(t)}$, the soil presents a bound for irreversible (plastic) deformations.
- In the behavior of a soil subjected to undrained cyclic shear tests we observe, Fig. 16.24,
  - The value of the dynamic modulus G_d decreases with the maximum amplitude of the applied shear strain and with the number of cycles.
  - The value of the damping coefficient λ increases with the maximum amplitude of the applied shear strain while it slightly decreases with the number of cycles.

The above listed characteristics are in accordance with the experimental behavior of clay soils, see for example [50, 84].

Appendix A

The Gradient of a Tensor Function

A.1 Introduction

Let $G = G(\sigma)$ be a generic function of a vector

$$\sigma = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{21}, \sigma_{23}, \sigma_{31}, \sigma_{32}\}^T$$

which we will refer to as stress vector. The gradient vector and the Hessian matrix of G with respect to σ can be generally indicated as

$$\begin{aligned} \nabla_\sigma G &= \left\{ \frac{\partial G}{\partial \sigma_{ij}} \right\}^T \\ \mathbf{H} &= \left[\frac{\partial^2 G}{\partial \sigma_{ij} \partial \sigma_{mn}} \right] \end{aligned}$$

In this Appendix we present the particular expressions that $\nabla_\sigma G$ and $\mathbf{H}$ take on in the case that G can be expressed as $G = G(p, q, \theta)$ where (p, q, θ) are the invariants of the symmetric stress tensor

$$\mathbf{T} = [\sigma_{ij}]$$

We recall from Section 7.11 that

$$\begin{aligned} p &= \frac{J_1}{3} \\ q &= (3J_2)^{1/2} \\ \theta &= \frac{1}{3} \arcsin\left(-\frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} \right) \end{aligned}$$

where

$$\begin{aligned} I_1 &= \sigma_{kk} \\ J_2 &= \frac{1}{2}s_{ij}s_{ij} \\ J_3 &= \det[s_{ij}] \end{aligned}$$

and

$$s_{ij} = \sigma_{ij} - \delta_{ij}p$$

A.2 The Gradient of a Generic Stress Function

The gradient of a generic stress function

$$G(\sigma) = 0 \tag{A.1}$$

can be expressed as

$$\nabla_\sigma G = \nabla_p G + \nabla_s G \tag{A.2}$$

where

$$\nabla_\sigma G = \left\{\frac{\partial G}{\partial \sigma_{ij}}\right\}^T \tag{A.3}$$

$$\nabla_p G = \left\{\frac{\partial G}{\partial p}\frac{\partial p}{\partial \sigma_{ij}}\right\}^T = c_1 \widehat{\mathbf{m}} \tag{A.4}$$

$$\nabla_s G = \left\{\frac{\partial G}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial \sigma_{ij}}\right\}^T = \left\{\frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3}\frac{\partial G}{\partial s_{kk}}\right\}^T \tag{A.5}$$

and

$$c_1 = \frac{1}{3}\frac{\partial G}{\partial p} = \frac{\partial G}{\partial I_1} \tag{A.6}$$

The vector $\nabla_p G$ has clearly the direction of the spatial diagonal $\widehat{\mathbf{m}}$; it is possible to prove that the vector $\nabla_s G$ is orthogonal to $\nabla_p G$, i.e.

$$\nabla_p G^T \nabla_s G = 0 \tag{A.7}$$

which implies that $\nabla_s G$ lies on the octahedral plane. The modulus of $\nabla_\sigma G$ is given by

$$\|\nabla_\sigma G\| = (\nabla_\sigma G^T \nabla_\sigma G)^{1/2} = \left(\|\nabla_p G\|^2 + \|\nabla_s G\|^2\right)^{1/2} \tag{A.8}$$

where

$$\|\nabla_p G\| = (\nabla_p G^T \nabla_p G)^{1/2} = \frac{1}{\sqrt{3}} \frac{\partial G}{\partial p} \tag{A.9}$$

$$\|\nabla_s G\| = (\nabla_s G^T \nabla_s G)^{1/2} = \left(\frac{\partial G}{\partial s_{ij}} \frac{\partial G}{\partial s_{ij}} - \frac{1}{3} \left(\frac{\partial G}{\partial s_{kk}} \right)^2 \right)^{1/2} \tag{A.10}$$

In fact, since

$$\sigma_{kl} = s_{kl} + \delta_{kl} p$$

we have

$$\frac{\partial G}{\partial \sigma_{ij}} = \frac{\partial G}{\partial p} \frac{\partial p}{\partial \sigma_{ij}} + \frac{\partial G}{\partial s_{kl}} \frac{\partial s_{kl}}{\partial \sigma_{ij}}$$

and

$$\begin{aligned} \frac{\partial G}{\partial p} \frac{\partial p}{\partial \sigma_{ij}} &= \frac{\partial G}{\partial p} \frac{\partial}{\partial \sigma_{ij}} \left(\frac{\sigma_{mm}}{3} \right) = \frac{\partial G}{\partial p} \frac{\delta_{im} \delta_{jm}}{3} = \\ &= \frac{\delta_{ij}}{3} \frac{\partial G}{\partial p} \end{aligned}$$

while

$$\begin{aligned} \frac{\partial G}{\partial s_{kl}} \frac{\partial s_{kl}}{\partial \sigma_{ij}} &= \frac{\partial G}{\partial s_{kl}} \frac{\partial}{\partial \sigma_{ij}} \left(\sigma_{kl} - \delta_{kl} \frac{\sigma_{mm}}{3} \right) = \\ &= \frac{\partial G}{\partial s_{kl}} \left(\delta_{ik} \delta_{jl} - \delta_{kl} \frac{\delta_{im} \delta_{jm}}{3} \right) = \\ &= \frac{\partial G}{\partial s_{kl}} \left(\delta_{ik} \delta_{jl} - \delta_{kl} \frac{\delta_{ij}}{3} \right) = \\ &= \frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3} \frac{\partial G}{\partial s_{kk}} \end{aligned}$$

The orthogonality condition between $\nabla_p G$ and $\nabla_s G$ is verified being

$$\frac{\delta_{ij}}{3} \frac{\partial G}{\partial p} \left(\frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3} \frac{\partial G}{\partial s_{kk}} \right) = \frac{1}{3} \frac{\partial G}{\partial p} \left(\frac{\partial G}{\partial s_{kk}} - \frac{\delta_{ij} \delta_{ij}}{3} \frac{\partial G}{\partial s_{kk}} \right) = 0$$

From this orthogonality it follows that

$$\begin{aligned} \nabla_\sigma G^T \nabla_\sigma G &= (\nabla_p G + \nabla_s G)^T (\nabla_p G + \nabla_s G) = \\ &= \nabla_p G^T \nabla_p G + 2 \nabla_p G^T \nabla_s G + \nabla_s G^T \nabla_s G = \\ &= \|\nabla_p G\|^2 + \|\nabla_s G\|^2 \end{aligned}$$

which proves the expression of $\|\nabla_\sigma G\|$. Finally, the expressions of $\|\nabla_p G\|$ and of $\|\nabla_s G\|$ are verified being

$$\left(\frac{\delta_{ij}}{3}\frac{\partial G}{\partial p}\right)\left(\frac{\delta_{ij}}{3}\frac{\partial G}{\partial p}\right) = \frac{\delta_{ij}\delta_{ij}}{9}\frac{\partial G}{\partial p} = \frac{1}{3}\frac{\partial G}{\partial p}$$

and

$$\begin{aligned}
&\left(\frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3}\frac{\partial G}{\partial s_{kk}}\right)\left(\frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3}\frac{\partial G}{\partial s_{kk}}\right) = \\
&= \frac{\partial G}{\partial s_{ij}}\frac{\partial G}{\partial s_{ij}} - 2\frac{\delta_{ij}}{3}\frac{\partial G}{\partial s_{ij}}\frac{\partial G}{\partial s_{kk}} + \frac{\delta_{ij}\delta_{ij}}{9}\frac{\partial G}{\partial s_{kk}}\frac{\partial G}{\partial s_{kk}} = \\
&= \frac{\partial G}{\partial s_{ij}}\frac{\partial G}{\partial s_{ij}} - \frac{2}{3}\left(\frac{\partial G}{\partial s_{kk}}\right)^2 + \frac{1}{3}\left(\frac{\partial G}{\partial s_{kk}}\right)^2 = \\
&= \frac{\partial G}{\partial s_{ij}}\frac{\partial G}{\partial s_{ij}} - \frac{1}{3}\left(\frac{\partial G}{\partial s_{kk}}\right)^2
\end{aligned}$$

