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ABSTRACT

In the context of multi-criteria decision-making, the ordered weighted averaging (OWA) operator and Choquet Integral (CI) emerge as two pivotal weighted aggregation operators. Their core features include the linear order structure and the monotonicity. However, there is currently no unified method for constructing monotonic OWA operators and CIs under a certain admissible order within the interval-valued fuzzy framework. Based on the complete lattice structure of the space of all closed subintervals of [0, 1] under the generated admissible order, we propose a unified method for constructing the monotonic interval-valued OWA operators and CIs. Moreover, we theoretically prove rigorously that the proposed OWA operators and CIs satisfy the axiomatic definition of aggregation operator, particularly the monotonicity under a certain admissible order. Finally, we establish a multi-expert decision-making algorithm based on the proposed operators, where the overall preference of each alternative is obtained by aggregating its interval-valued evaluations across admissible orders. The effectiveness of the proposed approach is illustrated through a practical decision-making example.

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1 Introduction

In multi-criteria decision-making (MCDM) frameworks, it is usually necessary to comprehensively consider the information of multiple criteria in order to make a final decision. The classic weighted average aggregation operator adopts a weighted average approach, where the weight coefficients are directly associated with specific criteria. This means that each criterion is assigned a weight that reflects its relative importance to the final decision. This method assumes that the contribution of each criterion to the decision is predetermined. Yager [1] introduced the ordered weighted averaging (OWA) operator, which brings forth a different way of assigning weights. In contrast to the classical average aggregation operator, the weight coefficients in this method are not directly related to the criteria but are associated with an ordered position. This means that when considering each criterion, not only is the importance of the criterion itself taken into account, but also its position in the overall ranking. This can make the algorithm more flexible and adaptable to changes in the relationships between criteria in some situations.

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As a special type of nonlinear aggregation operator and an extension of OWA operator, Choquet integral (CI) [2] was introduced by Schmeidler [3,4] into MCDM and has been widely applied [5–9]. A prominent advantage of the CI is that it quantifies the similarity or correlation between criteria by introducing fuzzy measures, thereby more comprehensively considering the interactions between criteria. Meanwhile, the introduction of fuzzy measures in the CI enables the incorporation of fuzziness into the decision-making process, aligning the decisions more closely with real-world scenarios.

Sometimes, due to incomplete information, limited knowledge, subjective judgment, or a lack of clear understanding of the complexity of decision factors, experts/decision-makers (DMs) find it challenging to provide accurate and reasonable assessments in some practical MCDM process. In order to better address such uncertainty and fuzziness of information in MCDM, interval-valued (IV) fuzzy sets have been introduced in MCDM [10,11]. This further propels the extension of various real-valued aggregation operators to accommodate decision modeling within the (IV) fuzzy framework [12–16]. Because the core of OWA operator and CI lies in linear order structure and monotonicity, there are two main difficulties in extending the ordered aggregation operators to the IV fuzzy framework: (D1) Arranging the evaluation information in a decreasing/increasing way according to a certain linear order; (D2) Maintaining the monotonicity in a certain linear order. This can ensure that the alternatives with higher scores in each criterion are ranked higher, thereby guaranteeing the fairness and impartiality of the decision.

To address (D1), a widely used and effective approach is to utilize the *admissible* orders on the space $L([0,1])$ of all closed subintervals of $[0,1]$. This concept was introduced by Bustince et al. [17]. Based on the admissible orders, Bustince et al. [12] proposed the IVOWA operators and the IVCI and showed that these two types of operations are extensions of real-valued OWA operator and CI. The other two attempts to define IVCIs proposed by Paternain et al. [15], based on Auman's set-valued approach [18] and integral decomposition approach [19,20], respectively. And they [15] also proved that these two IVCIs are equal. One significant drawback of these two IVCIs is the neglect of the ordinal structure of the intervals. Considering the finiteness of practical evaluation information, Paternain et al. [15] introduced a new IVCI based on admissible permutations, which is equal to the arithmetic mean of IVCIs under all admissible permutations. However, all the IV ordered operators mentioned above do not fulfill the monotonicity and, therefore, cannot be considered as aggregation operators. To the best of our knowledge, there is currently no IV ordered operator that satisfies the monotonicity under a certain admissible order. Consequently, there is currently a lack of effective methods to address (D2). The primary reason for this lies in the fact that the above IVOWA operators and IVCIs only consider the parallel/formal extension of the real-valued OWA operator and CI under the admissible order, i.e., only using an admissible order instead of standard order "$\leq$" on $[0,1]$, without fully considering the intrinsic properties and internal construction of the admissible order itself. However, the structure of the admissible order itself is crucial for the monotonicity of these operations, while noting that $L([0,1])$ is not isomorphic to $([0,1],\leq)$ under the generated admissible orders (see Theorem 5, [21]), this naturally leads to the loss of some desirable properties of the real-valued ordered aggregation operators.

In summary, this paper studies the construction of interval-valued OWA operators and interval-valued Choquet integrals under admissible orders. We consider ordered aggregation of interval-valued data under admissible orders and the associated monotonicity requirement. Existing interval-valued OWA operators and interval-valued Choquet integrals usually extend real-valued operators by replacing the natural order on $[0,1]$ with admissible orders on $L([0,1])$. However, such extensions do not, in general, preserve monotonicity. We show that this limitation is due to the fact that the order-theoretic structure induced by admissible orders is not explicitly exploited. By using the complete lattice structure of $(L([0,1]), \preccurlyeq)$ under generated admissible orders, we establish a unified construction framework for interval-valued OWA operators and interval-valued Choquet integrals that are monotonic with respect to a fixed admissible order. In particular, we construct the interval-valued OWA operator "$\mathsf{IIVOWA}_{\boldsymbol{\Omega}}^{\preccurlyeq}$" and the interval-valued Choquet integral "$\mathbf{IC}_{\lambda}^{\preccurlyeq}$", and prove that they satisfy the axiomatic properties of aggregation operators, including monotonicity, idempotency, boundedness, and consistency with their real-valued counterparts. Furthermore, we develop a multi-expert decision-making algorithm based on the proposed operators, where global interval-valued preferences are aggregated over all admissible orders $\preccurlyeq_{\alpha,1}(\alpha \in [0,1))$. The applicability of the proposed approach is illustrated through a broadband internet selection problem for students in the same dormitory.

2 Preliminaries

In this section, we recall some basic concepts on admissible orders for intervals.

Let $L([0,1])$ denote the family of all closed subintervals of $[0,1]$, i.e.,

$$L([0,1]) = \{\mathbf{y} \mid \mathbf{y} = [\underline{y}, \overline{y}] \text{ with } 0 \leq \underline{y} \leq \overline{y} \leq 1\}.$$

The *product partial order* $\preccurlyeq_{\mathbf{P}}$ is defined by

$$\mathbf{y} \preccurlyeq_{\mathbf{P}} \mathbf{x} \Longleftrightarrow \underline{y} \leq \underline{x} \text{ and } \overline{y} \leq \overline{x}. \tag{1}$$

Clearly, $(L([0,1]), \preccurlyeq_{\mathbf{P}})$ is a complete and bounded lattice with the minimum $[0,0]$ and the maximum $[1,1]$. We adopt the standard representation of closed subintervals of $[0,1]$ by ordered pairs $\mathbb{K}([0,1]) = \{(x,y) \in [0,1]^2 \mid x \leq y\}$, thereby establishing a natural correspondence between $L([0,1])$ and $\mathbb{K}([0,1])$. Without causing confusion, we indiscriminately use the elements in $L([0,1])$ and $\mathbb{K}([0,1])$.

Bustince et al. [17] introduced the following concept of admissible order for $L([0,1])$.

Definition 1 ([17]): *Let $(L([0,1]), \preccurlyeq)$ be a poset. The order $\preccurlyeq$ is admissible if it satisfies the following two properties:*

(1) $\preccurlyeq$ is a linear order on $L([0,1])$;

(2) For $\mathbf{x}, \mathbf{y} \in L([0,1])$, $\mathbf{x} \preccurlyeq \mathbf{y}$ whenever $\mathbf{x} \preccurlyeq_{\mathbf{P}} \mathbf{y}$. This means that $\preccurlyeq$ refines the product order $\preccurlyeq_{\mathbf{P}}$.

Example 1 ([22] (Example 2.1)): *The following two orders*

- **(Lexicographic-1 order $\leq_{\mathbf{lex1}}$)** $\mathbf{y} \leq_{lex1} \mathbf{x}$ *if and only if* $(\underline{y} < \underline{x})$ *or* $(\underline{y} = \underline{x}$ *and* $\overline{y} \leq \overline{x})$*;*
- **(Xu and Yager's order $\leq_{\mathbf{XY}}$ [23])** $\mathbf{y} \leq_{\mathrm{XY}} \mathbf{x}$ *if and only if* $(\underline{y} + \overline{y} < \underline{x} + \overline{x})$ *or* $(\underline{y} + \overline{y} = \underline{x} + \overline{x}$ *and* $\overline{y} - \underline{y} \leq \overline{x} - \underline{x})$ *are admissible.*

Bustince et al. [24] proposed the K_α-mapping $K_\alpha : L([0,1]) \longrightarrow [0,1]$ as follows: for $\mathbf{x} = [\underline{x}, \overline{x}] \in L([0,1])$,

$$K_\alpha(\mathbf{y}) = \alpha \cdot \overline{y} + (1-\alpha) \cdot \underline{y} \quad (\alpha \in [0,1]). \tag{2}$$

Then, by using the K_α-mapping, Bustince et al. [17] introduced the following order $\preccurlyeq_{\alpha,\beta}$ and proved that $\preccurlyeq_{\alpha,\beta}$ is an admissible order on $L([0,1])$.

For $\alpha, \beta \in [0,1]$ with $\alpha \neq \beta$,

$$\mathbf{y} \preccurlyeq_{\alpha,\beta} \mathbf{x} \Longleftrightarrow (K_\alpha(\mathbf{y}) < K_\alpha(\mathbf{x})) \text{ or } (K_\alpha(\mathbf{y}) = K_\alpha(\mathbf{x}) \text{ and } K_\beta(\mathbf{y}) \leq K_\beta(\mathbf{x})). \tag{3}$$

More generally, Bustince et al. [17] obtained a general method for constructing admissible orders on $L([0,1])$ through suitably chosen continuous mappings on $[0,1]$, thereby introducing the notion of generated admissible orders.

Definition 2 ([17] (Definition 3.2)): *An admissible order $\preccurlyeq$ on $L([0,1])$ is called a generated admissible order if there exist two continuous mappings $f, g : [0,1]^2 \longrightarrow \mathbb{R}$ such that, for $\mathbf{y}, \mathbf{x} \in L([0,1])$,*

$$\mathbf{y} \preccurlyeq \mathbf{x} \Longleftrightarrow (f(\underline{y}, \overline{y}) < f(\underline{x}, \overline{x})) \text{ or } (f(\underline{y}, \overline{y}) = f(\underline{x}, \overline{x}) \text{ and } g(\underline{y}, \overline{y}) \leq g(\underline{x}, \overline{x})). \tag{4}$$

Here, the mappings (f, g) is said to be a generating pair of the order $\preccurlyeq$.

Let $\preccurlyeq_{f,g}$ denote the admissible order on $L([0,1])$ induced by a generating pair (f, g). When there is no possibility of ambiguity of the values of (f, g), we use $\preccurlyeq$ instead of $\preccurlyeq_{f,g}$. It has been shown in [17] (Example 3.2) that the representation of a generated admissible order in terms of a generating pair is generally non-unique. Furthermore, according to [17] (Proposition 3.5), any admissible order generated by a pair of aggregation functions (f, g) admits an equivalent representation based on another pair $(\overline{f}, \overline{g})$ of continuous and idempotent aggregation functions, in the sense that both pairs induce the same order relation. Since the generating pair in Definition 2 is assumed to be continuous, we may, without loss of generality, restrict the ranges of f and g to the unit interval, i.e., $f([0,1]^2) \subseteq [0,1]$ and $g([0,1]^2) \subseteq [0,1]$.

Given a generated admissible order $\preccurlyeq$ on $L([0,1])$, define the (f, g)-*values mapping* $\mathbb{V}_{(f,g)} : L([0,1]) \longrightarrow [0,1]^2$ by

$$\mathbb{V}_{(f,g)}(\mathbf{y}) = (f(\underline{y}, \overline{y}), g(\underline{y}, \overline{y})). \tag{5}$$

Remark 1: *It can be readily verified that (1) $\mathbb{V}_{(f,g)}$ is injective by the antisymmetry of the order $\preccurlyeq$; (2) $\mathbf{y} \preccurlyeq \mathbf{x}$ if and only if $\mathbb{V}_{(f,g)}(\mathbf{y}) \leq_{lex1} \mathbb{V}_{(f,g)}(\mathbf{x})$.*

Noting that the generated admissible order in Definition 2 relies on the Lexicographic-1 order $\leq_{\text{lex1}}$ together with two continuous functions f and g defined on $\mathbb{K}([0,1])$, Santana et al. [25] put forward a more general scheme for constructing admissible orders on $L([0,1])$.

Definition 3 ([25]): *Let $\lesssim$ be an admissible order on $[0,1]^2$ and $f, g : \mathbb{K}([0,1]) \longrightarrow \mathbb{R}$ be two functions. The relation $\preccurlyeq_{f,g,\lesssim}$ on $L([0,1])$ is defined by*

$$\mathbf{y} \preccurlyeq_{f,g,\lesssim} \mathbf{x} \Longleftrightarrow (f(\underline{y}, \overline{y}), g(\underline{y}, \overline{y})) \lesssim (f(\underline{x}, \overline{x}), g(\underline{x}, \overline{x})). \tag{6}$$

Clearly, the generated admissible order $\preccurlyeq$ on $L([0,1])$ induced by a generating pair (f, g) in Definition 2 is equal to the order $\preccurlyeq_{f,g,\leq_{\text{lex1}}}$.

3 Monotonic Interval-Valued OWA Operators

Based on the linear order $\leq$ on $\mathbb{R}$, Yager [1] introduced one type of important aggregation functions, called ordered weighted aggregation (OWA) functions, as follows:

Definition 4 ([1]): *Let* $\mathbf{\Omega} = (\omega_1, \ldots, \omega_n)^\top$ *be the weight vector such that* $\omega_\gamma \in [0, 1]$ *and* $\sum_{\gamma=1}^{n} \omega_\gamma = 1$. *The OWA operator* $\mathsf{OWA}_{\mathbf{\Omega}} : [0, 1]^n \longrightarrow [0, 1]$, *associated with* $\mathbf{\Omega}$, *is defined by*

$$\mathsf{OWA}_{\mathbf{\Omega}}(x_1, \ldots, x_n) = \sum_{\gamma=1}^{n} \omega_\gamma x_{\sigma(\gamma)}, \tag{7}$$

where $(\sigma(1), \ldots, \sigma(n))$ *is a permutation of* $(1, \ldots, n)$ *such that* $x_{\sigma(n)} \leq \cdots \leq x_{\sigma(2)} \leq x_{\sigma(1)}$.

Similarly, based on the admissible orders, Bustince et al. [12] introduced the following form of interval-valued OWA operator to view the left and right bounds of the intervals as a whole.

Definition 5 ([12] (Definition 2)): *Let* $\preccurlyeq$ *be a generated admissible order on* $L([0, 1])$, *and* $\mathbf{\Omega} = (\omega_1, \ldots, \omega_n)^\top$ *be the weight vector such that* $\omega_\gamma \in [0, 1]$ *and* $\sum_{\gamma=1}^{n} \omega_\gamma = 1$. *Define the interval-valued OWA (IVOWA) operator* $\mathsf{IVOWA}_{\mathbf{\Omega}}^{\preccurlyeq} : (L([0, 1]))^n \longrightarrow L([0, 1])$, *associated with* $\preccurlyeq$ *and* $\mathbf{\Omega}$, *by*

$$\mathsf{IVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{\gamma=1}^{n} \omega_\gamma \mathbf{x}_{\sigma(\gamma)}, \tag{8}$$

where $(\sigma(1), \ldots, \sigma(n))$ *is a permutation of* $(1, \ldots, n)$ *such that* $\mathbf{x}_{\sigma(n)} \preccurlyeq \cdots \mathbf{x}_{\sigma(2)} \preccurlyeq \mathbf{x}_{\sigma(1)}$.

As an attempt to extend the usual OWA operators of Yager [1] on $\mathbb{R}$ (see Proposition 4, [12]), although the IVOWA operator defined in Definition 5 inherits some good properties (Proposition 5 and Corollary 1, [12]), as Bustince et al. (Example 4, [12]) pointed out, this operator does not have the monotonicity. To overcome this drawback, in this section, we propose a general construction method for interval-valued OWA operators satisfying the monotonicity under the generated admissible orders and the axiomatic definition of aggregation operators. Firstly, we review two basic lemmas proved by Wu et al. [21], which are crucial for the constructions of aggregation operators in this paper.

Lemma 1 (Lemma 7, [21]): *Let* $\preccurlyeq$ *be a generated admissible order on* $L([0, 1])$ *induced by a generating pair* (f, g). *Then,*

(1) *The function* f *is increasing on* $\mathbb{K}([0, 1])$.
(2) *The function* f *is strictly increasing on the diagonal* $\Delta = \{(x, x) \mid x \in [0, 1]\}$.

Lemma 2 (Theorem 7, [21]): *Let* $\preccurlyeq$ *be a generated admissible order on* $L([0, 1])$ *induced by a generating pair* (f, g). *Then,* $(L([0, 1]), \preccurlyeq)$ *is a complete lattice.*

Definition 6: *Let* $\preccurlyeq$ *be a generated admissible order on* $L([0, 1])$ *induced by a generating pair* (f, g), *and* $\mathbf{\Omega} = (\omega_1, \ldots, \omega_n)^\top$ *be the weight vector such that* $\omega_\gamma \in [0, 1]$ *and* $\sum_{\gamma=1}^{n} \omega_\gamma = 1$. *Define the mappings* $\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}$ *and* $\widetilde{\Psi}_{\mathbf{\Omega},g}^{\preccurlyeq} : (L([0, 1]))^n \longrightarrow \mathbb{R}$ *by*

$$\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{\gamma=1}^{n} \omega_\gamma f(\mathbf{x}_{\sigma(\gamma)}), \tag{9}$$

and

$$\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1,\ldots,\mathbf{x}_n)=\sum_{\gamma=1}^{n}\omega_\gamma g(\mathbf{x}_{\sigma(\gamma)}),\tag{10}$$

respectively, where $(\sigma(1),\ldots,\sigma(n))$ *is a permutation of* $(1,\ldots,n)$ *such that* $\mathbf{x}_{\sigma(n)}\preccurlyeq\cdots\preccurlyeq\mathbf{x}_{\sigma(2)}\preccurlyeq\mathbf{x}_{\sigma(1)}$.

Remark 2: *(1) Although the permutation* $(\cdot)$ *in Eqs. (9) and (10) need not be unique, which may happen only if some inputs are repeated, this does not affect the output results of Eqs. (9) and (10). Thus, Definition 6 is well-defined.*

(2) *By the definition of* $\preccurlyeq$, *we know that*

$$f(\mathbf{x}_{\sigma(n)})\le\cdots\le f(\mathbf{x}_{\sigma(2)})\le f(\mathbf{x}_{\sigma(1)}).\tag{11}$$

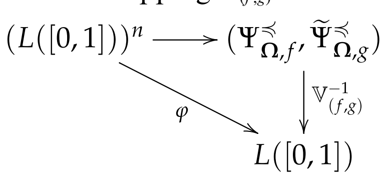
This, together with Eq. (9), implies that

$$\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1,\ldots,\mathbf{x}_n)=\mathsf{OWA}_{\mathbf{\Omega}}(f(\mathbf{x}_1),\ldots,f(\mathbf{x}_n)).\tag{12}$$

According to the monotonicity of the mapping $\mathsf{OWA}_{\mathbf{\Omega}}$, *we have*

$$f(\mathbf{x}_{\sigma(n)})\le\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\le f(\mathbf{x}_{\sigma(1)}).\tag{13}$$

The mappings $\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}$ and $\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}$ can be treated as the OWA operators of f-values and g-values of $\mathbf{x}_1,\ldots,\mathbf{x}_n$ under the admissible order $\preccurlyeq$, respectively. If for any $(\mathbf{x}_1,\ldots,\mathbf{x}_n)\in(L([0,1]))^n$, there exists a point in $L([0,1])$, denoted by $\varphi(\mathbf{x}_1,\ldots,\mathbf{x}_n)$, such that $\mathbb{V}_{(f,g)}(\varphi(\mathbf{x}_1,\ldots,\mathbf{x}_n))=(\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1,\ldots,\mathbf{x}_n),\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1,\ldots,\mathbf{x}_n))$. This means that $\varphi(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ is the inverse image of $(\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1,\ldots,\mathbf{x}_n),\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1,\ldots,\mathbf{x}_n))$ via the mapping $\mathbb{V}_{(f,g)}$ defined by Eq. (5), i.e., the diagram



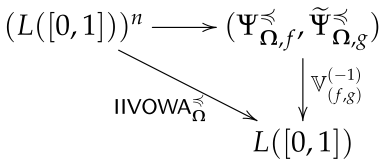
is commutative. It can be verified that the mapping $\varphi:(L([0,1]))^n\longrightarrow L([0,1])$ is increasing under the admissible order $\preccurlyeq$. Then, we can construct the interval-valued OWA operator associated with $\preccurlyeq$ based on the mapping φ. However, unfortunately, such a mapping φ does not always exist, since the inverse mapping $\mathbb{V}^{-1}_{(f,g)}$ of $\mathbb{V}_{(f,g)}$ does not always exist on $\mathbb{R}^2$. To overcome this, we need to construct a function similar to $\mathbb{V}^{-1}_{(f,g)}$ instead of it, in order to obtain a monotonic mapping φ, so that the above diagram is commutative. To achieve this, the pseudo-inverse $\mathbb{V}^{(-1)}_{(f,g)}$ (see Section 3, [26]) provides us with an effective tool as follows.

Definition 7: *Let* $\preccurlyeq$ *be a generated admissible order on* $L([0,1])$ *induced by a generating pair* (f,g), *and* $\mathbf{\Omega}=(\omega_1,\ldots,\omega_n)^\top$ *be the weight vector such that* $\omega_\gamma\in[0,1]$ *and* $\sum_{\gamma=1}^{n}\omega_\gamma=1$. *Define the improved interval-valued OWA (IIVOWA) operator* $\mathsf{IIVOWA}^{\preccurlyeq}_{\mathbf{\Omega}}:(L([0,1]))^n\longrightarrow L([0,1])$, *associated with* $\preccurlyeq$ *and* $\mathbf{\Omega}$, *by*

$$\begin{aligned}\mathsf{IIVOWA}^{\preccurlyeq}_{\mathbf{\Omega}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) &= \mathbb{V}^{(-1)}_{(f,g)}((\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1,\ldots,\mathbf{x}_n),\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1,\ldots,\mathbf{x}_n)))\\ &= \bigvee_{\preccurlyeq}\left\{\mathbf{x}\in L([0,1]) \mid \mathbb{V}_{(f,g)}(\mathbf{x}) \le_{\text{lex1}} (\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1,\ldots,\mathbf{x}_n),\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1,\ldots,\mathbf{x}_n))\right\},\end{aligned} \tag{14}$$

where $\bigvee_{\preccurlyeq}\{\cdot\}$ *denotes the smallest upper bound of the set* $\{\cdot\}$ *under the admissible* $\preccurlyeq$. *According to Lemma 2, this always exists.*

According to the above analysis, Definition A7 can be expressed as the following commutative diagram:



For convenience, denote $\left\{\mathbf{x}\in L([0,1]) \mid \mathbb{V}_{(f,g)}(\mathbf{x}) \le_{\text{lex1}} (\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1,\ldots,\mathbf{x}_n),\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1,\ldots,\mathbf{x}_n))\right\} = \mathscr{T}_{(f,g)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$. When there is no possibility of ambiguity of the values of (f,g) and $(\mathbf{x}_1,\ldots,\mathbf{x}_n)$, the notation $\mathscr{T}_{(f,g)}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$ will be simply denoted by $\mathscr{T}$.

To illustrate the pseudo-inverse construction $\mathbb{V}^{(-1)}_{(f,g)}$ and the practical computation of the proposed operator $\mathsf{IIVOWA}^{\preccurlyeq}_{\mathbf{\Omega}}$, we present the following example.

Example 2 (Example 4, [12]): *Consider the order* $\preccurlyeq$ *generated by* $f(x,y)=\dfrac{\sqrt{x}+\sqrt{y}}{2}$ *and* $g(x,y)=y$. *Let*

$\mathbf{x}_1=[0.25,0.25],\ \mathbf{x}_2=[0,1]$.

Taking $\mathbf{\Omega}=\left(\dfrac{1}{3},\dfrac{2}{3}\right)^{\top}$, *by direct calculation, we have* $\mathbf{x}\preccurlyeq\mathbf{y}$, *and thus*

$$\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1,\mathbf{x}_2)=\omega_1 f(\mathbf{x}_2)+\omega_2 f(\mathbf{x}_1)=\frac{1}{3}\times 0.5+\frac{2}{3}\times 0.5=0.5,$$

and

$$\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1,\mathbf{x}_2)=\omega_1 g(\mathbf{x}_2)+\omega_2 g(\mathbf{x}_1)=\frac{1}{3}\times 1+\frac{2}{3}\times 0.25=0.5.$$

This, together with Definition 7, implies that

$$\mathsf{IIVOWA}^{\preccurlyeq}_{\mathbf{\Omega}}(\mathbf{x},\mathbf{y})=\bigvee_{\preccurlyeq}\left\{[\overline{x},\underline{x}]\in L([0,1]) \mid \left(\frac{\sqrt{\underline{x}}+\sqrt{\overline{x}}}{2},\underline{x}\right)\le_{\text{lex1}}(0.5,0.5)\right\}=[1.5-2\sqrt{0.5},0.5].$$

Proposition 1: *For any* $(\mathbf{x}_1,\ldots,\mathbf{x}_n)\in(L([0,1]))^n$, *we have* $\mathbf{x}_{\sigma(n)}\in\mathscr{T}\neq\varnothing$.

Proof: By $\mathbf{x}_{\sigma(n)}\preccurlyeq\cdots\preccurlyeq\mathbf{x}_{\sigma(2)}\preccurlyeq\mathbf{x}_{\sigma(1)}$, we consider the following two cases:

(1) If $f(\mathbf{x}_{\sigma(1)}) = f(\mathbf{x}_{\sigma(n)})$, then $g(\mathbf{x}_{\sigma(1)}) \geq \cdots \geq g(\mathbf{x}_{\sigma(n)})$. This implies that

$\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = f(\mathbf{x}_{\sigma(n)})$,

and

$\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1, \ldots, \mathbf{x}_n) \geq g(\mathbf{x}_{\sigma(n)})$.

According to the definition of $\leq_{\text{lex1}}$, we have $\mathbb{V}_{(f,g)}(\mathbf{x}_{\sigma(n)}) \leq_{\text{lex1}} (\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1, \ldots, \mathbf{x}_n))$, i.e., $\mathbf{x}_{\sigma(n)} \in \mathscr{T}$.

(2) If $f(\mathbf{x}_{\sigma(1)}) > f(\mathbf{x}_{\sigma(n)})$, then $\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathsf{OWA}_{\mathbf{\Omega}}(f(\mathbf{x}_1), \ldots, f(\mathbf{x}_n)) \geq \mathsf{OWA}_{\mathbf{\Omega}}(f(\mathbf{x}_{\sigma(n)}), \ldots, f(\mathbf{x}_{\sigma(n)})) = \Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_{\sigma(n)}, \ldots, \mathbf{x}_{\sigma(n)}) = f(\mathbf{x}_{\sigma(n)})$ by Remark 2. Consider the following two subcases:

2-1) If $\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n) > f(\mathbf{x}_{\sigma(n)})$, it is clear that $\mathbb{V}_{(f,g)}(\mathbf{x}_{\sigma(n)}) \leq_{\text{lex1}} (\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1, \ldots, \mathbf{x}_n))$, i.e., $\mathbf{x}_{\sigma(n)} \in \mathscr{T}$.