A.3 The Gradient of a Stress Invariant Function

The gradient of a stress invariant function

$$G(p, q, \theta) = 0 \tag{A.11}$$

can be expressed as

$$\nabla_\sigma G = \nabla_p G + \nabla_s G \tag{A.12}$$

where

$$\nabla_\sigma G = \left\{\frac{\partial G}{\partial \sigma_{ij}}\right\}^T \tag{A.13}$$

$$\nabla_p G = \left\{\frac{\partial G}{\partial p}\frac{\partial p}{\partial \sigma_{ij}}\right\}^T = c_1\widehat{\mathbf{m}} \tag{A.14}$$

$$\nabla_s G = \left\{\frac{\partial G}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial \sigma_{ij}}\right\}^T = c_2\mathbf{s} + c_3\mathbf{a} \tag{A.15}$$

and

$$c_1 = \frac{1}{3}\frac{\partial G}{\partial p} = \frac{\partial G}{\partial I_1}$$

$$
\begin{aligned}
c_2 &= \frac{3}{2q}\left(\frac{\partial G}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial G}{\partial \theta}\right) = \frac{1}{2J_2^{1/2}}\left(\frac{\partial G}{\partial J_2^{1/2}} - \frac{\tan 3\theta}{J_2^{1/2}}\frac{\partial G}{\partial \theta}\right) \\
c_3 &= -\frac{9}{2q^3\cos 3\theta}\frac{\partial G}{\partial \theta} = -\frac{\sqrt{3}}{2J_2^{3/2}\cos 3\theta}\frac{\partial G}{\partial \theta}
\end{aligned}
$$

The elements of **a** are given by

$$
a_{ij} = b_{ij} - \frac{\delta_{ij}}{3} b_{kk} \tag{A.16}
$$

where

$$
[b_{ij}] = \left[\frac{\partial J_3}{\partial s_{ij}}\right] = \begin{bmatrix} (s_{22}s_{33} - s_{23}^2) & (s_{13}s_{23} - s_{33}s_{12}) & (s_{12}s_{23} - s_{22}s_{13}) \\ & (s_{11}s_{33} - s_{13}^2) & (s_{12}s_{13} - s_{11}s_{23}) \\ \text{Symmetric} & & (s_{11}s_{22} - s_{12}^2) \end{bmatrix}
$$

$$
b_{kk} = \frac{\partial J_3}{\partial s_{kk}} = -J_2
$$

The modulus of $\nabla_\sigma G$ is given by

$$
\|\nabla_\sigma G\| = (\nabla_\sigma G^T \nabla_\sigma G)^{1/2} = \left(\|\nabla_p G\|^2 + \|\nabla_s G\|^2\right)^{1/2} \tag{A.17}
$$

where

$$
\|\nabla_p G\| = (\nabla_p G^T \nabla_p G)^{1/2} = \frac{1}{\sqrt{3}}\frac{\partial G}{\partial p} \tag{A.18}
$$

$$
\|\nabla_s G\| = (\nabla_s G^T \nabla_s G)^{1/2} = \sqrt{\frac{3}{2}}\left[\left(\frac{\partial G}{\partial q}\right)^2 + \frac{1}{q^2}\left(\frac{\partial G}{\partial \theta}\right)^2\right]^{1/2} \tag{A.19}
$$

In fact, in the previous Section we have proved that in general for $G = G(\boldsymbol{\sigma})$

$$
\nabla_\sigma G = \nabla_p G + \nabla_s G
$$

where

$$
\begin{aligned}
\nabla_p G &= \left\{\frac{\partial G}{\partial p}\frac{\partial p}{\partial \sigma_{ij}}\right\}^T = c_1 \widehat{\mathbf{m}} \\
\nabla_s G &= \left\{\frac{\partial G}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial \sigma_{ij}}\right\}^T = \left\{\frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3}\frac{\partial G}{\partial s_{kk}}\right\}^T
\end{aligned}
$$

We now prove that for $G = G(p, q, \theta)$ we have

$$\frac{\partial G}{\partial s_{ij}} = c_2\mathbf{s} + c_3\mathbf{b} \tag{A.20}$$

$$\nabla_s G = \left\{\frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3}\frac{\partial G}{\partial s_{kk}}\right\}^T = c_2\mathbf{s} + c_3\mathbf{a} \tag{A.21}$$

The proof of this relationship is based on the following main mathematical steps:

a) Since p and s_{ij} are independent variables

$$\frac{\partial G}{\partial s_{ij}} = \frac{\partial G}{\partial q}\frac{\partial q}{\partial s_{ij}} + \frac{\partial G}{\partial \theta}\frac{\partial \theta}{\partial s_{ij}} \tag{A.22}$$

b) The q-invariant is by definition

$$q = \sqrt{3}J_2^{1/2} = \left(\frac{3}{2}s_{kl}s_{kl}\right)^{1/2} \tag{A.23}$$

from which it follows

$$\frac{\partial q}{\partial s_{ij}} = \frac{3}{2q}s_{ij} \tag{A.24}$$

c) The θ invariant is by definition

$$\sin 3\theta = -\frac{27}{2}\frac{J_3}{q^3} \tag{A.25}$$

from which it follows

$$\frac{\partial \theta}{\partial s_{ij}} = -\frac{3}{2q^2}\left(\tan 3\theta s_{ij} + \frac{3}{q\cos 3\theta}b_{ij}\right) \tag{A.26}$$

d) The above relationships allow to prove first the relationship in Eq. A.20 and then that in Eq. A.21.

In fact,

$$\begin{aligned}\frac{\partial q}{\partial s_{ij}} &= \frac{\partial}{\partial s_{ij}}\left(\frac{3}{2}s_{kl}s_{kl}\right)^{1/2} = \frac{1}{2}\sqrt{\frac{3}{2}}\left(s_{kl}s_{kl}\right)^{-1/2}\frac{\partial}{\partial s_{ij}}\left(s_{kl}s_{kl}\right) = \\ &= \frac{3}{4q}2\delta_{ik}\delta_{jl}s_{kl} = \frac{3}{2q}s_{ij}\end{aligned}$$

Since

$$\frac{\partial}{\partial s_{ij}}(\sin 3\theta) = 3\cos 3\theta \frac{\partial \theta}{\partial s_{ij}}$$

we have that

$$\begin{aligned}
\frac{\partial \theta}{\partial s_{ij}} &= \frac{1}{3\cos 3\theta}\frac{\partial}{\partial s_{ij}}(\sin 3\theta) = \\
&= \frac{1}{3\cos 3\theta}\frac{\partial}{\partial s_{ij}}\left(-\frac{27}{2}\frac{J_3}{q^3}\right) = \\
&= -\frac{9}{2\cos 3\theta}\left(J_3\frac{\partial q^{-3}}{\partial s_{ij}} + \frac{1}{q^3}\frac{\partial J_3}{\partial s_{ij}}\right) = \\
&= -\frac{9}{2\cos 3\theta}\left(-\frac{3J_3}{q^4}\frac{\partial q}{\partial s_{ij}} + \frac{1}{q^3}b_{ij}\right) = \\
&= \frac{81}{4\cos 3\theta}\frac{J_3}{q^5}s_{ij} - \frac{9}{2q^3\cos 3\theta}b_{ij} = \\
&= -\frac{3}{2q^2}\left(\tan 3\theta s_{ij} + \frac{3}{q\cos 3\theta}b_{ij}\right)
\end{aligned}$$

where

$$[b_{ij}] = \left[\frac{\partial J_3}{\partial s_{ij}}\right]$$

The J_3 invariant is by definition

$$J_3 = s_{11}s_{22}s_{33} + s_{12}s_{23}s_{31} + s_{13}s_{21}s_{32} - (s_{31}s_{22}s_{13} + s_{32}s_{23}s_{11} + s_{33}s_{21}s_{12})$$

consequently

$$\begin{aligned}
b_{11} &= \frac{\partial J_3}{\partial s_{11}} = s_{22}s_{33} - s_{23}^2 \\
b_{12} &= \frac{\partial J_3}{\partial s_{12}} = -s_{33}s_{12} + s_{13}s_{23} \\
\ldots &= \ldots
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial J_3}{\partial s_{kk}} &= \frac{\partial J_3}{\partial s_{11}} + \frac{\partial J_3}{\partial s_{22}} + \frac{\partial J_3}{\partial s_{33}} = \\
&= \left(s_{22}s_{33} - s_{23}^2\right) + \left(s_{11}s_{33} - s_{13}^2\right) + \left(s_{11}s_{22} - s_{12}^2\right) =
\end{aligned}$$

$$\begin{aligned}
&= \left(s_{22}s_{33} + s_{11}s_{33} + s_{11}s_{22}\right) - \left(s_{12}^2 + s_{13}^2 + s_{23}^2\right) = \\
&= -\frac{1}{2}\left(s_{11}^2 + s_{22}^2 + s_{33}^2\right) - \left(s_{12}^2 + s_{13}^2 + s_{23}^2\right) = \\
&= -\frac{1}{2}s_{ij}s_{ij} = -J_2
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\partial G}{\partial s_{ij}} &= \frac{\partial G}{\partial q}\frac{\partial q}{\partial s_{ij}} + \frac{\partial G}{\partial \theta}\frac{\partial \theta}{\partial s_{ij}} = \\
&= \frac{\partial G}{\partial q}\frac{3}{2q}s_{ij} - \frac{\partial G}{\partial \theta}\frac{3}{2q^2}\left(\tan 3\theta s_{ij} + \frac{3}{q\cos 3\theta}b_{ij}\right) \\
&= \frac{3}{2q}\left(\frac{\partial G}{\partial q} - \frac{\tan 3\theta}{q^2}\frac{\partial G}{\partial \theta}\right)s_{ij} - \frac{9}{2q^3\cos 3\theta}\frac{\partial G}{\partial \theta}b_{ij} \\
&= c_2 s_{ij} + c_3 b_{ij}
\end{aligned}$$

and consequently

$$\begin{aligned}
\frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3}\frac{\partial G}{\partial s_{kk}} &= \left(c_2 s_{ij} + c_3 b_{ij}\right) - \frac{\delta_{ij}}{3}\left(c_2 s_{kk} + c_3 b_{kk}\right) \\
&= c_2 s_{ij} + c_3\left(b_{ij} - \frac{\delta_{ij}}{3}b_{kk}\right) \\
&= c_2 s_{ij} + c_3 a_{ij}
\end{aligned}$$

Finally, in order to derive the expression of $\|\nabla_s G\|$ it is necessary to remark that

$$\begin{aligned}
b_{ij}b_{ij} &= b_{kk}^2 = J_2^2 = \frac{1}{9}q^4 \\
s_{ij}b_{ij} &= 3J_3 = -\frac{2}{9}q^3\sin 3\theta
\end{aligned}$$

and consequently

$$\begin{aligned}
a_{ij}a_{ij} &= \left(b_{ij} - \frac{\delta_{ij}}{3}b_{kk}\right)\left(b_{ij} - \frac{\delta_{ij}}{3}b_{kk}\right) = \\
&= \frac{2}{3}b_{kk}^2 = \frac{2}{27}q^4 \\
s_{ij}a_{ij} &= s_{ij}\left(b_{ij} - \frac{\delta_{ij}}{3}b_{kk}\right) = \\
&= s_{ij}b_{ij} = -\frac{2}{9}q^3\sin 3\theta
\end{aligned}$$

It follows that

$$\begin{aligned} \nabla_s G^T \nabla_s G &= (c_2 s_{ij} + c_3 a_{ij})(c_2 s_{ij} + c_3 a_{ij}) = \\ &= c_2^2 s_{ij} s_{ij} + 2 c_2 c_3 s_{ij} a_{ij} + c_3^2 a_{ij} a_{ij} = \\ &= \frac{2}{3} q^2 c_2^2 - \frac{4}{9} q^3 \sin 3\theta c_2 c_3 + \frac{2}{27} q^4 c_3^2 \end{aligned}$$

and substituting the coefficient values we eventually obtain the expression of $||\nabla_s G||$.

A.4 Deviatoric Strain and Strain Invariants

If the strains are defined as

$$\epsilon = \lambda \nabla_\sigma G = \lambda \frac{\partial G}{\partial \sigma} \tag{A.27}$$

the deviatoric strains and the strain invariants result to be defined as

$$\mathbf{e} = \epsilon - \widehat{\mathbf{m}} \frac{\epsilon_v}{3} = \lambda \nabla_s G \tag{A.28}$$

$$\epsilon_v = \epsilon_{kk} = \lambda \widehat{\mathbf{m}}^T \nabla_p G = \lambda \frac{\partial G}{\partial p} \tag{A.29}$$

$$\epsilon_s = \left(\frac{2}{3} \mathbf{e}^T \mathbf{e}\right)^{1/2} = \lambda \sqrt{\frac{2}{3}} ||\nabla_s G|| \tag{A.30}$$

If $G = G(p, q, \theta)$ then

$$\epsilon = \lambda \nabla_\sigma G = \lambda (c_1 \widehat{\mathbf{m}} + c_2 \mathbf{s} + c_3 \mathbf{a}) \tag{A.31}$$

$$\mathbf{e} = \lambda \nabla_s G = \lambda (c_2 \mathbf{s} + c_3 \mathbf{a}) \tag{A.32}$$

$$\epsilon_s = \lambda \left[\left(\frac{\partial G}{\partial q}\right)^2 + \frac{1}{q} \left(\frac{\partial G}{\partial \theta}\right)^2 \right]^{1/2} \tag{A.33}$$

and if $G = G(p, q)$

$$\mathbf{e} = \lambda \frac{\partial G}{\partial \mathbf{s}} = \lambda c_2 \mathbf{s} \tag{A.34}$$

$$\epsilon_s = \lambda \frac{\partial G}{\partial q} \tag{A.35}$$

In fact

$$
\begin{aligned}
\epsilon_v &= \epsilon_{kk} = \widehat{\mathbf{m}}^T \boldsymbol{\epsilon} = \lambda \widehat{\mathbf{m}}^T \boldsymbol{\nabla}_\sigma G = \lambda \widehat{\mathbf{m}}^T \left(\boldsymbol{\nabla}_p G + \boldsymbol{\nabla}_s G \right) = \\
&= \lambda \widehat{\mathbf{m}}^T \boldsymbol{\nabla}_p G = \lambda \widehat{\mathbf{m}}^T \widehat{\mathbf{m}} c_1 = \lambda \frac{\partial G}{\partial p} \\
\mathbf{e} &= \boldsymbol{\epsilon} - \widehat{\mathbf{m}} \frac{\epsilon_v}{3} = \lambda \boldsymbol{\nabla}_\sigma G - \lambda \widehat{\mathbf{m}} \frac{1}{3} \frac{\partial G}{\partial p} = \\
&= \lambda \left(\boldsymbol{\nabla}_p G + \boldsymbol{\nabla}_s G - \boldsymbol{\nabla}_p G \right) = \lambda \boldsymbol{\nabla}_s G \\
\epsilon_s &= \left(\frac{2}{3} \mathbf{e}^T \mathbf{e} \right)^{1/2} = \lambda \left(\frac{2}{3} \boldsymbol{\nabla}_s G^T \boldsymbol{\nabla}_s G \right)^{1/2} = \\
&= \lambda \sqrt{\frac{2}{3}} \| \boldsymbol{\nabla}_s G \|
\end{aligned}
$$

We recall that in case $G = G(p, q, \theta)$ we have that

$$
\begin{aligned}
\boldsymbol{\nabla}_s G &= c_2 \mathbf{s} + c_3 \mathbf{a} \\
\| \boldsymbol{\nabla}_s G \| &= \sqrt{\frac{3}{2}} \left[\left(\frac{\partial G}{\partial q} \right)^2 + \frac{1}{q^2} \left(\frac{\partial G}{\partial \theta} \right)^2 \right]^{1/2}
\end{aligned}
$$