2-2) If $\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = f(\mathbf{x}_{\sigma(n)})$, let us take $\mathbf{N} = \max\{\gamma \mid 1 \leq \gamma \leq n \text{ and } f(\mathbf{x}_{\sigma(\gamma)}) > f(\mathbf{x}_{\sigma(n)})\}$. Then, by Eq. (12), $\omega_\gamma = 0$ for $1 \leq \gamma \leq \mathbf{N}$ and $f(\mathbf{x}_{\sigma(\gamma)}) = f(\mathbf{x}_{\sigma(n)})$ for $\mathbf{N} < \gamma \leq n$. Together with $\mathbf{x}_{\sigma(\mathbf{N}+1)} \succcurlyeq \cdots \succcurlyeq \mathbf{x}_{\sigma(n)}$, we have $g(\mathbf{x}_{\sigma(\mathbf{N}+1)}) \geq \cdots \geq g(\mathbf{x}_{\sigma(n)})$. Thus,

$$\widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{\gamma=1}^{n} \omega_\gamma g(\mathbf{x}_{\sigma(\gamma)}) = \sum_{\gamma=\mathbf{N}+1}^{n} \omega_\gamma g(\mathbf{x}_{\sigma(\gamma)}) \geq \sum_{\gamma=\mathbf{N}+1}^{n} \omega_\gamma g(\mathbf{x}_{\sigma(n)}) = g(\mathbf{x}_{\sigma(n)}).$$

This implies that $\mathbb{V}_{(f,g)}(\mathbf{x}_{\sigma(n)}) \leq_{\text{lex1}} (\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1, \ldots, \mathbf{x}_n))$, i.e., $\mathbf{x}_{\sigma(n)} \in \mathscr{T}$. □

By the completeness of $(L([0,1]), \preccurlyeq)$ (see Lemma 2) and Proposition 1, we know that the right part $\bigvee_{\preccurlyeq} \mathscr{T}$ of Eq. (14) uniquely exists. Thus, the IIVOWA operator in Definition 7 is well-defined. The following theorem indicates that the f-value of the output aggregated data obtained by the IIVOWA operator is equal to the aggregated value of f-values of inputs obtained the OWA operator.

Theorem 1: *For any* $(\mathbf{x}_1, \ldots, \mathbf{x}_n) \in (L([0,1]))^n$, *we have* $f(\mathit{IIVOWA}^{\preccurlyeq}_{\mathbf{\Omega}}(\mathbf{x}_1, \ldots, \mathbf{x}_n)) = \Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathit{OWA}_{\mathbf{\Omega}}(f(\mathbf{x}_1), \ldots, f(\mathbf{x}_n))$.

Proof: For simplicity, denote $\mathsf{IIVOWA}^{\preccurlyeq}_{\mathbf{\Omega}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \boldsymbol{\zeta}$, $f(\mathsf{IIVOWA}^{\preccurlyeq}_{\mathbf{\Omega}}(\mathbf{x}_1, \ldots, \mathbf{x}_n)) = \xi$, and $\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \eta$. Suppose that $\xi \neq \eta$, we consider the following cases:

(1) If $\xi < \eta$, then $[\xi, \eta] \subseteq [\xi, f(\mathbf{x}_{\sigma(1)})] \subseteq [f(0,0), f(1,1)]$ by $\eta \leq f(\mathbf{x}_{\sigma(1)})$ and Lemma 1 (1). Since f is continuous and $\mathbb{K}([0,1])$ is compact and connected, according to [27] (Theorem 4.16 and 4.22), there exists $\boldsymbol{\zeta}' \in L([0,1])$ such that $f(\boldsymbol{\zeta}') = \dfrac{\xi + \eta}{2}$. Then, by the definition of $\leq_{\text{lex1}}$, we have $\mathbb{V}_{(f,g)}(\boldsymbol{\zeta}') \leq_{\text{lex1}} (\Psi^{\preccurlyeq}_{\mathbf{\Omega},f}(\mathbf{x}_1, \ldots, \mathbf{x}_n), \widetilde{\Psi}^{\preccurlyeq}_{\mathbf{\Omega},g}(\mathbf{x}_1, \ldots, \mathbf{x}_n))$, and thus $\boldsymbol{\zeta}' \in \mathscr{T}$. This, together with $\bigvee_{\preccurlyeq} \mathscr{T} = \boldsymbol{\zeta}$, implies that $\boldsymbol{\zeta}' \preccurlyeq \boldsymbol{\zeta}$. This is a contradiction due to $\boldsymbol{\zeta} \prec \boldsymbol{\zeta}'$ by $f(\boldsymbol{\zeta}) < f(\boldsymbol{\zeta}')$.

(2) If $\xi > \eta$, then $[\eta, \xi] \subseteq [f(\mathbf{x}_{\sigma(n)}), \xi] \subseteq [f(0,0), f(1,1)]$ by $\eta \geq f(\mathbf{x}_{\sigma(n)})$ and Lemma 1 (1). Since f is continuous and $\mathbb{K}([0,1])$ is compact and connected, according to [27] (Theorem 4.16 and 4.22), there exists $\tilde{\boldsymbol{\zeta}}' \in L([0,1])$ such that $f(\tilde{\boldsymbol{\zeta}}') = \dfrac{\xi + \eta}{2}$. Then, by the definition of $\leq_{\text{lex1}}$, we have that, for any $\mathbf{x} \in \mathscr{T}$, $f(\mathbf{x}) \leq \eta < f(\tilde{\boldsymbol{\zeta}}')$, i.e., $\tilde{\boldsymbol{\zeta}}'$ is an upper bound. This, together with $\bigvee_{\preccurlyeq} \mathscr{T} = \boldsymbol{\zeta}$, implies that $\boldsymbol{\zeta} \preccurlyeq \tilde{\boldsymbol{\zeta}}'$. This is a contradiction due to $\tilde{\boldsymbol{\zeta}}' \prec \boldsymbol{\zeta}$ by $f(\tilde{\boldsymbol{\zeta}}') < \xi = f(\boldsymbol{\zeta})$.

Summing up (1) and (2), $f(\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)) = \xi = \eta = \Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$. □

Proposition 2: *If there exists* $\mathbf{x} \in L([0,1])$ *such that* $\mathbb{V}_{(f,g)}(\mathbf{x}) = (\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n), \widetilde{\Psi}_{\mathbf{\Omega},g}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n))$, *then* $\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \mathbf{x}$.

Proof: It follows from Remark 1 and $\mathbb{V}_{(f,g)}(\mathbf{x}) = (\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n), \widetilde{\Psi}_{\mathbf{\Omega},g}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n))$ that $\mathbf{x} = \max \mathscr{T}$, and thus $\mathbf{x} = \bigvee_{\preccurlyeq} \mathscr{T} = \mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)$. □

Corollary 1: *For* $\alpha, \beta \in [0,1]$ *with* $\alpha \neq \beta$, *we have*

$$\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq_{\alpha,\beta}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \left[\sum_{\gamma=1}^{n} \omega_\gamma \underline{x_{\sigma(\gamma)}}, \sum_{\gamma=1}^{n} \omega_\gamma \overline{x_{\sigma(\gamma)}}\right],$$

where $(\sigma(1),\ldots,\sigma(n))$ *is a permutation of* $(1,\ldots,n)$ *such that* $\mathbf{x}_{\sigma(n)} \preccurlyeq_{\alpha,\beta} \mathbf{x}_{\sigma(n-1)} \preccurlyeq_{\alpha,\beta} \cdots \preccurlyeq_{\alpha,\beta} \mathbf{x}_{\sigma(1)}$ *and* $\mathbf{x}_\gamma = [\underline{x_\gamma}, \overline{x_\gamma}]$ *for* $1 \leq \gamma \leq n$.

Proof: Denote $\left[\sum_{\gamma=1}^{n} \omega_\gamma \underline{x_{\sigma(\gamma)}}, \sum_{\gamma=1}^{n} \omega_\gamma \overline{x_{\sigma(\gamma)}}\right] = [\underline{x}, \overline{x}] = \mathbf{x}$. By direct calculation, we have

$$\begin{aligned} K_\alpha(\mathbf{x}) &= \alpha \sum_{\gamma=1}^{n} \omega_\gamma \overline{x_{\sigma(\gamma)}} + (1-\alpha) \sum_{\gamma=1}^{n} \omega_\gamma \underline{x_{\sigma(\gamma)}} \\ &= \sum_{\gamma=1}^{n} \omega_\gamma (\alpha \cdot \overline{x_{\sigma(\gamma)}} + (1-\alpha) \cdot \underline{x_{\sigma(\gamma)}}) \\ &= \sum_{\gamma=1}^{n} \omega_\gamma \cdot K_\alpha(\mathbf{x}_{\sigma(\gamma)}) \\ &= \Psi_{\mathbf{\Omega},K_\alpha}^{\preccurlyeq_{\alpha,\beta}}(\mathbf{x}_1,\ldots,\mathbf{x}_n), \end{aligned}$$

and

$$\begin{aligned} K_\beta(\mathbf{x}) &= \beta \sum_{\gamma=1}^{n} \omega_\gamma \overline{x_{\sigma(\gamma)}} + (1-\beta) \sum_{\gamma=1}^{n} \omega_\gamma \underline{x_{\sigma(\gamma)}} \\ &= \sum_{\gamma=1}^{n} \omega_\gamma (\beta \cdot \overline{x_{\sigma(\gamma)}} + (1-\beta) \cdot \underline{x_{\sigma(\gamma)}}) \\ &= \sum_{\gamma=1}^{n} \omega_\gamma \cdot K_\beta(\mathbf{x}_{\sigma(\gamma)}) \\ &= \widetilde{\Psi}_{\mathbf{\Omega},K_\beta}^{\preccurlyeq_{\alpha,\beta}}(\mathbf{x}_1,\ldots,\mathbf{x}_n), \end{aligned}$$

implying that $\mathbb{V}_{(K_\alpha,K_\beta)}(\mathbf{x}) = (\Psi_{\mathbf{\Omega},K_\alpha}^{\preccurlyeq_{\alpha,\beta}}(\mathbf{x}_1,\ldots,\mathbf{x}_n), \widetilde{\Psi}_{\mathbf{\Omega},K_\beta}^{\preccurlyeq_{\alpha,\beta}}(\mathbf{x}_1,\ldots,\mathbf{x}_n))$. Thus, $\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq_{\alpha,\beta}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \mathbf{x}$ by Proposition 2. □

The mapping $\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}$ has the following desirable properties.

Property 1: *Let $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g), and $\mathbf{\Omega} = (\omega_1, \ldots, \omega_n)^\top$ be the weight vector such that $\omega_\gamma \in [0,1]$ and $\sum_{\gamma=1}^{n} \omega_\gamma = 1$. If f and g are idempotent, then,*

$$\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}([x_1,x_1],\ldots,[x_n,x_n]) = \mathsf{OWA}_{\mathbf{\Omega}}(x_1,\ldots,x_n).$$

In particular,

$$\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq_{\alpha,\beta}}([x_1,x_1],\ldots,[x_n,x_n]) = \mathsf{OWA}_{\mathbf{\Omega}}(x_1,\ldots,x_n),$$

for α, $\beta \in [0,1]$ with $\alpha \neq \beta$.

Proof: Let $(\sigma(1),\ldots,\sigma(n))$ be a permutation of $(1,\ldots,n)$ such that $x_{\sigma(n)} \leq \cdots \leq x_{\sigma(2)} \leq x_{\sigma(1)}$. Then, $\mathbf{x}_{\sigma(n)} \preccurlyeq \cdots \preccurlyeq \mathbf{x}_{\sigma(2)} \preccurlyeq \mathbf{x}_{\sigma(1)}$. Since f and g are idempotent, by direct calculation, we have

$$\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum_{\gamma=1}^{n} \omega_\gamma f(\mathbf{x}_{\sigma(\gamma)}) = \sum_{\gamma=1}^{n} \omega_\gamma x_{\sigma(\gamma)} = \mathsf{OWA}_{\mathbf{\Omega}}(x_1,\ldots,x_n),$$

and

$$\widetilde{\Psi}_{\mathbf{\Omega},g}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum_{\gamma=1}^{n} \omega_\gamma g(\mathbf{x}_{\sigma(\gamma)}) = \sum_{\gamma=1}^{n} \omega_\gamma x_{\sigma(\gamma)} = \mathsf{OWA}_{\mathbf{\Omega}}(x_1,\ldots,x_n).$$

This, together with Proposition 2, implies that $\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}([x_1,x_1],\ldots,[x_n,x_n]) = [\mathsf{OWA}_{\mathbf{\Omega}}(x_1,\ldots,x_n), \mathsf{OWA}_{\mathbf{\Omega}}(x_1,\ldots,x_n)] = \mathsf{OWA}_{\mathbf{\Omega}}(x_1,\ldots,x_n)$. □

Based on [17] (Proposition 3.5), we know that each generated admissible order induced by a generating pair (f,g) of continuous aggregation functions is equal to an admissible order generated by two idempotent and continuous aggregation functions. This shows that the assumption of idempotency for f and g does not lose its generality. Meanwhile, Proposition 1 indicates that our proposed IIVOWA operator is indeed an extension of the original OWA operator introduced by Yager [1].

Property 2: *Let $\mathbf{\Omega} = (\omega_1,\ldots,\omega_n)^\top$ be the weight vector such that $\omega_\gamma \in [0,1]$ and $\sum_{\gamma=1}^{n} \omega_\gamma = 1$. Then, for α, $\beta \in [0,1]$ with $\alpha \neq \beta$,*

$$\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq_{\alpha,\beta}}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \mathsf{IVOWA}_{\mathbf{\Omega}}^{\preccurlyeq_{\alpha,\beta}}(\mathbf{x}_1,\ldots,\mathbf{x}_n).$$

Proof: It follows directly from Definition 5 and Corollary 1. □

Property 3 (Idempotency): *Let $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g), and $\mathbf{\Omega} = (\omega_1,\ldots,\omega_n)^\top$ be the weight vector such that $\omega_j \in [0,1]$ and $\sum_{j=1}^{n} \omega_j = 1$. If $\mathbf{x}_1 = \cdots = \mathbf{x}_n = \mathbf{x} \in L([0,1])$, then $\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \mathbf{x}$.*

Proof: By direct calculation, we have

$$\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum_{\gamma=1}^{n} \omega_\gamma f(\mathbf{x}_{\sigma(\gamma)}) = f(\mathbf{x}),$$

and

$$\widetilde{\Psi}_{\mathbf{\Omega},g}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \sum_{\gamma=1}^{n} \omega_\gamma g(\mathbf{x}_{\sigma(\gamma)}) = g(\mathbf{x}),$$

and thus $\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n) = \mathbf{x}$ by Proposition 2. □

Property 4 (Monotonicity): Let $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g), and $\mathbf{\Omega}=(\omega_1,\ldots,\omega_n)^\top$ be the weight vector such that $\omega_\gamma\in[0,1]$ and $\sum_{\gamma=1}^n\omega_\gamma=1$. If $\mathbf{x}_\gamma\preccurlyeq\mathbf{y}_\gamma$ for $1\le\gamma\le n$, then

$\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\preccurlyeq\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{y}_1,\ldots,\mathbf{y}_n)$.