Finally, in case $G = G(p, q)$, i.e. θ-independent, we have that $c_3 = 0$ and consequently

$$
\begin{aligned}
\frac{\partial G}{\partial s_{kk}} &= c_3 \frac{\partial J_3}{\partial s_{kk}} = 0 \\
\boldsymbol{\nabla}_s G &= \left\{ \frac{\partial G}{\partial s_{ij}} - \frac{\delta_{ij}}{3} \frac{\partial G}{\partial s_{kk}} \right\} = \left\{ \frac{\partial G}{\partial s_{ij}} \right\}
\end{aligned}
$$

A.5 The Hessian Matrix

The Hessian matrix associated to stress invariant function, is defined as

$$
\mathbf{H} = \left[\frac{\partial^2 G}{\partial \sigma_{ij} \partial \sigma_{mn}} \right] \tag{A.36}
$$

If $G(p, q, \theta) = 0$ then the Hessian matrix can be expressed as

$$
\begin{aligned}
\mathbf{H} &= c_{11} \mathbf{H}^{(11)} + c_{22} \mathbf{H}^{(22)} + c_{33} \mathbf{H}^{(33)} + \\
&+ c_{21} \mathbf{H}^{(21)} + c_{31} \mathbf{H}^{(31)} + c_{23} \mathbf{H}^{(32)} + \\
&+ c_2 \widetilde{\mathbf{H}}^{(11)} + c_3 \widetilde{\mathbf{H}}^{(22)}
\end{aligned} \tag{A.37}
$$

where

$$
\begin{aligned}
c_{11} &= \frac{1}{3}\frac{\partial c_1}{\partial p} = \frac{1}{9}\frac{\partial^2 G}{\partial p^2} \\
c_{22} &= \frac{3}{2q}\left(\frac{\partial c_2}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial c_2}{\partial \theta}\right) = \frac{9}{4q^2}\left\{\left(\frac{\partial^2 G}{\partial q^2} - \frac{1}{q}\frac{\partial G}{\partial q}\right) + \right. \\
&\left. + \frac{\tan 3\theta}{q}\left[\frac{1}{q}\left(2 + \frac{3}{\cos^2 3\theta}\right)\frac{\partial G}{\partial \theta} - 2\frac{\partial^2 G}{\partial q \partial \theta} + \frac{\tan 3\theta}{q}\frac{\partial^2 G}{\partial \theta^2}\right]\right\} \\
c_{33} &= -\frac{9}{2q^3 \cos 3\theta}\frac{\partial c_3}{\partial \theta} = \frac{81}{4q^6 \cos^2 3\theta}\left(3\tan 3\theta \frac{\partial G}{\partial \theta} + \frac{\partial^2 G}{\partial \theta^2}\right) \\
c_{21} &= \frac{1}{3}\frac{\partial c_2}{\partial p} = \frac{1}{2q}\left(\frac{\partial^2 G}{\partial p \partial q} - \frac{\tan 3\theta}{q}\frac{\partial^2 G}{\partial p \partial \theta}\right) \\
c_{31} &= \frac{1}{3}\frac{\partial c_3}{\partial p} = -\frac{3}{2q^3 \cos 3\theta}\frac{\partial^2 G}{\partial p \partial \theta} \\
c_{23} &= -\frac{9}{2q^3 \cos 3\theta}\frac{\partial c_2}{\partial \theta} = \\
&= -\frac{27}{4q^4 \cos 3\theta}\left[\frac{\partial^2 G}{\partial q \partial \theta} - \frac{1}{q}\left(\frac{3}{\cos^2 3\theta}\frac{\partial G}{\partial \theta} + \tan 3\theta \frac{\partial^2 G}{\partial \theta^2}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{H}^{(11)} &= [\delta_{ij}\delta_{mn}] \\
\mathbf{H}^{(22)} &= [s_{ij}s_{mn}] \\
\mathbf{H}^{(33)} &= [a_{ij}a_{mn}] \\
\mathbf{H}^{(21)} &= [s_{ij}\delta_{mn} + s_{mn}\delta_{ij}] \\
\mathbf{H}^{(31)} &= [a_{ij}\delta_{mn} + a_{mn}\delta_{ij}] \\
\mathbf{H}^{(23)} &= [s_{ij}a_{mn} + s_{mn}a_{ij}] \\
\tilde{\mathbf{H}}^{(11)} &= \left[\delta_{im}\delta_{jn} - \frac{\delta_{ij}\delta_{mn}}{3}\right] \\
\tilde{\mathbf{H}}^{(22)} &= \left[\frac{\partial^2 J_3}{\partial s_{ij}\partial s_{mn}} + \frac{1}{3}\left(s_{ij}\delta_{mn} + s_{mn}\delta_{ij}\right)\right]
\end{aligned}
$$

In fact, we have already proved that

$$
\frac{\partial p}{\partial \sigma_{mn}} = \frac{\delta_{mn}}{3}
$$

$$
\begin{aligned}
\frac{\partial s_{kl}}{\partial \sigma_{mn}} &= \left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) \\
\frac{\partial q}{\partial s_{mn}} &= \frac{3}{2q}s_{mn} \\
\frac{\partial J_3}{\partial s_{kk}} &= -J_2 \\
\frac{\partial \theta}{\partial s_{mn}} &= -\left(\frac{\tan 3\theta}{q^2}s_{mn} + \frac{9}{2q^3\cos 3\theta}b_{mn}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial J_2}{\partial s_{ij}} &= \frac{\partial}{\partial s_{ij}}\left(\frac{1}{2}s_{mn}s_{mn}\right) = \delta_{im}\delta_{jn}s_{mn} = \\
&= s_{ij}
\end{aligned}
$$

It follows

$$
\begin{aligned}
\frac{\partial a_{ij}}{\partial s_{mn}} &= \frac{\partial}{\partial s_{mn}}\left(b_{ij} - \frac{\delta_{ij}}{3}b_{kk}\right) = \frac{\partial}{\partial s_{mn}}\left(\frac{\partial J_3}{\partial s_{ij}} + \frac{\delta_{ij}}{3}J_2\right) \\
&= \frac{\partial^2 J_3}{\partial s_{mn}\partial s_{ij}} + \frac{\delta_{ij}}{3}s_{mn} \\
\frac{\partial a_{ij}}{\partial s_{kk}} &= \frac{\partial}{\partial s_{kk}}\left(\frac{\partial J_3}{\partial s_{ij}}\right) + \frac{\delta_{ij}}{3}s_{kk} = -\frac{\partial J_2}{\partial s_{ij}} = \\
&= -s_{ij}
\end{aligned}
$$

The gradient of an isotropic stress function is given by

$$
\frac{\partial G}{\partial \sigma_{ij}} = c_1\delta_{ij} + c_2 s_{ij} + c_3 a_{ij} = w_{ij}
$$

and consequently its derivative may be calculated as

$$
\begin{aligned}
\frac{\partial^2 G}{\partial \sigma_{ij}\partial \sigma_{mn}} &= \frac{\partial w_{ij}}{\partial \sigma_{mn}} = \frac{\partial w_{ij}}{\partial p}\frac{\partial p}{\partial \sigma_{mn}} + \frac{\partial w_{ij}}{\partial s_{kl}}\frac{\partial s_{kl}}{\partial \sigma_{mn}} = \\
&= \frac{\partial}{\partial p}\left(c_1\delta_{ij} + c_2 s_{ij} + c_3 a_{ij}\right)\frac{\partial p}{\partial \sigma_{mn}} + \\
&\quad + \frac{\partial}{\partial s_{kl}}\left(c_1\delta_{ij} + c_2 s_{ij} + c_3 a_{ij}\right)\frac{\partial s_{kl}}{\partial \sigma_{mn}} = \\
&= \left[\left(\frac{\partial c_1}{\partial p}\delta_{ij} + \frac{\partial c_2}{\partial p}s_{ij} + \frac{\partial c_3}{\partial p}a_{ij}\right) + \right.
\end{aligned}
$$

$$
\begin{aligned}
& + \left(c_1 \frac{\partial \delta_{ij}}{\partial p} + c_2 \frac{\partial s_{ij}}{\partial p} + c_3 \frac{\partial a_{ij}}{\partial p}\right)\Big] \frac{\partial p}{\partial \sigma_{mn}} + \\
& + \left[\left(\frac{\partial c_1}{\partial s_{kl}} \delta_{ij} + \frac{\partial c_2}{\partial s_{kl}} s_{ij} + \frac{\partial c_3}{\partial s_{kl}} a_{ij}\right) + \right. \\
& + \left(c_1 \frac{\partial \delta_{ij}}{\partial s_{kl}} + c_2 \frac{\partial s_{ij}}{\partial s_{kl}} + c_3 \frac{\partial a_{ij}}{\partial s_{kl}}\right)\Big] \frac{\partial s_{kl}}{\partial \sigma_{mn}} = \\
= \; & \left[\frac{\partial c_1}{\partial p} \delta_{ij} + \frac{\partial c_2}{\partial p} s_{ij} + \frac{\partial c_3}{\partial p} a_{ij}\right] \frac{\delta_{mn}}{3} + \\
& + \left[\frac{\partial c_1}{\partial s_{kl}} \delta_{ij} + \frac{\partial c_2}{\partial s_{kl}} s_{ij} + \frac{\partial c_3}{\partial s_{kl}} a_{ij}\right] \left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) + \\
& + \left[c_2 \delta_{ik}\delta_{jl} + c_3 \frac{\partial a_{ij}}{\partial s_{kl}}\right] \left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) \\
= \; & \frac{1}{3}\frac{\partial c_1}{\partial p} [\delta_{ij}\delta_{mn}] + \frac{1}{3}\frac{\partial c_2}{\partial p} [s_{ij}\delta_{mn}] + \frac{1}{3}\frac{\partial c_3}{\partial p} [a_{ij}\delta_{mn}] + \\
& + \frac{\partial c_1}{\partial s_{kl}} \left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) \delta_{ij} + \frac{\partial c_2}{\partial s_{kl}} \left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) s_{ij} + \\
& + \frac{\partial c_3}{\partial s_{kl}} \left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) a_{ij} + c_2 \delta_{ik}\delta_{jl} \left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) + \\
& + c_3 \frac{\partial a_{ij}}{\partial s_{kl}} \left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right)
\end{aligned}
$$

Let us indicate

$$
\begin{aligned}
c_{a1} &= \frac{1}{3}\frac{\partial c_a}{\partial p} \\
c_{a2} &= \frac{3}{2q}\left(\frac{\partial c_a}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial c_a}{\partial \theta}\right) \\
c_{a3} &= -\frac{9}{2q^3 \cos 3\theta}\frac{\partial c_a}{\partial \theta}
\end{aligned}
$$

where $a = 1, 2, 3$. In general the derivative of the coefficients c_a with respect to s_{mn} can be calculated as

$$
\begin{aligned}
\frac{\partial c_a}{\partial s_{mn}} &= \frac{\partial c_a}{\partial q}\frac{\partial q}{\partial s_{mn}} + \frac{\partial c_a}{\partial \theta}\frac{\partial \theta}{\partial s_{mn}} = \\
&= \frac{\partial c_a}{\partial q}\frac{3}{2q} s_{mn} - \frac{\partial c_a}{\partial \theta}\frac{3}{2q^2}\left(\tan 3\theta s_{mn} + \frac{3}{q \cos 3\theta} b_{mn}\right) = \\
&= \frac{3}{2q}\left(\frac{\partial c_a}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial c_a}{\partial \theta}\right) s_{mn} - \frac{9}{2q^3 \cos 3\theta}\frac{\partial c_a}{\partial \theta} b_{mn} = \\
&= c_{a2} s_{mn} + c_{a3} b_{mn}
\end{aligned}
$$

In particular

$$\frac{\partial c_a}{\partial s_{kk}} = c_{a2}s_{kk} + c_{a3}b_{kk} = c_{a3}b_{kk}$$

and

$$\begin{aligned}
\frac{\partial c_a}{\partial s_{kl}}\left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) &= \frac{\partial c_a}{\partial s_{mn}} - \frac{\delta_{mn}}{3}\frac{\partial c_a}{\partial s_{kk}} = \\
&= (c_{a2}s_{mn} + c_{a3}b_{mn}) - \frac{\delta_{mn}}{3}c_{a3}b_{kk} = \\
&= c_{a2}s_{mn} + c_{a3}\left(b_{mn} - \frac{\delta_{mn}}{3}b_{kk}\right) = \\
&= c_{a2}s_{mn} + c_{a3}a_{mn}
\end{aligned}$$

We also note that

$$\begin{aligned}
\delta_{ik}\delta_{jl}\left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) &= \left[\delta_{im}\delta_{jn} - \frac{\delta_{ij}\delta_{mn}}{3}\right] = \tilde{\mathbf{H}}^{(11)} \\
\frac{\partial a_{ij}}{\partial s_{kl}}\left(\delta_{mk}\delta_{nl} - \frac{\delta_{kl}\delta_{mn}}{3}\right) &= \left[\frac{\partial a_{ij}}{\partial s_{mn}} - \frac{\delta_{mn}}{3}\frac{\partial a_{ij}}{\partial s_{kk}}\right] = \\
&= \left[\frac{\partial^2 J_3}{\partial s_{mn}\partial s_{ij}} + \frac{\delta_{ij}}{3}s_{mn} + \frac{\delta_{mn}}{3}s_{ij}\right] = \\
&= \left[\frac{\partial^2 J_3}{\partial s_{mn}\partial s_{ij}} + \frac{1}{3}(\delta_{ij}s_{mn} + s_{ij}\delta_{mn})\right] = \tilde{\mathbf{H}}^{(22)}
\end{aligned}$$

With these positions we have that

$$\begin{aligned}
\frac{\partial^2 G}{\partial\sigma_{ij}\partial\sigma_{mn}} &= c_{11}[\delta_{ij}\delta_{mn}] + c_{21}[s_{ij}\delta_{mn}] + c_{31}[a_{ij}\delta_{mn}] + \\
&\quad + (c_{12}s_{mn} + c_{13}a_{mn})\delta_{ij} + (c_{22}s_{mn} + c_{23}a_{mn})s_{ij} + \\
&\quad + (c_{32}s_{mn} + c_{33}a_{mn})a_{ij} + c_2\tilde{\mathbf{H}}^{(11)} + c_3\tilde{\mathbf{H}}^{(22)} = \\
&= c_{11}\mathbf{H}^{(11)} + c_{22}\mathbf{H}^{(22)} + c_{33}\mathbf{H}^{(33)} + \\
&\quad + [c_{21}s_{ij}\delta_{mn} + c_{12}s_{mn}\delta_{ij}] + [c_{31}a_{ij}\delta_{mn} + c_{13}a_{mn}\delta_{ij}] + \\
&\quad + [c_{23}s_{ij}a_{mn} + c_{32}s_{mn}a_{ij}] + c_2\tilde{\mathbf{H}}^{(11)} + c_3\tilde{\mathbf{H}}^{(22)}
\end{aligned}$$

Finally

$$c_{11} = \frac{1}{3}\frac{\partial c_1}{\partial p} = \frac{1}{9}\frac{\partial^2 G}{\partial p^2}$$