Proof: For simplicity, denote $\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)=\mathbf{x}$, $\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{y}_1,\ldots,\mathbf{y}_n)=\mathbf{y}$, and $f(\mathbf{x})=\xi$, $f(\mathbf{y})=\eta$. By the definition of the order $\preccurlyeq$ and $\mathbf{x}_\gamma\preccurlyeq\mathbf{y}_\gamma$, we get $f(\mathbf{x}_\gamma)\le f(\mathbf{y}_\gamma)$ for $1\le\gamma\le n$. Then, $\xi=\mathsf{OWA}_{\mathbf{\Omega}}(f(\mathbf{x}_1),\ldots,f(\mathbf{x}_n))\le\mathsf{OWA}_{\mathbf{\Omega}}(f(\mathbf{y}_1),\ldots,f(\mathbf{y}_n))=\eta$ by Theorem 1. Without loss of generality, assume that $\mathsf{OWA}_{\mathbf{\Omega}}(f(\mathbf{x}_1),\ldots,f(\mathbf{x}_n))=\sum_{\gamma=1}^n\omega_\gamma f(\mathbf{x}_{\sigma(\gamma)})$ and $\mathsf{OWA}_{\mathbf{\Omega}}(f(\mathbf{y}_1),\ldots,f(\mathbf{y}_n))=\sum_{\gamma=1}^n\omega_\gamma f(\mathbf{y}_{\sigma'(\gamma)})$. By $\xi\le\eta$, we consider the following two cases:

(1) If $\xi<\eta$, it is clear that $\mathbf{x}\prec\mathbf{y}$.

(2) If $\xi=\eta$, i.e., $\eta-\xi=\mathsf{OWA}_{\mathbf{\Omega}}(f(\mathbf{y}_1),\ldots,f(\mathbf{y}_n))-\mathsf{OWA}_{\mathbf{\Omega}}(f(\mathbf{x}_1),\ldots,f(\mathbf{x}_n))=\sum_{\gamma=1}^n\omega_\gamma(f(\mathbf{y}_{\sigma'(\gamma)})-f(\mathbf{x}_{\sigma(\gamma)}))=0$, noting that $f(\mathbf{y}_{\sigma'(j)})\ge f(\mathbf{x}_{\sigma(j)})$, we have $\omega_\gamma=0$ for $f(\mathbf{y}_{\sigma'(\gamma)})>f(\mathbf{x}_{\sigma(\gamma)})$. Let $\Lambda=\{\gamma\in\{1,2,\ldots,n\}\mid f(\mathbf{y}_{\sigma'(\gamma)})>f(\mathbf{x}_{\sigma(\gamma)})\}$ and $\Lambda^c=\{1,2,\ldots,n\}\backslash\Lambda$. Clearly, for any $\gamma\in\Lambda^c$, $f(\mathbf{y}_{\sigma'(\gamma)})=f(\mathbf{x}_{\sigma(\gamma)})$ and $\mathbf{x}_{\sigma(\gamma)}\preccurlyeq\mathbf{y}_{\sigma'(\gamma)}$. This implies that, for any $\gamma\in\Lambda^c$, $g(\mathbf{x}_{\sigma(\gamma)})\le g(\mathbf{y}_{\sigma'(\gamma)})$. Then,

$$\begin{aligned}\widetilde{\Psi}_{\mathbf{\Omega},g}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)&=\sum_{\gamma=1}^n\omega_\gamma g(\mathbf{x}_{\sigma(\gamma)})=\sum_{\gamma\in\Lambda}\omega_\gamma g(\mathbf{x}_{\sigma(\gamma)})+\sum_{\gamma\in\Lambda^c}\omega_\gamma g(\mathbf{x}_{\sigma(\gamma)})=\sum_{\gamma\in\Lambda^c}\omega_\gamma g(\mathbf{x}_{\sigma(\gamma)})\\&\le\sum_{\gamma\in\Lambda^c}\omega_\gamma g(\mathbf{y}_{\sigma'(\gamma)})=\sum_{\gamma\in\Lambda}\omega_\gamma g(\mathbf{y}_{\sigma'(\gamma)})+\sum_{\gamma\in\Lambda^c}\omega_\gamma g(\mathbf{y}_{\sigma'(\gamma)})=\sum_{\gamma=1}^n\omega_\gamma g(\mathbf{y}_{\sigma'(\gamma)})\\&=\widetilde{\Psi}_{\mathbf{\Omega},g}^{\preccurlyeq}(\mathbf{y}_1,\ldots,\mathbf{y}_n).\end{aligned}$$

This, together with $\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)=\xi=\eta=\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{y}_1,\ldots,\mathbf{y}_n)$ by Theorem 1, yields that $\mathscr{T}_1\subseteq\mathscr{T}_2$, where $\mathscr{T}_1=\left\{\tilde{\mathbf{x}}\in L([0,1])\mid\mathbb{V}_{(f,g)}(\tilde{\mathbf{x}})\le_{\text{lex1}}(\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n),\widetilde{\Psi}_{\mathbf{\Omega},g}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n))\right\}$ and $\mathscr{T}_2=\left\{\tilde{\mathbf{x}}\in L([0,1])\mid\mathbb{V}_{(f,g)}(\tilde{\mathbf{x}})\le_{\text{lex1}}(\Psi_{\mathbf{\Omega},f}^{\preccurlyeq}(\mathbf{y}_1,\ldots,\mathbf{y}_n),\widetilde{\Psi}_{\mathbf{\Omega},g}^{\preccurlyeq}(\mathbf{y}_1,\ldots,\mathbf{y}_n))\right\}$, and thus $\mathbf{x}=\bigvee_{\preccurlyeq}\mathscr{T}_1\preccurlyeq\bigvee_{\preccurlyeq}\mathscr{T}_2=\mathbf{y}$. Summing up (1) and (2), $\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\preccurlyeq\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{y}_1,\ldots,\mathbf{y}_n)$. □

Property 5 (Boundedness): *Let $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g), and $\mathbf{\Omega}=(\omega_1,\ldots,\omega_n)^\top$ be the weight vector such that $\omega_\gamma\in[0,1]$ and $\sum_{\gamma=1}^n\omega_\gamma=1$. Then, $\min\{\mathbf{x}_\gamma\mid 1\le\gamma\le n\}\preccurlyeq\mathit{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\preccurlyeq\max\{\mathbf{x}_\gamma\mid 1\le\gamma\le n\}$.*

Proof: For simplicity, denote $\min\{\mathbf{x}_\gamma\mid 1\le\gamma\le n\}=\xi$ and $\max\{\mathbf{x}_\gamma\mid 1\le\gamma\le n\}=\boldsymbol{\zeta}$. Clearly, $\xi\preccurlyeq\mathbf{x}_\gamma\preccurlyeq\boldsymbol{\zeta}$ holds for all $1\le\gamma\le n$. Combining with Properties 3 and 4, we have $\xi=\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\xi,\ldots,\xi)\preccurlyeq\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\mathbf{x}_1,\ldots,\mathbf{x}_n)\preccurlyeq\mathsf{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}(\boldsymbol{\zeta},\ldots,\boldsymbol{\zeta})=\boldsymbol{\zeta}$. □

Theorem 2: *Let $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g), and $\mathbf{\Omega}=(\omega_1,\ldots,\omega_n)^\top$ be the weight vector such that $\omega_\gamma\in[0,1]$ and $\sum_{\gamma=1}^n\omega_\gamma=1$. Then, the mapping $\mathit{IIVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}$ is an aggregation function on $(L([0,1]),\preccurlyeq)$.*

Proof: It follows directly from Properties 3 and 4. □

Directly form Properties 2 and 4, it follows that [12] (Corollary 1) holds trivially.

Example 3 (Example 4, [12]): *Consider the order* $\preccurlyeq$ *generated by* $f(x,y) = \dfrac{\sqrt{x}+\sqrt{y}}{2}$ *and* $g(x,y) = y$. *Let*

$\mathbf{x} = [0.25, 0.25],\ \mathbf{y} = [0, 1],\ \mathbf{z} = [0.25, 0.28].$

Taking $\boldsymbol{\Omega} = \left(\frac{1}{3}, \frac{2}{3}\right)^{\top}$, *by direct calculation and Proposition 2, we have* $\mathbf{x} \preccurlyeq \mathbf{y} \preccurlyeq \mathbf{z}$ *and*

$$\mathsf{IIVOWA}_{\boldsymbol{\Omega}}^{\preccurlyeq}(\mathbf{x},\mathbf{y}) = [1.5 - 2\sqrt{0.5}, 0.5] \approx [0.0858, 0.5],$$

$$\mathsf{IIVOWA}_{\boldsymbol{\Omega}}^{\preccurlyeq}(\mathbf{z},\mathbf{y}) = \left[\left(\frac{2.5+\sqrt{0.28}}{3} - \sqrt{0.76}\right)^2, 0.76\right] \approx [0.0190, 0.76].$$

Then, $\mathsf{IIVOWA}_{\boldsymbol{\Omega}}^{\preccurlyeq}(\mathbf{x},\mathbf{y}) \prec \mathsf{IIVOWA}_{\boldsymbol{\Omega}}^{\preccurlyeq}(\mathbf{z},\mathbf{y})$ *by* $f(\mathsf{IIVOWA}_{\boldsymbol{\Omega}}^{\preccurlyeq}(\mathbf{x},\mathbf{y})) = 0.5 < \dfrac{2.5+\sqrt{0.28}}{6} = f(\mathsf{IIVOWA}_{\boldsymbol{\Omega}}^{\preccurlyeq}(\mathbf{z},\mathbf{y}))$. *This example further confirms Property 4 and demonstrates the monotonicity of IIVOWA. Meanwhile, according to [12] (Example 4), this example also shows that the IVOWA introduced by Bustince et al. [12] does not have the monotonicity.*

4 Monotonic Interval-Valued Choquet Integrals

Since Choquet [2] defined an integration operation, called *Choquet integral* later, by using the non-additive "measures", this type of integral has been extended to various settings, such as set-valued functions [18], interval-valued fuzzy [12,15,19,20], intuitionistic fuzzy [28–31], interval-valued intuitionistic fuzzy [32], and interval-valued intuitionistic hesitant fuzzy [33]. However, all of these extensions fail to inherit an essential property of Choquet integral–"monotonicity". In view of this, this section propose a general construction of IV Choquet integral based on the generated admissible orders and prove that it satisfies various aggregate operator properties, especially monotonicity.

Definition 8 ([2,34]): *Let* $X = \{x_1, \ldots, x_n\}$ *and* 2^X *be the power set of* X. *A mapping* $\boldsymbol{\lambda} : 2^X \longrightarrow [0,1]$ *is called a fuzzy measure on* X, *if it satisfies the following properties:*

(M1) $\boldsymbol{\lambda}(\varnothing) = 0$ *and* $\boldsymbol{\lambda}(X) = 1$;

(M2) If $A, B \in 2^X$ *and* $A \subseteq B$, *then* $\boldsymbol{\lambda}(A) \leq \boldsymbol{\lambda}(B)$.

A fuzzy measure $\boldsymbol{\lambda} : 2^X \longrightarrow [0,1]$ *is said to be strictly monotonic if it satisfies the following Property (M3):*

(M3) If $A, B \in 2^X$ *and* $A \subsetneqq B$, *then* $\boldsymbol{\lambda}(A) < \boldsymbol{\lambda}(B)$.

For a fuzzy measure $\boldsymbol{\lambda}$ on $X = \{x_1, \ldots, x_n\}$, we define the *gap* $G(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$ by

$$G(\boldsymbol{\lambda}) = \min\{\boldsymbol{\lambda}(B) - \boldsymbol{\lambda}(A) \mid A \subsetneqq B \subseteq X\}. \tag{15}$$

Clearly, the fuzzy measure $\boldsymbol{\lambda}$ is strictly monotonic if and only if $G(\boldsymbol{\lambda}) > 0$.

Let $X = \{x_1, \ldots, x_n\}$ and $\boldsymbol{\lambda} : 2^X \longrightarrow [0, 1]$ be a fuzzy measure. The *discrete Choquet integral* $\mathbf{C}_\lambda(f)$ of a fuzzy set $f : X \longrightarrow [0, 1]$ with respect to $\boldsymbol{\lambda}$ is defined by

$$\mathbf{C}_\lambda(f) = \sum_{\gamma=1}^{n} f(x_{\sigma(\gamma)})(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})), \tag{16}$$

where $(\sigma(1), \ldots, \sigma(n))$ is a permutation of $(1, \ldots, n)$ such that $f(x_{\sigma(1)}) \leq f(x_{\sigma(2)}) \leq \cdots \leq f(x_{\sigma(n)})$ and $X_{\sigma(\gamma)} = \{x_{\sigma(\gamma)}, \ldots, x_{\sigma(n)}\}$, $X_{\sigma(n+1)} = \varnothing$.

From [2] (see also [35]), the discrete Choquet integral $\mathbf{C}_\lambda(f)$ has the following Riemann integral expression:

$$\mathbf{C}_\lambda(f) = \int_0^{+\infty} \boldsymbol{\lambda}(\{x \in X \mid f(x) \geq \alpha\})\mathrm{d}\alpha. \tag{17}$$

For two fuzzy sets f_1 and f_2 on X with $f_1 \leq f_2$, it is clear that $\boldsymbol{\lambda}(\{x \in X \mid f_1(x) \geq \alpha\}) \leq \boldsymbol{\lambda}(\{x \in X \mid f_2(x) \geq \alpha\})$. According to Eq. (17), we have $\mathbf{C}_\lambda(f_1) \leq \mathbf{C}_\lambda(f_2)$. This means that the discrete Choquet integral $\mathbf{C}_\lambda(\cdot)$ has the property of monotonicity. Next, we shall further investigate the monotonicity of $\mathbf{C}_\lambda(\cdot)$.