$$
\begin{aligned}
c_{22} &= \frac{3}{2q}\left(\frac{\partial c_2}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial c_2}{\partial \theta}\right) = \\
&= \frac{3}{2q}\left\{\frac{\partial}{\partial q}\left[\frac{3}{2q}\left(\frac{\partial G}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial G}{\partial \theta}\right)\right] - \right. \\
&\quad \left. -\frac{\tan 3\theta}{q}\frac{\partial}{\partial \theta}\left[\frac{3}{2q}\left(\frac{\partial G}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial G}{\partial \theta}\right)\right]\right\} \\
&= \frac{9}{4q}\left\{\left[\left(-\frac{1}{q^2}\frac{\partial G}{\partial q} + \frac{1}{q}\frac{\partial^2 G}{\partial q^2}\right) - \tan 3\theta\left(-\frac{2}{q^3}\frac{\partial G}{\partial \theta} + \frac{1}{q^2}\frac{\partial^2 G}{\partial q \partial \theta}\right)\right] - \right. \\
&\quad \left. - \frac{\tan 3\theta}{q^2}\left[\frac{\partial^2 G}{\partial q \partial \theta} - \frac{1}{q}\left(\frac{3}{\cos^2 3\theta}\frac{\partial G}{\partial \theta} + \tan 3\theta \frac{\partial^2 G}{\partial \theta^2}\right)\right]\right\} = \\
&= \frac{9}{4q^2}\left\{\left(\frac{\partial^2 G}{\partial q^2} - \frac{1}{q}\frac{\partial G}{\partial q}\right) + \right. \\
&\quad \left. + \frac{\tan 3\theta}{q}\left[\frac{1}{q}\left(2 + \frac{3}{\cos^2 3\theta}\right)\frac{\partial G}{\partial \theta} - 2\frac{\partial^2 G}{\partial q \partial \theta} + \frac{\tan 3\theta}{q}\frac{\partial^2 G}{\partial \theta^2}\right]\right\} \\
c_{33} &= -\frac{9}{2q^3 \cos 3\theta}\frac{\partial c_3}{\partial \theta} = \\
&= -\frac{9}{2q^3 \cos 3\theta}\frac{\partial}{\partial \theta}\left(-\frac{9}{2q^3 \cos 3\theta}\frac{\partial G}{\partial \theta}\right) = \\
&= \frac{81}{4q^6 \cos 3\theta}\frac{\partial}{\partial \theta}\left(\frac{1}{\cos 3\theta}\frac{\partial G}{\partial \theta}\right) = \\
&= \frac{81}{4q^6 \cos 3\theta}\left(\frac{3 \sin 3\theta}{\cos^2 3\theta}\frac{\partial G}{\partial \theta} + \frac{1}{\cos 3\theta}\frac{\partial^2 G}{\partial \theta^2}\right) = \\
&= \frac{81}{4q^6 \cos^2 3\theta}\left(3 \tan 3\theta \frac{\partial G}{\partial \theta} + \frac{\partial^2 G}{\partial \theta^2}\right) \\
c_{21} &= \frac{1}{3}\frac{\partial c_2}{\partial p} = \frac{1}{3}\frac{\partial}{\partial p}\left[\frac{3}{2q}\left(\frac{\partial G}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial G}{\partial \theta}\right)\right] = \\
&= \frac{1}{2q}\left(\frac{\partial^2 G}{\partial p \partial q} - \frac{\tan 3\theta}{q}\frac{\partial^2 G}{\partial p \partial \theta}\right) \\
c_{12} &= \frac{3}{2q}\left[\frac{\partial c_1}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial c_1}{\partial \theta}\right] = \\
&= \frac{3}{2q}\left[\frac{\partial}{\partial q}\left(\frac{1}{3}\frac{\partial G}{\partial p}\right) - \frac{\tan 3\theta}{q}\frac{\partial}{\partial \theta}\left(\frac{1}{3}\frac{\partial G}{\partial p}\right)\right] = c_{21} \\
c_{31} &= \frac{1}{3}\frac{\partial c_3}{\partial p} = \frac{1}{3}\frac{\partial}{\partial p}\left(-\frac{9}{2q^3 \cos 3\theta}\frac{\partial G}{\partial \theta}\right) =
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{3}{2q^3\cos 3\theta}\frac{\partial^2 G}{\partial p \partial\theta} \\
c_{13} &= -\frac{9}{2q^3\cos 3\theta}\frac{\partial c_1}{\partial\theta} = -\frac{9}{2q^3\cos 3\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{3}\frac{\partial G}{\partial p}\right) = c_{31} \\
c_{23} &= -\frac{9}{2q^3\cos 3\theta}\frac{\partial c_2}{\partial\theta} = \\
&= -\frac{9}{2q^3\cos 3\theta}\frac{\partial}{\partial\theta}\left[\frac{3}{2q}\left(\frac{\partial G}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial G}{\partial\theta}\right)\right] = \\
&= -\frac{27}{4q^4\cos 3\theta}\left[\frac{\partial^2 G}{\partial q\partial\theta} - \frac{1}{q}\left(\frac{3}{\cos^2 3\theta}\frac{\partial G}{\partial\theta} + \tan 3\theta\frac{\partial^2 G}{\partial\theta^2}\right)\right] \\
c_{32} &= \frac{3}{2q}\left[\frac{\partial c_3}{\partial q} - \frac{\tan 3\theta}{q}\frac{\partial c_3}{\partial\theta}\right] = \\
&= \frac{3}{2q}\left[\frac{\partial}{\partial q}\left(-\frac{9}{2q^3\cos 3\theta}\frac{\partial G}{\partial\theta}\right) - \frac{\tan 3\theta}{q}\frac{\partial}{\partial\theta}\left(-\frac{9}{2q^3\cos 3\theta}\frac{\partial G}{\partial\theta}\right)\right] = \\
&= -\frac{3}{2q}\left[\frac{9}{2\cos 3\theta}\frac{\partial}{\partial q}\left(\frac{1}{q^3}\frac{\partial G}{\partial\theta}\right) - \frac{9\tan 3\theta}{2q^4}\frac{\partial}{\partial\theta}\left(\frac{1}{\cos 3\theta}\frac{\partial G}{\partial\theta}\right)\right] = \\
&= -\frac{27}{4q^4\cos 3\theta}\left[q^3\left(-\frac{3}{q^4}\frac{\partial G}{\partial\theta} + \frac{1}{q^3}\frac{\partial^2 G}{\partial q\partial\theta}\right) - \right. \\
&\quad \left. -\frac{\sin 3\theta}{q}\left(\frac{3\sin 3\theta}{\cos^2 3\theta}\frac{\partial G}{\partial\theta} + \frac{1}{\cos 3\theta}\frac{\partial^2 G}{\partial\theta^2}\right)\right] = \\
&= -\frac{27}{4q^4\cos 3\theta}\left[-\frac{3}{q}\frac{\partial G}{\partial\theta} + \frac{\partial^2 G}{\partial q\partial\theta} - \frac{3\tan^2 3\theta}{q}\frac{\partial G}{\partial\theta} - \frac{\tan 3\theta}{q}\frac{\partial^2 G}{\partial\theta^2}\right] = \\
&= -\frac{27}{4q^4\cos 3\theta}\left\{\frac{\partial^2 G}{\partial q\partial\theta} - \frac{1}{q}\left[3\left(1+\tan^2 3\theta\right)\frac{\partial G}{\partial\theta} - \tan 3\theta\frac{\partial^2 G}{\partial\theta^2}\right]\right\} = \\
&= -\frac{27}{4q^4\cos 3\theta}\left\{\frac{\partial^2 G}{\partial q\partial\theta} - \frac{1}{q}\left[\frac{3}{\cos^2 3\theta}\frac{\partial G}{\partial\theta} - \tan 3\theta\frac{\partial^2 G}{\partial\theta^2}\right]\right\} = c_{23}
\end{aligned}
$$

Bibliography

[1] Aboshi, H., *Measurement of Pore Water Pressure During Consolidation of Fine-Grained Soil*, Bulletin of Faculty of Engineers, Hiroshima, Vol. 4, No. 1, pp.1-12, 1955.

[2] Aravin, V.I., and Numerov, S.N., *Theory of Fluid Flow in Undeformable Porous Media*, Israel Programme for Scientific Translaction, Jerusalem, 1965.

[3] Banerjee, P.K., and Stipho, A.S., *Associated and Non-associated Constitutive Relations for Undrained Behaviour of Isotropic Soft Clays*, Int. J. Num. Anal. Meth. Geomech., Vol. 2, pp. 35-56, 1978.

[4] Biot, M.A., *General Theory of Three Dimensional Consolidation*, J. Appl. Phys., Vol. 12, pp. 155-164, 1941.

[5] Biot, M.A., *Consolidation Settlement Under Rectangular Load Distribution* J. Appl. Phys., Vol. 12, pp. 426-430, 1941.

[6] Biot, M.A., and Clingan, F.M., *Consolidation Settlement of a Soil With an Impervious Top Surface*, J. Appl. Phys., Vol. 12, pp. 578-581, 1941.

[7] Biot, M.A., *Theory of Propagation of Elastic Waves in a Fluid Saturated Porous Solid*, J. Acoust. Soc. of America, Vol. 28, pp. 168-191, 1956.

[8] Biot, M.A., *Mechanics of Deformation and Acoustic Propagation in Porous Media*, J. Appl. Phys., Vol. 33, pp. 1482-1498, 1962.

[9] Biot, M.A., *Generalized Theory of Acoustic Propagation in Porous Dissipation Media*, J. Acoust. Soc. of America, Vol. 28, pp. 168-191, 1956.

[10] Bingham, E.C., *Fluidity and Plasticity*, Chapter 8, pp. 215-218, McGraw-Hill, New York, 1922.

[11] Bowen, R.M., *Theory of Mixture*, In: Continuum Physics Vol.3, Ed. Eringen, Accademic Press, New York, 1976.

[12] Booker, J.R., *A Numerical Method for the Solution of Biot's Consolidation Theory*, Quart. J. Mech. Appl. Math., Vol. 26, p. 457, 1973.

[13] Burland, G.B., *Deformation of Soft Clay*, Ph.D. Thesis, Cambridge University.

[14] Calladine, C.R., *The yielding of Clay*, Correspondence, Geotechnique, Vol. 13, pp. 250-255, 1963.

[15] Casagrande, A., *Seepage Through Dams*, In Contribution to Soil Mechanics 1925-1940, Boston Society of Civil Engineers, 1940.

[16] Cedergen, H.R., *Seepage, Drainage and Flow Nets*, John Wiley, New York, 1967.

[17] Chen, W.F., *Limit Analysis and Soil Plasticity*, Elsevier Scientific Publishing Company, 1975.

[18] Chen, W.F., and Saleeb, A.F., *Constitutive Equations For Engineering Material. Volume 1: Elasticity and Modeling*, John Wiley & Sons, Inc., New York, 1982.

[19] Coburn, N., *Vector and Tensor Analysis*, The Macmillan Company, New York, 1955.

[20] Cryer, C.W., *A Comparison of the Three-Dimensional Theories of Biot and Terzaghi*, Quart. J. Mech. Appl. Math., Vol. 16, p. 401-412, 1963.

[21] Dafalias, Y.F. and Popov, E.P., *A Model of Nonlinearly Hardening Materials for Complex Loadings*, Acta Mech. Vol. 21, pp. 173-192, 1975.

[22] Dafalias, Y.F. and Popov, E.P., *Cyclic Loading for Materials with a Vanishing Elastic Region*, Nucl.Eng. and Design, Vol. 41, pp. 293-302, 1977.

[23] De Crescenzo, A., and Fusco, A., *The Theoretical Framework for a New Class of Plastic Models*, Quaderni ISMES, 1993.

[24] De Crescenzo, A., and Fusco, A., *A Mathematical Description of the Limit State Theory for Soils*, Quaderni ISMES, 1993.

[25] De Crescenzo, A., and Fusco, A., *A Constitutive Equation for Soils*, Quaderni ISMES, 1993.

[26] De Crescenzo, A., and Fusco, A., *SUOLO/STANDARD V1: A Constitutive Equation for Soils*, Research Report, N. , CIMNE, 1993.

[27] Desai, C.S., and Wu, T.H., *A General Function for Stress-Strain Curves*, Proc. 2nd Int. Conf. Numer. Methods Geomech., C.S. Desai (Ed.), Blacksburg, Va., ASCE, June 1976.

[28] Desai, C.S., and Siriwardane, H.J, *Constitutive Laws For Engineering Material*, Prentice Hall, Inc., Englewood Cliffs, New Jersey, 1984.

[29] Derski, W., *Thermomecanique Lineaire des Milieux Poreux*, E.N.S.M., Nantes, 1982.

[30] Duncan, J.M., and Chang, C.Y., *Nonlinear Analysis of Stress and Strain in Soils*, J. Soil Mech. Found. Div., ASCE, Vol. 96, No. SM5, pp. 1629-1653, Sept. 1970.

[31] Drucker, D.C., *A More Fundamental Approach to Plastic Stress-Strain Relations*, Proc. 1st U.S. National Congress for Apllied Mechanics, New York, ASME, pp. 487-491, 1951.

[32] Drucker, D.C., and Prager, W., *Soil Mechanics and Plastic Analysis or Limit Design*, Quart. Appl. Math., Vol. 10, N. 2, pp. 157-165, 1952.

[33] Drucker, D.C., Gibson, R.E., and Henkel, D.J., *Soil Mechanics and Work-Hardening Theories of Plasticity*, ASCE, J. Soil Mech. Found. Div., Vol. 122, pp. 338-346, 1957.

[34] Drucker, D.C., *On Stress-Strain Relations for Soil and Load Carrying Capacity*, Proc. 1st Int. Conference on the Mechanics of Soil-vehicle Systems, Torino, Edizioni Minerva Tecnica, pp. 15-23, 1961.

[35] Drucker, D.C., *Concept of Path Independence and Material Stability for Soils*, In: Rheology et Mecanique des Sols. Proc. IUTAM Symposium, Grenoble. Eds. J. Kravtchenko and P.M. Sirieys, Apr. 1-8, 1964.

[36] Finnie, I., and Heller, W.R., *Creep of Enginnering Materials*, McGraw-Hill, 1959.

[37] Fusco, A., *Continuum Mechanics and Finite Element Numerical Solutions in Geotechnique*, Ph.D. Thesis, University of Ottawa, 1985.

[38] Fusco, A., *Linear Algebra for Engineers*, Monograph CIMNE, N. 18, 1993.

[39] Fusco, A., *Numerical Analysis for Engineers* Monograph CIMNE, N. 19, 1993.

[40] Fusco, A., *The Finite Element Method for Engineers*, Monograph CIMNE, N. 21, 1993.

[41] Fusco, A., and Cervera, M., *Coupled Solid-Fluid Problems Solved by Finite Elements*, Monograph CIMNE, N. 22, 1993.

[42] Gibson, R.E, and McNamee, J., *A Three-Dimensional Problem of Consolidation of a Semi-Infinite Clay Stratum*, Quart. J. Mech. Appl. Math., Vol. 16, p. 34-50, 1963.

[43] Gibson, R.E, Schiffman, R.L., and Pu, S.L., *Plain Strain and Axially Symmetric Consolidation of a Clay Layer on a Smooth Impervious Base*, Quart. J. Mech. Appl. Math., Vol. 16, p. 34-50, 1963.

[44] Gray, W.G., and O'Neil, K., *On the General Equation of Flow in Porous Media and Their Reduction to Darcy's Law*, Wat. Res. Research, Vol. 12, pp. 148-154, 1976.

[45] Green, A.E., and Naghdi, P.M., *A Dynamical Theory of Interacting Continua*, Int. J. Eng. Sci., Vol. 3, pp. 231-241, 1965.

[46] Green, A.E., and Naghdi, P.M., *A Note on Mixture*, Int. J. Eng. Sci., Vol. 6, pp. 631-635, 1968.

[47] Green, A.E., *On the Basic Equations for Mixture*, Quart. J. Mech. Appl. Math., Vol. 22, pp. 427-438, 1969.

[48] Green, A.E., *On the Flow of Fluids Through an Elastic Solid*, Act. Mech., Vol. 9, pp. 329-340, 1970.

[49] Hansen, J.B., *Discussion of Hyperbolic Stress-Strain Response: Cohesive Soils*, J. Soil Mech. Found. Div., ASCE, Vol. 89, No. SM4, pp. 241-242, July 1963.

[50] Hardin, O.B., and Drnevich V.P., *Shear Modulus and Damping in Soils: Measurements and Parameter Effects*, J. of the Soil Mech. and Found. Div., ASCE, 1972.

[51] Harr, M.E., *Ground Water and Seepage*, McGraw-Hill, New York, 1962.

[52] Hill, R., *The Mathematical Theory of Plasticity*, Oxford University Press, 1950.

[53] Hvorslev, M.J., *Ueber die Festigkeitseigenschaften Gestoerter Bindiger Boden*, (On the physical properties of disturbed cohesive soil), Ingenieurdenskabelige Skriften, A. No. 45, p. 159, 1937.

[54] Josselin de Jong, G. DE., *Application of the Stress Function to Consolidation Problems*, Proc. 4st Int. Conference of Soil Mechanics, London, Vol. 1, pp. 320-323, 1957.

[55] Kondner, R.L., *Hyperbolic Stress-Strain Response: Cohesive Soils*, J. Soil Mech. Found. Div., ASCE, Vol. 89, No. SM1, pp. 115-143, Jan. 1963.

[56] Krieg, R.D., *A Practical Two-surface Plasticity Theory*, J. Appl. Mech. Trans., ASCE, Series E42, pp. 641-646, 1975.