Proposition 3: *Let $X = \{x_1, \ldots, x_n\}$ and $\boldsymbol{\lambda} : 2^X \longrightarrow [0, 1]$ be a fuzzy measure. Then, the following statements are equivalent:*

(1) *For any fuzzy sets f_1, f_2 on X with $f_1 \leq f_2$ and $f_1 \neq f_2$, $\mathbf{C}_\lambda(f_1) < \mathbf{C}_\lambda(f_2)$;*

(2) *The fuzzy measure $\boldsymbol{\lambda}$ is strictly monotonic.*

Proof: (2)$\Longrightarrow$(1). Let f_1 and f_2 be two fuzzy sets on X satisfying that $f_1 \leq f_2$ and $f_1 \neq f_2$. Then, there exists some $x_\ell \in X$ such that $f_1(x_\ell) < f_2(x_\ell)$ and $f_1(x_j) \leq f_2(x_j)$ for $j \neq \ell$. Noting that, for any $\alpha \in (f_1(x_\ell), f_2(x_\ell)]$,

$$x_\ell \notin \{x \in X \mid f_1(x) \geq \alpha\} \subseteq \{x \in X \mid f_2(x) \geq \alpha\} \ni x_\ell,$$

i.e., $\{x \in X \mid f_1(x) \geq \alpha\} \subsetneqq \{x \in X \mid f_2(x) \geq \alpha\}$, since $\boldsymbol{\lambda}$ is strictly monotonic, we have

$$\boldsymbol{\lambda}(\{x \in X \mid f_2(x) \geq \alpha\}) - \boldsymbol{\lambda}(\{x \in X \mid f_1(x) \geq \alpha\}) \geq G(\boldsymbol{\lambda}) > 0.$$

Together with Eq. (17) and $\boldsymbol{\lambda}(\{x \in X \mid f_1(x) \geq \alpha\}) \leq \boldsymbol{\lambda}(\{x \in X \mid f_2(x) \geq \alpha\})$ for $\alpha \in [0, +\infty)$, we get

$$\begin{aligned}\mathbf{C}_\lambda(f_2) - \mathbf{C}_\lambda(f_1) &\geq \int_{f_1(x_\ell)}^{f_2(x_\ell)} (\boldsymbol{\lambda}(\{x \in X \mid f_2(x) \geq \alpha\}) - \boldsymbol{\lambda}(\{x \in X \mid f_1(x) \geq \alpha\}))\mathrm{d}\alpha \\ &\geq \int_{f_1(x_\ell)}^{f_2(x_\ell)} G(\boldsymbol{\lambda})\mathrm{d}\alpha = (f_2(x_\ell) - f_1(x_\ell)) \cdot G(\boldsymbol{\lambda}) > 0,\end{aligned}$$

i.e., $\mathbf{C}_\lambda(f_1) < \mathbf{C}_\lambda(f_2)$.

(1)$\Longrightarrow$(2). Suppose, on the contrary, that $\boldsymbol{\lambda}$ is not strictly monotonic, i.e., there exist two subsets A and B of X such that $A \subsetneqq B$ and $\boldsymbol{\lambda}(A) = \boldsymbol{\lambda}(B)$. Let us take

$$f_1(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in X \backslash A, \end{cases}$$

and

$$f_2(x) = \begin{cases} 1, & x \in B, \\ 0, & x \in X \backslash B. \end{cases}$$

Clearly, $f_1 \leq f_2$ and $f_1 \neq f_2$ by $A \subsetneq B$. By direct calculation, we have $\mathbf{C}_\lambda(f_1) = \boldsymbol{\lambda}(A) = \boldsymbol{\lambda}(B) = \mathbf{C}_\lambda(f_2)$, which is a contradiction. □

Based on the admissible order, Bustince et al. [12] introduced the following form of interval-valued Choquet integral.

Definition 9 (Definition 4, [12]): *Let $\boldsymbol{\lambda} : 2^X \longrightarrow [0,1]$ be a fuzzy measure on $X = \{x_1, \ldots, x_n\}$ and $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g). Define the interval-valued Choquet integral (IVCI) of $F \in IVF(X)$, denoted by $\mathbf{C}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F)$, with respect to the order $\preccurlyeq$ and the fuzzy measure $\boldsymbol{\lambda}$, by*

$$\mathbf{C}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F) = \sum_{\gamma=1}^{n} F(x_{\sigma(\gamma)})(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})), \tag{18}$$

where $(\sigma(1), \ldots, \sigma(n))$ is a permutation of $(1, \ldots, n)$ such that $F(x_{\sigma(1)}) \preccurlyeq F(x_{\sigma(2)}) \preccurlyeq \cdots \preccurlyeq F(x_{\sigma(n)})$ and $X_{\sigma(\gamma)} = \{x_{\sigma(\gamma)}, \ldots, x_{\sigma(n)}\}$, $X_{\sigma(n+1)} = \varnothing$, and $IVF(X)$ is the set of all interval-valued fuzzy sets on X.

As an attempt to extend the usual real-valued Choquet integral, the IVCI defined in Definition 9 inherits some good properties (Proposition 6, [12]), as Bustince et al. (Example 4, [12]) pointed out, this operator does not have the monotonicity. To overcome this drawback, similarly to IIVOWA operator in Section 3, we provide a general construction method for IVCI, which we call the improved IVCI, satisfying the monotonicity under the generated admissible orders and the axiomatic definition of aggregation operators.

Definition 10: *Let $\boldsymbol{\lambda} : 2^X \longrightarrow [0,1]$ be a fuzzy measure on $X = \{x_1, \ldots, x_n\}$ and $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g). Define the mappings $\mathbf{C}_{\boldsymbol{\lambda},f}^{\preccurlyeq}$ and $\tilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}$: $IVF(X) \longrightarrow \mathbb{R}$ by*

$$\mathbf{C}_{\boldsymbol{\lambda},f}^{\preccurlyeq}(F) = \sum_{\gamma=1}^{n} f(F(x_{\sigma(\gamma)}))(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})), \tag{19}$$

and

$$\widetilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}(F) = \sum_{\gamma=1}^{n} g(F(x_{\sigma(\gamma)}))(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})), \tag{20}$$

respectively, where $(\sigma(1), \ldots, \sigma(n))$ is a permutation of $(1, \ldots, n)$ such that $F(x_{\sigma(1)}) \preccurlyeq F(x_{\sigma(2)}) \preccurlyeq \cdots \preccurlyeq F(x_{\sigma(n)})$ and $X_{\sigma(\gamma)} = \{x_{\sigma(\gamma)}, \ldots, x_{\sigma(n)}\}$, $X_{\sigma(n+1)} = \varnothing$.

Remark 3: *By the definition of $\preccurlyeq$, we know that*

$$f(F(x_{\sigma(1)})) \leq f(F(x_{\sigma(2)})) \leq \cdots \leq f(F(x_{\sigma(n)})). \tag{21}$$

This, together with Eq. (19), implies that

$$\mathbf{C}^{\preccurlyeq}_{\boldsymbol{\lambda},f}(F) = \mathbf{C}_{\boldsymbol{\lambda}}(f \circ F), \tag{22}$$

i.e., $\mathbf{C}^{\preccurlyeq}_{\boldsymbol{\lambda},f}(F)$ *is equal to the discrete Choquet integral of the fuzzy set* $f \circ F$ *with respect to* $\boldsymbol{\lambda}$.

The mappings $\mathbf{C}^{\preccurlyeq}_{\boldsymbol{\lambda},f}(F)$ and $\tilde{\mathbf{C}}^{\preccurlyeq}_{\boldsymbol{\lambda},g}(F)$ can be treated as the the discrete Choquet integral of the fuzzy sets $f \circ F$ and $g \circ F$ under the admissible order $\preccurlyeq$, respectively. Similarly to Definition 7, we introduce the following improved interval-valued Choquet integral $\mathbf{IC}^{\preccurlyeq}_{\boldsymbol{\lambda}}(\cdot)$ to ensure that the diagram

$$\begin{array}{ccc} \mathrm{IVF}(X) & \longrightarrow & (\mathbf{C}^{\preccurlyeq}_{\lambda,f}, \tilde{\mathbf{C}}^{\preccurlyeq}_{\lambda,g}) \\ & \searrow^{\mathbf{IC}^{\preccurlyeq}_{\lambda}} & \downarrow \mathbb{V}^{(-1)}_{(f,g)} \\ & & L([0,1]) \end{array}$$

is commutative.

Definition 11: *Let* $F : X \longrightarrow L([0,1])$ *be an interval-valued fuzzy set on* $X = \{x_1, \ldots, x_n\}$ *and* $\boldsymbol{\lambda} : 2^X \longrightarrow [0,1]$ *be a fuzzy measure. For a generated admissible order* $\preccurlyeq$ *on* $L([0,1])$ *induced by a generating pair* (f, g)*, define the improved interval-valued Choquet integral of* F*, denoted by* $\mathbf{IC}^{\preccurlyeq}_{\boldsymbol{\lambda}}(F)$*, with respect to the order* $\preccurlyeq$ *and the fuzzy measure* $\boldsymbol{\lambda}$*, by*

$$\begin{aligned} \mathbf{IC}^{\preccurlyeq}_{\boldsymbol{\lambda}}(F) &= \mathbb{V}^{(-1)}_{(f,g)}((\mathbf{C}^{\preccurlyeq}_{\boldsymbol{\lambda},f}(F), \widetilde{\mathbf{C}}^{\preccurlyeq}_{\boldsymbol{\lambda},g}(F))) \\ &= \bigvee_{\preccurlyeq} \left\{ \mathbf{x} \in L([0,1]) \mid \mathbb{V}_{(f,g)}(\mathbf{x}) \leq_{\mathrm{lex1}} (\mathbf{C}^{\preccurlyeq}_{\boldsymbol{\lambda},f}(F), \widetilde{\mathbf{C}}^{\preccurlyeq}_{\boldsymbol{\lambda},g}(F)) \right\}, \end{aligned} \tag{23}$$

where $\bigvee_{\preccurlyeq}\{\cdot\}$ *denotes the smallest upper bound of the set* $\{\cdot\}$ *under the admissible* $\preccurlyeq$*. According to Lemma 2, this always exists.*

Given a weighted vector $\boldsymbol{\Omega} = (\omega_1, \ldots, \omega_n)^\top \in [0,1]^n$ with $\sum_{\gamma=1}^{n} \omega_\gamma = 1$, define the fuzzy measure $\boldsymbol{\lambda}_{\boldsymbol{\Omega}}$ associated with $\boldsymbol{\Omega}$ by

$$\boldsymbol{\lambda}_{\boldsymbol{\Omega}}(A) = \begin{cases} \sum_{\gamma=1}^{k} \omega_\gamma, & |A| = k, \\ 0, & A = \varnothing. \end{cases}$$

It is easy to see that, for any $F \in \mathrm{IVF}(X)$, $\mathbf{IC}^{\preccurlyeq}_{\boldsymbol{\lambda}_{\boldsymbol{\Omega}}}(F) = \mathsf{IIVOWA}^{\preccurlyeq}_{\boldsymbol{\Omega}}(F(x_1), \ldots, F(x_n))$. This means that the IIVOWA operator proposed in previous section is a special form of improved interval-valued Choquet integral.

For convenience, denote $\left\{\mathbf{x} \in L([0,1]) \mid \mathbb{V}_{(f,g)}(\mathbf{x}) \leq_{\mathrm{lex1}} (\mathbf{C}^{\preccurlyeq}_{\boldsymbol{\lambda},f}(F), \widetilde{\mathbf{C}}^{\preccurlyeq}_{\boldsymbol{\lambda},g}(F))\right\} = \tilde{\mathscr{T}}(F)$.

Proposition 4: *For any interval-valued fuzzy set* F*, we have* $F(x_{\sigma(1)}) \in \tilde{\mathscr{T}}(F) \neq \varnothing$.

Proof: Similar to the proof of Proposition 1, we only need to replace the symbols $\Psi^{\preccurlyeq}_{\boldsymbol{\Omega},f}$ and $\widetilde{\Psi}^{\preccurlyeq}_{\boldsymbol{\Omega},g}$ in Proposition 1 with $\mathbf{C}^{\preccurlyeq}_{\boldsymbol{\lambda},f}$ and $\widetilde{\mathbf{C}}^{\preccurlyeq}_{\boldsymbol{\lambda},g}$, respectively. □

By the completeness of $(L([0,1]),\preccurlyeq)$ (see Lemma 2) and Proposition 4, we know that the right part $\bigvee_{\preccurlyeq}\tilde{\mathscr{T}}(F)$ of Eq. (23) uniquely exists. Thus, the improved IVCI in Definition 11 is well-defined.

Similarly to the proof of Theorem 1, by Remark 3, we have the following result.

Theorem 3: *For any interval-valued fuzzy set F, we have $f(\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F)) = \mathbf{C}_{\boldsymbol{\lambda},f}^{\preccurlyeq}(F) = \mathbf{C}_{\boldsymbol{\lambda}}(f\circ F)$.*

Proposition 5: *If there exists $\mathbf{x}\in L([0,1])$ such that $\mathbb{V}_{(f,g)}(\mathbf{x}) = (\mathbf{C}_{\boldsymbol{\lambda},f}^{\preccurlyeq}(F),\widetilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}(F))$, then $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F)=\mathbf{x}$.*

Proof: It follows from Remark 1 and $\mathbb{V}_{(f,g)}(\mathbf{x}) = (\mathbf{C}_{\boldsymbol{\lambda},f}^{\preccurlyeq}(F),\widetilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}(F))$ that $\mathbf{x}=\max\tilde{\mathscr{T}}(F)$, and thus $\mathbf{x} = \bigvee_{\preccurlyeq}\tilde{\mathscr{T}}(F)=\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F)$. □

Corollary 2: *For $\alpha,\ \beta\in[0,1]$ with $\alpha\neq\beta$, we have*

$$\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,\beta}}(F) = \left[\sum_{\gamma=1}^{n}\underline{x_{\sigma(\gamma)}}\cdot(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)})),\sum_{\gamma=1}^{n}\overline{x_{\sigma(\gamma)}}\cdot(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))\right],$$

where $(\sigma(1),\ldots,\sigma(n))$ is a permutation of $(1,\ldots,n)$ such that $F(x_{\sigma(1)})\preccurlyeq F(x_{\sigma(2)})\preccurlyeq\cdots\preccurlyeq F(x_{\sigma(n)})$, $X_{\sigma(\gamma)}=\{x_{\sigma(\gamma)},\ldots,x_{\sigma(n)}\}$, $X_{\sigma(n+1)}=\varnothing$, and $F(x_\gamma)=[\underline{x_\gamma},\overline{x_\gamma}]$ for $1\le\gamma\le n$.

Proof: Denote $\left[\sum_{\gamma=1}^{n}\underline{x_{\sigma(\gamma)}}\cdot(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)})),\sum_{\gamma=1}^{n}\overline{x_{\sigma(\gamma)}}\cdot(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))\right]=[\underline{x},\overline{x}]=\mathbf{x}$. By direct calculation, we have

$$\begin{aligned}K_\alpha(\mathbf{x}) &= \alpha\sum_{\gamma=1}^{n}\overline{x_{\sigma(\gamma)}}\cdot(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))+(1-\alpha)\sum_{\gamma=1}^{n}\underline{x_{\sigma(\gamma)}}\cdot(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))\\
&=\sum_{\gamma=1}^{n}(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))\cdot(\alpha\cdot\overline{x_{\sigma(\gamma)}}+(1-\alpha)\cdot\underline{x_{\sigma(\gamma)}})\\
&=\sum_{\gamma=1}^{n}(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))\cdot K_\alpha(\mathbf{x}_{\sigma(\gamma)})\\
&=\mathbf{C}_{\boldsymbol{\lambda},K_\alpha}^{\preccurlyeq}(F),\end{aligned}$$

and

$$\begin{aligned}K_\beta(\mathbf{x}) &= \beta\sum_{\gamma=1}^{n}\overline{x_{\sigma(\gamma)}}\cdot(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))+(1-\beta)\sum_{\gamma=1}^{n}\underline{x_{\sigma(\gamma)}}\cdot(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))\\
&=\sum_{\gamma=1}^{n}(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))\cdot(\beta\cdot\overline{x_{\sigma(\gamma)}}+(1-\beta)\cdot\underline{x_{\sigma(\gamma)}})\\
&=\sum_{\gamma=1}^{n}(\boldsymbol{\lambda}(X_{\sigma(\gamma)})-\boldsymbol{\lambda}(X_{\sigma(\gamma+1)}))\cdot K_\beta(\mathbf{x}_{\sigma(\gamma)})\end{aligned}$$

$= \mathbf{C}^{\preccurlyeq}_{\lambda,K_\beta}(F),$

implying that $\mathbb{V}_{(K_\alpha,K_\beta)}(\mathbf{x}) = (\mathbf{C}^{\preccurlyeq}_{\lambda,K_\alpha}(F), \mathbf{C}^{\preccurlyeq}_{\lambda,K_\beta}(F))$. Thus, $\mathbf{IC}^{\preccurlyeq_{\alpha,\beta}}_{\lambda}(F) = \mathbf{x}$ by Proposition 5. □

The improved interval-valued Choquet integral $\mathbf{IC}^{\preccurlyeq}_{\lambda}(\cdot)$ has the following desirable properties.

Property 6: *Let $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g), and $\boldsymbol{\lambda} : 2^X \longrightarrow [0,1]$ be a fuzzy measure on $X = \{x_1,\ldots,x_n\}$. If f and g are idempotent, then, for any interval-valued fuzzy set $F : X \longrightarrow L([0,1])$ with $\underline{F(x_i)} = \overline{F(x_i)}$ for all $1 \le i \le n$,*

$\mathbf{IC}^{\preccurlyeq}_{\lambda}(F) = \mathbf{C}_{\lambda}(F).$

In particular,

$\mathbf{IC}^{\preccurlyeq_{\alpha,\beta}}_{\lambda}(F) = \mathbf{C}_{\lambda}(F),$

for $\alpha,\ \beta \in [0,1]$ with $\alpha \neq \beta$.

Proof: Let $(\sigma(1),\ldots,\sigma(n))$ be a permutation of $(1,\ldots,n)$ such that $F(x_{\sigma(1)}) \ge F(x_{\sigma(2)}) \ge \cdots F(\ge x_{\sigma(n)})$. Since f and g are idempotent, by direct calculation, we have

$$\mathbf{C}^{\preccurlyeq}_{\lambda,f}(F) = \sum_{\gamma=1}^{n} f(F(x_{\sigma(\gamma)}))(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})) = \sum_{\gamma=1}^{n} F(x_{\sigma(\gamma)})(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})) = \mathbf{C}_{\lambda}(F),$$

and

$$\widetilde{\mathbf{C}}^{\preccurlyeq}_{\lambda,g}(F) = \sum_{\gamma=1}^{n} g(F(x_{\sigma(\gamma)}))(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})) = \sum_{\gamma=1}^{n} F(x_{\sigma(\gamma)})(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})) = \mathbf{C}_{\lambda}(F).$$

This, together with Proposition 5, implies that $\mathbf{IC}^{\preccurlyeq}_{\lambda}(F) = [\mathbf{C}_{\lambda}(F), \mathbf{C}_{\lambda}(F)] = \mathbf{C}_{\lambda}(F)$. □

Property 7: *Let $F : X \longrightarrow L([0,1])$ be an interval-valued fuzzy set on $X = \{x_1,\ldots,x_n\}$, and $\boldsymbol{\lambda} : 2^X \longrightarrow [0,1]$ be a fuzzy measure. Then, for $\alpha,\ \beta \in [0,1]$ with $\alpha \neq \beta$,*

$\mathbf{IC}^{\preccurlyeq_{\alpha,\beta}}_{\lambda}(F) = \mathbf{C}^{\preccurlyeq_{\alpha,\beta}}_{\lambda}(F).$

Proof: It follows directly from Definition 9 and Corollary 2. □

Property 8 (Idempotency): *Let $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g), $F : X \longrightarrow L([0,1])$ be an interval-valued fuzzy set on $X = \{x_1,\ldots,x_n\}$, and $\boldsymbol{\lambda} : 2^X \longrightarrow [0,1]$ be a fuzzy measure. If $F(x_1) = F(x_2) = \cdots = F(x_n) = \mathbf{x} \in L([0,1])$, then $\mathbf{IC}^{\preccurlyeq}_{\lambda}(F) = \mathbf{x}$.*

Proof: By direct calculation, we have

$$\mathbf{C}^{\preccurlyeq}_{\lambda,f}(F) = \sum_{\gamma=1}^{n} f(F(x_{\sigma(\gamma)}))(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})) = f(\mathbf{x}),$$

and

$$\tilde{\mathbf{C}}^{\preccurlyeq}_{\lambda,g}(F) = \sum_{\gamma=1}^{n} g(F(x_{\sigma(\gamma)}))(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})) = g(\mathbf{x}),$$

and thus $\mathbf{IC}^{\preccurlyeq}_{\lambda}(F) = \mathbf{x}$ by Proposition 5. □

Property 9 (Monotonicity): *Let $\preccurlyeq$ be a generated admissible order on $L([0,1])$ induced by a generating pair (f,g), F_1 and $F_2 : X \longrightarrow L([0,1])$ be two interval-valued fuzzy sets on $X = \{x_1,\ldots,x_n\}$, and $\boldsymbol{\lambda} : 2^X \longrightarrow [0,1]$ be a strictly monotonic fuzzy measure. If $F_1(x_j) \preccurlyeq F_2(x_j)$ holds for all $1 \le j \le n$, then $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_1) \preccurlyeq \mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_2)$.*

Proof: By the definition of the order $\preccurlyeq$ and $F_1(x_\gamma) \preccurlyeq F_2(x_\gamma)$, we get $f(F_1(x_\gamma)) \le f(F_2(x_\gamma))$ for $1 \le \gamma \le n$, i.e., $f \circ F_1 \le f \circ F_2$. Then, $f(\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_1)) = \mathbf{C}_{\boldsymbol{\lambda}}(f \circ F_1) \le \mathbf{C}_{\boldsymbol{\lambda}}(f \circ F_2) = f(\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_2))$ by Theorem 3. Let us consider the following two cases:

(1) If $f(\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_1)) < f(\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_2))$, it is clear that $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_1) \preccurlyeq \mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_2)$.

(2) If $f(\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_1)) = f(\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_2))$, by $f \circ F_1 \le f \circ F_2$, it follows from Proposition 3 and Theorem 3 that $f \circ F_1 = f \circ F_2$. Since $F_1(x_\gamma) \preccurlyeq F_2(x_\gamma)$ holds for all $1 \le \gamma \le n$, by the definition of $\preccurlyeq$, we have $g \circ F_1 \le g \circ F_2$. Let $(\sigma(1),\ldots,\sigma(n))$ and $(\sigma'(1),\ldots,\sigma'(n))$ be two permutations of $(1,\ldots,n)$ such that $F_1(x_{\sigma(1)}) \preccurlyeq F_1(x_{\sigma(2)}) \preccurlyeq \cdots \preccurlyeq F_1(x_{\sigma(n)})$, $F_2(x_{\sigma'(1)}) \preccurlyeq F_2(x_{\sigma'(2)}) \preccurlyeq \cdots \preccurlyeq F_2(x_{\sigma'(n)})$, and set $X_{\sigma(\gamma)} = \{x_{\sigma(\gamma)},\ldots,x_{\sigma(n)}\}$, $X'_{\sigma'(\gamma)} = \{x_{\sigma'(\gamma)},\ldots,x_{\sigma'(n)}\}$, and $X_{\sigma(n+1)} = X'_{\sigma'(n+1)} = \varnothing$. Then,

$$\widetilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}(F_1) = \sum_{\gamma=1}^{n} g(F_1(x_{\sigma(\gamma)}))(\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})), \tag{24}$$

and

$$\widetilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}(F_2) = \sum_{\gamma=1}^{n} g(F_2(x_{\sigma'(j)}))(\boldsymbol{\lambda}(X'_{\sigma'(j)}) - \boldsymbol{\lambda}(X'_{\sigma'(j+1)})). \tag{25}$$

We consider the following two subcases:

2-1) If $f \circ F_1$ is constant, by $F_1(x_{\sigma(1)}) \preccurlyeq F_1(x_{\sigma(2)}) \preccurlyeq \cdots \preccurlyeq F_1(x_{\sigma(n)})$, then $g(F_1(x_{\sigma(1)})) \le g(F_1(x_{\sigma(2)})) \le \cdots \le g(F_1(x_{\sigma(n)}))$. Similarly, we have $g(F_2(x_{\sigma'(1)})) \le g(F_2(x_{\sigma'(2)})) \le \cdots \le g(F_2(x_{\sigma'(n)}))$. As a consequence of Eqs. (24) and (25), this implies that $\widetilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}(F_1) = \mathbf{C}_{\boldsymbol{\lambda}}(g \circ F_1)$ and $\widetilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}(F_2) = \mathbf{C}_{\boldsymbol{\lambda}}(g \circ F_2)$. Together with $g \circ F_1 \le g \circ F_2$, since $\mathbf{C}_{\boldsymbol{\lambda}}(\cdot)$ has the property of monotonicity, we obtain $\widetilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}(F_1) \le \widetilde{\mathbf{C}}_{\boldsymbol{\lambda},g}^{\preccurlyeq}(F_2)$, and thus $\tilde{\mathscr{T}}(F_1) \subseteq \tilde{\mathscr{T}}(F_2)$ by $f \circ F_1 = f \circ F_2$. Therefore,

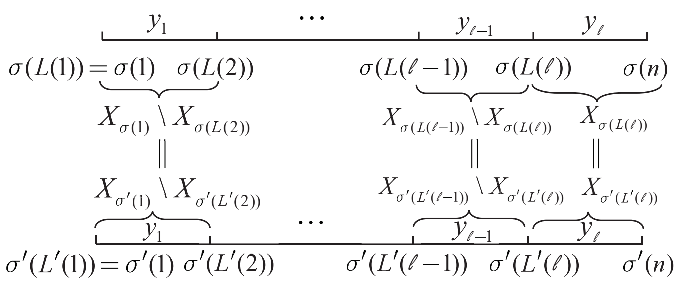
$$\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_1) = \bigvee_{\preccurlyeq} \tilde{\mathscr{T}}(F_1) \preccurlyeq \bigvee_{\preccurlyeq} \tilde{\mathscr{T}}(F_2) = \mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}(F_2).$$

2-2) If $f \circ F_1$ is not constant, by $f \circ F_1 = f \circ F_2$, without loss of generality, assume that $\{f \circ F_1(x_\gamma) \mid 1 \le \gamma \le n\} = \{f \circ F_2(x_\gamma) \mid 1 \le \gamma \le n\} = \{y_1, y_2, \ldots, y_\ell\}$ with $y_1 < y_2 < \cdots < y_\ell$. For any $1 \le k \le \ell$, let $L(k) = \min\{\gamma \mid 1 \le \gamma \le n, f(F_1(x_{\sigma(\gamma)})) = y_k\}$, $L'(k) = \min\{\gamma \mid 1 \le \gamma \le n, f(F_2(x_{\sigma'(\gamma)})) = y_k\}$, and $L(\ell+1) = L'(\ell+1) = n$. According to the constructions of $(\sigma(1),\ldots,\sigma(n))$ and $(\sigma'(1),\ldots,\sigma'(n))$, it can be verified that, for $1 \le k \le \ell$,

(i) $L(k) = L'(k)$ and $L(1) = L'(1) = 1$;

(ii) $X_{\sigma(L(k))} = \{x \in X \mid f(F_1(x)) \ge y_k\}$, $X_{\sigma'(L'(k))} = \{x \in X \mid f(F_2(x)) \ge y_k\}$, and $X_{\sigma(L(k))} = X_{\sigma'(L'(k))}$.

(iii) $\{x_{\sigma(\gamma)} \mid L(k) \le \gamma < L(k+1)\} = X_{\sigma(L(k))} \backslash X_{\sigma(L(k+1))} = \{x \in X \mid f(F_1(x)) = y_k\} \triangleq \mathcal{X}_k^{(1)}$ and $\{x_{\sigma'(\gamma)} \mid L'(k) \le \gamma < L'(k+1)\} = X'_{\sigma'(L'(k))} \backslash X'_{\sigma'(L'(k+1))} = \{x \in X \mid f(F_2(x)) = y_k\} \triangleq \mathcal{X}_k^{(2)}$. Thus, $\mathcal{X}_k^{(1)} = \mathcal{X}_k^{(2)} \triangleq \mathcal{X}_k$ by $f \circ F_1 = f \circ F_2$.