[57] Kulhawy, F.H., Duncan, J.M., and Seed, H.B., *Finite Element Analyses of Stresses and Movements in Embankments During Construction*, Geotechnical Engineering Report No. TE 69-4, Dept. of Civil Engineering, University of California, Berkeley, 1969.

[58] Jambu, N., *Soil Compressibility as Determined by Odemeter and Triaxial Test*, Proc. Eur. Conf. Soil Mech. Found. Eng., Wiesbaden, Vol. 1, pp. 19-25, 1963.

[59] Johnson, A.E., *Complex Stress Creep*, Met. Rev., Vol. 5, 447, 1960.

[60] Leckei, F.A., and Martin, J.B., *Deformation Bounds for Bodies in a State of Creep*, J. Appl. Mech., Am. Soc. Mech. Eng., pp. 411-417, June 1967.

[61] Lewis, R.W., and Schrefler, E., *The Finite Element Method in the Deformation and Consolidation of Porous Media*, John Wiley & Sons, New York, 1987.

[62] Malvern, L.E., *Introduction to the Mechanics of a Continuous Medium* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1969.

[63] Mandel, L.E., *Consolidation des Couches d'Argile*, Proc. 4st Int. Conference Soil Mechanics Foundations Engineering, Vol. 1, pp. 360-367, 1957.

[64] Mase, G.E., *Theory and Problems of Continuun Mechanics*, Schaum's Outline Series, McGraw-Hill, Inc., 1970.

[65] McNamee, J., and Gibson, R.E, *Plain Strain and Axially Symmetric Problems of Consolidation of a Semi-infinite Clay Stratum*, Quart. J. Mech. Appl. Math., Vol. 13, p. 210-227, 1960.

[66] McNamee, J., and Gibson, R.E, *Displacement Functions and Linear Transforms Applied to Diffusion Through Porous Elastic Media*, Quart. J. Mech. Appl. Math., Vol. 13, p. 98-111, 1960.

[67] Melan, E., Ingenieur-Archiv, Vol. 9, pp. 116, 1938.

[68] Mroz, Z., Norris, V.A., and Zienkiewicz, O.C., *An Anisotropic Hardening Model for Soils and its Application to Cyclic Loading*, Int. J. for Numer. Anal. Meth. in Geomechanics, Vol. 2, pp. 203-221, 1978.

[69] Mroz, Z., Norris, V.A., and Zienkiewicz, O.C., *Simulation of Behaviour of Soils Under Cyclic Loading by using a More General Hardening Rule*, University College of Swansea Report, 1978.

[70] Nayak, G.C., and Zienkiewicz, O.C., *Elastoplastic Stress Analysis. A Generalization of Various Constitutive Relations Including Strain Softening*, Int. J. Num. Meth. Eng., Vol. 5, pp. 113-135, 1972.

[71] Noll, W., *A Mathematical Theory of the Mechanical Behavior of Continuous Media*, Arch. Rational Mech. Anal., Vol. 2, 1958. (reprinted in Rational Mechanics of Materials, ed. C. Truesdell, New York: Gordon & Breach, 1965).

[72] Olszak, W., and Perzyna, P., *On Elasto-Viscoplastic Soil*, In: Rheology and Soil Mechanics, IUTAM Symposium, Springer-Verlag, Grenoble, 1966.

[73] Pender, M.J., *A Model for the Behaviour of Over-Consolidated Soil*, Geotechnique, Vol. 28, No. 1, pp. 1-25, 1978.

[74] Perzyna, P., *Fundamental Problems in Visco-Plasticity*, Adv. Appl. Mech., Vol. 9, pp. 243-377, 1966.

[75] Polubarinova-Kochina, P.Y., *The Theory of Ground Water Movement*, Princeton University Press, Princeton, New Jersey, 1962.

[76] Ramberg, W., and Osgood, W.R., *Description of Stress-Strain Curves by Three Parameters*, Tech. Note 902, National Advisory Comm. Aeronaut., Washington D.C., 1943.

[77] Rendulic, L., *Porenziffer und Porenwasser Drueck in Tonen*, (Porosity and pore water pressure in Clay), Der Bauingenieur, 17(51/53), pp. 559-564, 1933

[78] Rendulic, L., *Relation between Void Ration and Effective Principal Stress for a Remolded Silty Clay*, Proc. 1st Int. Conference on Soil Mechanics and Foundation Engineering, Cambridge, Mass., Vol. 3, pp. 48-51, 1936.

[79] Roscoe, K.H., Schofield, A.N., and Wroth, C.P., *On the Yielding of Soils*, Geotechnique, Vol. 8, pp. 22-53, 1958.

[80] Roscoe, K.H., Schofield, A.N., and Thurairajah, A., *Yielding of Soils in States Wetter than Critical*, Geotechnique, Vol. 8, pp. 22-53, 1963.

[81] Roscoe, K.H., Pooorooshab, H.B., *A Theoreatical and Experimental Study of Strains in Triaxial Compression Test on Normally Consolidated Clays*, Geotechnique, Vol. 13(1), pp. 12-38, 1963.

[82] Roscoe, K.H., and Burland, G.B., *On the Generalised Stress-Strain Behaviour of Wet Clay*, In: Engineering Plasticity, Eds. J. Heyman and F.A. Lekie, Cambridge University Press, London, pp. 535-609, 1968.

[83] Rivlin, R.S., and Ericksen, J.L., *Stress-Deformation Relations for Isotropic Materials*, J. Rat. Mech. Anal., Vol. 4, pp. 323-425, 1955. (reprinted in Rational Mechanics of Materials, ed. C. Truesdell. New York: Gordon & Breach, 1965).

[84] Sangrey D.A., Henkel D.J. and Esrig M.I., *The Effective Stress response of a Saturated Clay Soil to Repeated Loading*, Canadian Geotechnical Journal, Vol. 6, pp. 241-252, 1969.

[85] Schiffman, R.L., Chen, A.T.F., and Jordan, J.C., *An Analyisis of Consolidation Theories*, ASCE, J. Soils Mech. Found. Div., Vol. 95(SM1), pp. 285-311, 1969.

[86] Schofield, A.N., and Wroth, C.P., *Critical State Soil Theory*, Mc. Graw-Hill, New York, 1968.

[87] Signorini, A., *Sulle deformazioni Finite dei Sistemi a Trasformazioni Reversibili*, Rend. Accad. Naz. Lincei, Ser. 6(18), pp. 388-394, 1933.

[88] Taylor, D.W., *Fundamentals of Soil Mechanics*, Wiley & Sons, New York, 1948.

[89] Terzaghi, K., *Die Physichalischen Grundlagen des Technisch-Geologischen Gutachtens*, In: Archit. Verein Z., Vol. 73, No. 36/37, pp. 237-241, 1921.

[90] Terzaghi, K., *Die Berechnung der Durchlaessigkeitsziffer des Tones aus dem Verlauf der hydrodynamischen Spannungserscheinungen*. Mathematische Naturwissenschaftliche Klass, Vol. 8, pp. 125-138, 1923.

[91] Terzaghi, K., *Theoretical Soil Mechanics*, Wiley, New York, 1943.

[92] Truesdell, C., *Sulle Basi della Termomeccanica*, Rend. Linc., Vol 22, pp. 33-38, pp. 158-166, 1957.

[93] Truesdell, C., and Noll, W., *The Non-Linear Field Theories of Mechanics*, In: Encyclopedia of Physics, Ed. S. Flugge, Vol. 3/3, Berlin, Springer-Verlag, 1965.

[94] Truesdell, C., and Toupin, R.A., *The Classical Field Theories*, In: Encyclopedia of Physics, Ed. S. Flugge, Vol. 3/3, Berlin, Springer-Verlag, 1960.

[95] Zienkiewicz, O.C., Humpheson, C., and Lewis, R.W., *A Unified Approach to Soil Mechanics*, In: Finite Element in Geomechanics, Ch. 4, Ed. G. Gudehus, Wiley, London, 1977.

[96] Zienkiewicz, O.C., and Shiomi, T., *A Dynamic Behaviour of Saturated Porous Media, The Generalized Biot Formulation and Its Numerical Solution*, Int. J. Num. Anal. Meth. Geomech., Vol. 8, pp. 71-96, 1983.