For $\mathcal{X}_\ell$, define the fuzzy measure $\boldsymbol{\lambda}_\ell : 2^{\mathcal{X}_\ell} \longrightarrow [0,1]$ on $\mathcal{X}_\ell$ by $\boldsymbol{\lambda}_\ell(A) = \dfrac{\boldsymbol{\lambda}(A)}{\boldsymbol{\lambda}(\mathcal{X}_\ell)}$. Then, by (i) and (iii), it can be verified that

$$\mathbf{C}_{\boldsymbol{\lambda}_\ell}(g \circ F_1|_{\mathcal{X}_\ell}) = \sum_{\gamma=L(\ell)}^{n} g(F_1(x_{\sigma(\gamma)})) \cdot \left[\frac{\boldsymbol{\lambda}(X_{\sigma(\gamma)})}{\boldsymbol{\lambda}(\mathcal{X}_\ell)} - \frac{\boldsymbol{\lambda}(X_{\sigma(\gamma+1)})}{\boldsymbol{\lambda}(\mathcal{X}_\ell)} \right],$$

and

$$\mathbf{C}_{\boldsymbol{\lambda}_\ell}(g \circ F_2|_{\mathcal{X}_\ell}) = \sum_{\gamma=L'(\ell)}^{n} g(F_2(x_{\sigma'(j)})) \cdot \left[\frac{\boldsymbol{\lambda}(X'_{\sigma'(j)})}{\boldsymbol{\lambda}(\mathcal{X}_\ell)} - \frac{\boldsymbol{\lambda}(X'_{\sigma'(j+1)})}{\boldsymbol{\lambda}(\mathcal{X}_\ell)} \right].$$

Together with $g \circ F_1 \leq g \circ F_2$, since $\mathbf{C}_{\boldsymbol{\lambda}_\ell}(\cdot)$ has the property of monotonicity, we obtain

$$\sum_{\gamma=L(\ell)}^{n} g(F_1(x_{\sigma(\gamma)})) \cdot \left[\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})\right] \leq \sum_{\gamma=L'(\ell)}^{n} g(F_2(x_{\sigma'(\gamma)})) \cdot \left[\boldsymbol{\lambda}(X'_{\sigma'(\gamma)}) - \boldsymbol{\lambda}(X'_{\sigma'(\gamma+1)})\right].$$

For $\mathcal{X}_k$ $(1 \leq k < \ell)$, define the fuzzy measure $\boldsymbol{\lambda}_k : 2^{\mathcal{X}_k} \longrightarrow [0,1]$ on $\mathcal{X}_k$ by $\boldsymbol{\lambda}_k(A) = \dfrac{\boldsymbol{\lambda}(A \cup \mathcal{X}_{k+1} \cup \cdots \cup \mathcal{X}_\ell)}{\boldsymbol{\lambda}(\mathcal{X}_k \cup \mathcal{X}_{k+1} \cup \cdots \cup \mathcal{X}_\ell)}$. It can be verified that, for any $L(k) \leq \gamma \leq L(k+1) - 1$,

$$\boldsymbol{\lambda}_k(\{x_{\sigma(\gamma)}, \ldots, x_{\sigma(L(k+1)-1)}\}) = \frac{\boldsymbol{\lambda}(X_{\sigma(\gamma)})}{\boldsymbol{\lambda}(\mathcal{X}_k \cup \mathcal{X}_{k+1} \cup \cdots \cup \mathcal{X}_\ell)},$$

and

$$\boldsymbol{\lambda}_k(\{x_{\sigma'(\gamma)}, \ldots, x_{\sigma'(L(k+1)-1)}\}) = \frac{\boldsymbol{\lambda}(X'_{\sigma'(\gamma)})}{\boldsymbol{\lambda}(\mathcal{X}_k \cup \mathcal{X}_{k+1} \cup \cdots \cup \mathcal{X}_\ell)}.$$

Then, by (i) and (iii),

$$\mathbf{C}_{\boldsymbol{\lambda}_k}(g \circ F_1|_{\mathcal{X}_k}) = \sum_{\gamma=L(k)}^{L(k+1)-1} g(F_1(x_{\sigma(\gamma)})) \cdot \left[\frac{\boldsymbol{\lambda}(X_{\sigma(\gamma)})}{\boldsymbol{\lambda}(\mathcal{X}_k \cup \mathcal{X}_{k+1} \cup \cdots \cup \mathcal{X}_\ell)} - \frac{\boldsymbol{\lambda}(X_{\sigma(\gamma+1)})}{\boldsymbol{\lambda}(\mathcal{X}_k \cup \mathcal{X}_{k+1} \cup \cdots \cup \mathcal{X}_\ell)} \right],$$

and

$$\mathbf{C}_{\boldsymbol{\lambda}_k}(g \circ F_2|_{\mathcal{X}_k}) = \sum_{\gamma=L(k)}^{L(k+1)-1} g(F_2(x_{\sigma(\gamma)})) \cdot \left[\frac{\boldsymbol{\lambda}(X'_{\sigma'(j)})}{\boldsymbol{\lambda}(\mathcal{X}_k \cup \mathcal{X}_{k+1} \cup \cdots \cup \mathcal{X}_\ell)} - \frac{\boldsymbol{\lambda}(X'_{\sigma'(\gamma+1)})}{\boldsymbol{\lambda}(\mathcal{X}_k \cup \mathcal{X}_{k+1} \cup \cdots \cup \mathcal{X}_\ell)} \right].$$

Together with $g \circ F_1 \leq g \circ F_2$, since $\mathbf{C}_{\lambda_k}(\cdot)$ has the property of monotonicity, we obtain

$$\sum_{\gamma=L(k)}^{L(k+1)-1} g(F_1(x_{\sigma(\gamma)})) \cdot \left[\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})\right] \leq \sum_{\gamma=L(k)}^{L(k+1)-1} g(F_2(x_{\sigma(\gamma)})) \cdot \left[\boldsymbol{\lambda}(X'_{\sigma'(j)}) - \boldsymbol{\lambda}(X'_{\sigma'(\gamma+1)})\right].$$

By Eqs. (24) and (25), we get

$$\begin{aligned}
\widetilde{\mathbf{C}}^{\preccurlyeq}_{\boldsymbol{\lambda},g}(F_1) &= \sum_{k=1}^{\ell-1} \sum_{\gamma=L(k)}^{L(k+1)-1} g(F_1(x_{\sigma(\gamma)})) \cdot \left[\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})\right] \\
&\quad + \sum_{\gamma=L(\ell)}^{n} g(F_1(x_{\sigma(\gamma)})) \cdot \left[\boldsymbol{\lambda}(X_{\sigma(\gamma)}) - \boldsymbol{\lambda}(X_{\sigma(\gamma+1)})\right] \\
&\quad \leq \sum_{k=1}^{\ell-1} \sum_{\gamma=L(k)}^{L(k+1)-1} g(F_2(x_{\sigma(\gamma)})) \cdot \left[\boldsymbol{\lambda}(X'_{\sigma'(\gamma)}) - \boldsymbol{\lambda}(X'_{\sigma'(\gamma+1)})\right] \\
&\quad + \sum_{\gamma=L'(\ell)}^{n} g(F_2(x_{\sigma'(j)})) \cdot \left[\boldsymbol{\lambda}(X'_{\sigma'(\gamma)}) - \boldsymbol{\lambda}(X'_{\sigma'(\gamma+1)})\right] \\
&= \widetilde{\mathbf{C}}^{\preccurlyeq}_{\boldsymbol{\lambda},g}(F_2),
\end{aligned}$$

and thus $\tilde{\mathscr{T}}(F_1) \subseteq \tilde{\mathscr{T}}(F_2)$ by $f \circ F_1 = f \circ F_2$. Therefore,

$$\mathbf{IC}^{\preccurlyeq}_{\boldsymbol{\lambda}}(F_1) = \bigvee_{\preccurlyeq} \tilde{\mathscr{T}}(F_1) \preccurlyeq \bigvee_{\preccurlyeq} \tilde{\mathscr{T}}(F_2) = \mathbf{IC}^{\preccurlyeq}_{\boldsymbol{\lambda}}(F_2).$$

Summing up (1) and (2), it follows that $\mathbf{IC}^{\preccurlyeq}_{\boldsymbol{\lambda}}(F_1) \preccurlyeq \mathbf{IC}^{\preccurlyeq}_{\boldsymbol{\lambda}}(F_2)$. □

Example 4: *Let the fuzzy measure* $\boldsymbol{\lambda}$ *on* $X = \{1, 2, 3\}$ *be given by* $\boldsymbol{\lambda}(\varnothing) = 0$, $\boldsymbol{\lambda}(\{1\}) = 0.1$, $\boldsymbol{\lambda}(\{2\}) = 0.5$, $\boldsymbol{\lambda}(\{3\}) = 0.2$, $\boldsymbol{\lambda}(\{1, 2\}) = 0.5$, $\boldsymbol{\lambda}(\{1, 3\}) = 0.3$, $\boldsymbol{\lambda}(\{2, 3\}) = 0.9$, $\boldsymbol{\lambda}(X) = 1$. *Consider the order* $\preccurlyeq$ *generated by* $f(x, y) = \dfrac{\sqrt{x} + \sqrt{y}}{2}$ *and* $g(x, y) = y$. *For* $F : X \longrightarrow L([0, 1])$ *given by* $F(1) = [0.1, 1]$, $F(2) = [0.3, 0.9]$, *and* $F(3) = [0.4, 0.4]$, *by direct calculation, we have*

$$\mathbf{C}^{\preccurlyeq}_{\boldsymbol{\lambda},f}(F) = \frac{\sqrt{0.3} + \sqrt{0.9}}{2} \cdot 0.5 + \frac{\sqrt{0.1} + 1}{2} \cdot (0.5 - 0.5) + \frac{\sqrt{0.4} + \sqrt{0.4}}{2} \cdot (1 - 0.5) = 0.6903,$$

and

$$\widetilde{\mathbf{C}}^{\preccurlyeq}_{\boldsymbol{\lambda},g}(F) = 0.9 \cdot 0.5 + 1 \cdot (0.5 - 0.5) + 0.4 \cdot (1 - 0.5) = 0.65.$$

This, together with Proposition 5, implies that $\mathbf{IC}^{\preccurlyeq}_{\boldsymbol{\lambda}}(F) = [0.33, 0.65]$.

5 Applications

In practical problems, due to various reasons, it is difficult for experts to accurately determine the preference values between the alternatives, so the IV preference relationship is more in line with the actual situation and can better handle uncertainty. Let $\mathfrak{A} = \{A_1, \ldots, A_n\}$ be the set of n alternatives. An IV fuzzy binary relation R_{IV} on $\mathfrak{A}$ is an IV fuzzy subset on $\mathfrak{A} \times \mathfrak{A}$, i.e., a mapping

$\mathsf{R}_{\mathrm{IV}} : \mathfrak{A} \times \mathfrak{A} \longrightarrow L([0,1])$. In particular, in preference analysis, $\mathsf{R}_{\mathrm{IV}}(A_i, A_j)$ represents the degree of preference of an alternative A_i relative to A_j. Considering ℓ experts $E = \{\mathsf{e}_1, \ldots, \mathsf{e}_\ell\}$, each expert e_k expresses the preference by means of an IV fuzzy preference relation matrix (IVFPRM) $\mathsf{R}_{\mathrm{IV}}^{(\mathsf{e}_k)} = \left(\mathsf{R}_{\mathrm{IV}}^{(\mathsf{e}_k)}(A_i, A_j) = [\underline{\mathsf{R}}_{ij}^{(\mathsf{e}_k)}, \overline{\mathsf{R}}_{ij}^{(\mathsf{e}_k)}]\right)_{n\times n}$, which is shown in Table 1.

Table 1: The IV fuzzy preference relation matrix $\mathsf{R}_{\mathrm{IV}}^{(\mathsf{e}_i)}$

		A_1	A_2	$\cdots$	A_n
$\mathsf{R}_{\mathrm{IV}}^{(\mathsf{e}_i)}=$	A_1	$[0.5, 0.5]$	$[\underline{\mathsf{R}}_{12}^{(\mathsf{e}_k)}, \overline{\mathsf{R}}_{12}^{(\mathsf{e}_k)}]$	$\cdots$	$[\underline{\mathsf{R}}_{1n}^{(\mathsf{e}_k)}, \overline{\mathsf{R}}_{1n}^{(\mathsf{e}_k)}]$
	A_2	$[\underline{\mathsf{R}}_{21}^{(\mathsf{e}_k)}, \overline{\mathsf{R}}_{21}^{(\mathsf{e}_k)}]$	$[0.5, 0.5]$	$\cdots$	$[\underline{\mathsf{R}}_{2n}^{(\mathsf{e}_k)}, \overline{\mathsf{R}}_{2n}^{(\mathsf{e}_k)}]$
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	A_n	$[\underline{\mathsf{R}}_{n1}^{(\mathsf{e}_k)}, \overline{\mathsf{R}}_{n1}^{(\mathsf{e}_k)}]$	$[\underline{\mathsf{R}}_{n2}^{(\mathsf{e}_k)}, \overline{\mathsf{R}}_{n2}^{(\mathsf{e}_k)}]$	$\cdots$	$[0.5, 0.5]$

Our goal here is to obtain the most accepted alternative by ℓ experts. To this end, and also to demonstrate the importance of the IVOWA operator $\mathsf{IVOWA}_{\mathbf{\Omega}}^{\preccurlyeq}$ and the improved IVCI $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}$ introduced in the above two sections, we propose the following algorithm based on [12] (Algorithms 1 and 2).

Step 1. Determine the weight vector $\mathbf{\Omega} = (\omega_1, \ldots, \omega_\ell)^\top$.

Step 2. Aggregate the collective IVFPRM $\mathsf{R}_{\mathrm{IV}}^{(\mathsf{e}_k)}$ $(k = 1, 2, \ldots, \ell)$ to the group IVFPRM $\mathsf{R}_{\mathrm{IV}} = ([\underline{\mathsf{R}}_{ij}, \overline{\mathsf{R}}_{ij}])_{n\times n}$ by using the IVOWA operator $\mathsf{IVOWA}_{\mathbf{\Omega}}^{\preccurlyeq_{\alpha,1}}$ for $\alpha \in [0, 1)$.

Step 3. Calculate the fuzzy measure $\boldsymbol{\lambda}$ by the following Eq. (26):

$$\boldsymbol{\lambda}(A) = \left(\frac{\sum_{j=1}^{n}\sum_{i\in A}(\underline{\mathsf{R}}_{ij} + \overline{\mathsf{R}}_{ij})}{\sum_{j=1}^{n}\sum_{i=1}^{n}(\underline{\mathsf{R}}_{ij} + \overline{\mathsf{R}}_{ij})}\right)^2, \text{ for } A \subseteq \{1, 2, \ldots, n\}, \tag{26}$$

where $[\underline{\mathsf{R}}_{ij}, \overline{\mathsf{R}}_{ij}]$ is the element of the group IVFPRM obtained by Step 2.

Step 4. Compute the overall IV preference value $\mathbf{\Phi}(A_i)$ of each alternative A_i by using the improved IVCI $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}$ and the Eq. (27):

$$\mathbf{\Phi}(A_i) = \int_0^1 \alpha \cdot \mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}(F(A_i))\mathrm{d}\alpha, \tag{27}$$

where $F(A_i)(j) = [\underline{\mathsf{R}}_{ij}, \overline{\mathsf{R}}_{ij}]$ $(\gamma = 1, 2, \ldots, n)$ and $\boldsymbol{\lambda}$ is the fuzzy measure obtained by Step 3.

Step 5. Rank the alternatives $A_1, A_2, \ldots, A_n$ according to the descending order of $\dfrac{\underline{\mathbf{\Phi}(A_i)} + \overline{\mathbf{\Phi}(A_i)}}{2}$ $(i = 1, 2, \ldots, n)$.

Remark 4.

(i) *For two subsets* A, $B \subseteq \{1, 2, \ldots, n\}$ *with* $A \subsetneq B$, *according to Eq. (26), we have*

$$\boldsymbol{\lambda}(B) = \left(\frac{\sum_{j=1}^{n}\sum_{i\in B}(\underline{R}_{ij} + \overline{R}_{ij})}{\sum_{j=1}^{n}\sum_{i=1}^{n}(\underline{R}_{ij} + \overline{R}_{ij})}\right)^2 = \left(\frac{\sum_{j=1}^{n}\sum_{i\in A}(\underline{R}_{ij} + \overline{R}_{ij})}{\sum_{j=1}^{n}\sum_{i=1}^{n}(\underline{R}_{ij} + \overline{R}_{ij})} + \frac{\sum_{j=1}^{n}\sum_{i\in B\setminus A}(\underline{R}_{ij} + \overline{R}_{ij})}{\sum_{j=1}^{n}\sum_{i=1}^{n}(\underline{R}_{ij} + \overline{R}_{ij})}\right)^2$$

$$> \left(\frac{\sum_{j=1}^{n}\sum_{i\in A}(\underline{R}_{ij} + \overline{R}_{ij})}{\sum_{j=1}^{n}\sum_{i=1}^{n}(\underline{R}_{ij} + \overline{R}_{ij})}\right)^2 = \boldsymbol{\lambda}(A)(\text{by } \underline{R}_{ij},\ \overline{R}_{ij} \geq 0 \text{ and } \underline{R}_{ii} = \overline{R}_{ii} = 0.5),$$

implying that the fuzzy measure $\boldsymbol{\lambda}$ *is strictly monotonic.*

(ii) *By the superadditivity of the quadratic function* x^2, *it can be verified that the fuzzy measure* $\boldsymbol{\lambda}$ *obtained by Eq. (26) in Step 3 is superadditive, i.e., for two subsets* A, $B \subseteq \{1, 2, \ldots, n\}$ *with* $A \cap B = \varnothing$,

$\boldsymbol{\lambda}(A \cup B) \geq \boldsymbol{\lambda}(A) + \boldsymbol{\lambda}(B)$.

(iii) *The fuzzy measure* $\boldsymbol{\lambda}$ *obtained by Eq. (26) in Step 3 is different from the fuzzy measures in [12] (Eq. (17)). The fuzzy measure obtained for each row in [12] may have different values in each row, and the IVCI used in [12] is not monotonic, so the monotonicity of [12] (Algorithms 1 and 2) cannot be guaranteed. Since our entire algorithm utilizes a unified fuzzy measure and the IVOWA operator* $\mathit{IVOWA}_{\Omega}^{\preccurlyeq}$ *and the improved IVCI* $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq}$ *have monotonicity (see Properties 3 and 9), it is not difficult to prove that our proposed algorithm is monotonic, i.e., if* $[\underline{R}_{ij}, \overline{R}_{ij}] \succcurlyeq [\underline{R}_{i'j}, \overline{R}_{i'j}]$ *holds for all* $1 \leq j \leq n$, *then* $\boldsymbol{\Phi}(A_i) \succcurlyeq \boldsymbol{\Phi}(A_{i'})$. *This theoretically ensures the effectiveness and rationality of our proposed algorithm.*

(iv) *The overall IV preference value* $\boldsymbol{\Phi}(A_i)$ *obtained by Eq. (27) in Step 4 can be considered as the average of the global IV preference values of* A_i *under all admissible orders* $\preccurlyeq_{\alpha,1}$. *This can avoid the disadvantage of insufficient persuasiveness in the ranking exported by a single order.*

Being the end of this section, we use a practical example to illustrate the effectiveness of the algorithm proposed above.

Example 5. *Four university students* US_1, US_2, US_3, *and* US_4 *from Guizhou University of Finance and Economics, sharing the same dormitory, are exploring options for broadband internet connection from three different internet service providers: China Mobile, China Telecom, and China Unicom. The providers offer four choices:*

- *Option 1 (O1): Broadband with a speed of 10 Mb/s;*
- *Option 2 (O2): Broadband with a speed of 20 Mb/s;*
- *Option 3 (O3): Broadband with a speed of 40 Mb/s;*
- *Option 4 (O4): Broadband with a speed of 50 Mb/s.*

They plan to evaluate these options to determine the most suitable broadband plan for their dormitory. Based on individual bandwidth needs and economic conditions, the four students express their preference relation for the options independently and anonymously by using the 0.1∼0.9 ratio scale in Table 2, which are shown in Tables A1–A4.

Table 2: 0.1∼0.9 ratio scale

0.1∼0.9 scale	Meaning
0.1	Extremely disfavored
0.2	Very strongly disfavored
0.3	Strongly disfavored
0.4	Moderately disfavored
0.5	Indifference
0.6	Moderately favored
0.7	Strongly favored
0.8	Very strongly favored
0.9	Extremely favored
Other values	Intermediate degrees reflecting compromise

Step 1. *After four students' discussion, they obtain an acceptable weight vector* $\mathbf{\Omega} = (0.3, 0.3, 0.2, 0.2)^\top$.

Step 2. *Aggregate the four students' IVFPRMs to the group IVFPRM* R_{IV}, *which is shown in* Table 3.

Table 3: The group IV fuzzy preference relation matrix R_{IV}

		O1	O2	O3	O4
$\mathsf{R}_{\text{IV}} =$	O1	[0.5, 0.5]	[0.425, 0.5]	[0.43, 0.49]	[0.445, 0.545]
	O2	[0.525, 0.6]	[0.5, 0.5]	[0.315, 0.39]	[0.385, 0.445]
	O3	[0.515, 0.58]	[0.615, 0.69]	[0.5, 0.5]	[0.455, 0.515]
	O4	[0.47, 0.555]	[0.57, 0.635]	[0.49, 0.555]	[0.5, 0.5]

Step 3. *Calculate the fuzzy measure* $\boldsymbol{\lambda}$ *by* Eq. (26) *as follows:* $\boldsymbol{\lambda}(\varnothing) = 0$, $\boldsymbol{\lambda}(\{1\}) = 0.0565$, $\boldsymbol{\lambda}(\{2\}) = 0.0514$, $\boldsymbol{\lambda}(\{3\}) = 0.0733$, $\boldsymbol{\lambda}(\{4\}) = 0.0702$, $\boldsymbol{\lambda}(\{1,2\}) = 0.2156$, $\boldsymbol{\lambda}(\{1,3\}) = 0.2584$, $\boldsymbol{\lambda}(\{1,4\}) = 0.2525$, $\boldsymbol{\lambda}(\{2,3\}) = 0.2475$, $\boldsymbol{\lambda}(\{2,4\}) = 0.2417$, $\boldsymbol{\lambda}(\{3,4\}) = 0.2869$, $\boldsymbol{\lambda}(\{1,2,3\}) = 0.5404$, $\boldsymbol{\lambda}(\{1,2,4\}) = 0.5318$, $\boldsymbol{\lambda}(\{1,3,4\}) = 0.5979$, $\boldsymbol{\lambda}(\{2,3,4\}) = 0.5812$, $\boldsymbol{\lambda}(\{1,2,3,4\}) = 1$.

Step 4. *Compute the overall IV preference value* $\boldsymbol{\Phi}(Oi)$ *of each option* Oi.

- $\boldsymbol{\Phi}(\mathrm{O1})$. *Noting that*
  - $F(\mathrm{O1})(2) \preccurlyeq_{\alpha,1} F(\mathrm{O1})(3) \preccurlyeq_{\alpha,1} F(\mathrm{O1})(4) \preccurlyeq_{\alpha,1} F(\mathrm{O1})(1)$ for $\alpha \in \left[0, \frac{1}{3}\right)$;
  - $F(\mathrm{O1})(3) \preccurlyeq_{\alpha,1} F(\mathrm{O1})(2) \preccurlyeq_{\alpha,1} F(\mathrm{O1})(4) \preccurlyeq_{\alpha,1} F(\mathrm{O1})(1)$ for $\alpha \in \left[\frac{1}{3}, 0.55\right)$;
  - $F(\mathrm{O1})(3) \preccurlyeq_{\alpha,1} F(\mathrm{O1})(2) \preccurlyeq_{\alpha,1} F(\mathrm{O1})(1) \preccurlyeq_{\alpha,1} F(\mathrm{O1})(4)$ for $\alpha \in [0.55, 1)$;
  - *we have*
  - $\mathbf{IC}_{\lambda}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O1})) = 0.0565 \cdot [0.5, 0.5] + (0.2525 - 0.0565) \cdot [0.445, 0.545] + (0.5979 - 0.2525) \cdot [0.43, 0.49] + (1 - 0.5979) \cdot [0.425, 0.5] = [0.4349, 0.5054]$ for $\alpha \in \left[0, \frac{1}{3}\right]$;

– $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O1})) = 0.0565 \cdot [0.5, 0.5] + (0.2525 - 0.0565) \cdot [0.445, 0.545] + (0.5318 - 0.2525) \cdot [0.425, 0.5] + (1 - 0.5318) \cdot [0.43, 0.49] = [0.4355, 0.5041]$ for $\alpha \in \left[\frac{1}{3}, 0.55\right]$;

– $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O1})) = 0.0702 \cdot [0.445, 0.545] + (0.2525 - 0.0702) \cdot [0.5, 0.5] + (0.5318 - 0.2525) \cdot [0.425, 0.5] + (1 - 0.5318) \cdot [0.43, 0.49] = [0.4424, 0.4985]$ for $\alpha \in [0.55, 1)$.

α	$\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O1}))$
$\left[0, \frac{1}{3}\right)$	$[0.4349, 0.5054]$
$\left[\frac{1}{3}, 0.55\right)$	$[0.4355, 0.5041]$
$[0.55, 1)$	$[0.4424, 0.4985]$

Then,

$$\begin{aligned}\mathbf{\Phi}(\mathrm{O1}) &= \int_0^1 \alpha \cdot \mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O1}))\mathrm{d}\alpha \\ &= \int_0^{\frac{1}{3}} \alpha \cdot [0.4349, 0.5054]\mathrm{d}\alpha + \int_{\frac{1}{3}}^{0.55} \alpha \cdot [0.4355, 0.5041]\mathrm{d}\alpha + \int_{0.55}^{1} \alpha \cdot [0.4424, 0.4985]\mathrm{d}\alpha \\ &= \frac{1}{18} \cdot [0.4349, 0.5054] + \frac{1.7225}{18} \cdot [0.4355, 0.5041] + \frac{0.6975}{2} \cdot [0.4424, 0.4985] \\ &= [0.2201, 0.2502].\end{aligned}$$

- $\mathbf{\Phi}(\mathrm{O2})$. *For any* $\alpha \in [0, 1)$, *noting that* $F(O2)(3) \preccurlyeq_{\alpha,1} F(O2)(4) \preccurlyeq_{\alpha,1} F(O2)(2) \preccurlyeq_{\alpha,1} F(O2)(1)$, *we have* $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O2})) = 0.0565 \cdot [0.525, 0.6] + (0.2156 - 0.0565) \cdot [0.5, 0.5] + (0.5318 - 0.2156) \cdot [0.385, 0.445] + (1 - 0.5318) \cdot [0.315, 0.39] = [0.3784, 0.4368]$. *Then,*

$$\mathbf{\Phi}(\mathrm{O2}) = \int_0^1 \alpha \cdot \mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O2}))\mathrm{d}\alpha = \int_0^1 \alpha \cdot [0.3784, 0.4368]\mathrm{d}\alpha = [0.1892, 0.2184].$$

- $\mathbf{\Phi}(\mathrm{O3})$. *Noting that*
  – $F(\mathrm{O3})(4) \preccurlyeq_{\alpha,1} F(\mathrm{O3})(3) \preccurlyeq_{\alpha,1} F(\mathrm{O3})(1) \preccurlyeq_{\alpha,1} F(\mathrm{O3})(2)$ for $\alpha \in [0, 0.75]$;
  – $F(\mathrm{O3})(3) \preccurlyeq_{\alpha,1} F(\mathrm{O3})(4) \preccurlyeq_{\alpha,1} F(\mathrm{O3})(1) \preccurlyeq_{\alpha,1} F(\mathrm{O3})(2)$ for $\alpha \in [0.75, 1)$;
  – *we have*
  – $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O3})) = 0.0514 \cdot [0.615, 0.69] + (0.2156 - 0.0514) \cdot [0.515, 0.58] + (0.5404 - 0.2156) \cdot [0.5, 0.5] + (1 - 0.5404) \cdot [0.455, 0.515] = [0.4877, 0.5298]$ for $\alpha \in [0, 0.75)$;
  – $\mathbf{IC}_{\boldsymbol{\lambda}}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O3})) = 0.0514 \cdot [0.615, 0.69] + (0.2156 - 0.0514) \cdot [0.515, 0.58] + (0.5318 - 0.2156) \cdot [0.455, 0.515] + (1 - 0.5318) \cdot [0.5, 0.5] = [0.4941, 0.5276]$ for $\alpha \in [0.75, 1)$.

α	$\mathbf{IC}_{\lambda}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O3}))$
[0, 0.75)	[0.4877, 0.5298]
[0.75, 1)	[0.4941, 0.5276]

Then,

$$\begin{aligned}\mathbf{\Phi}(\mathrm{O3}) &= \int_0^1 \alpha \cdot \mathbf{IC}_{\lambda}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O3}))\mathrm{d}\alpha \\ &= \int_0^{0.75} \alpha \cdot [0.4877, 0.5298]\mathrm{d}\alpha + \int_{0.75}^{1} \alpha \cdot [0.4941, 0.5276]\mathrm{d}\alpha \\ &= \frac{9}{32} \cdot [0.4877, 0.5298] + \frac{7}{32} \cdot [0.4941, 0.5276] \\ &= [0.2453, 0.2644].\end{aligned}$$

- $\mathbf{\Phi}(\mathrm{O4})$. *Noting that*
  - $F(\mathrm{O4})(1) \preccurlyeq_{\alpha,1} F(\mathrm{O4})(3) \preccurlyeq_{\alpha,1} F(\mathrm{O4})(4) \preccurlyeq_{\alpha,1} F(\mathrm{O4})(2)$ for $\alpha \in \left[0, \frac{2}{13}\right)$;
  - $F(\mathrm{O4})(1) \preccurlyeq_{\alpha,1} F(\mathrm{O4})(4) \preccurlyeq_{\alpha,1} F(\mathrm{O4})(3) \preccurlyeq_{\alpha,1} F(\mathrm{O4})(2)$ for $\alpha \in \left[\frac{2}{13}, \frac{6}{17}\right)$;
  - $F(\mathrm{O4})(4) \preccurlyeq_{\alpha,1} F(\mathrm{O4})(1) \preccurlyeq_{\alpha,1} F(\mathrm{O4})(3) \preccurlyeq_{\alpha,1} F(\mathrm{O4})(2)$ for $\alpha \in \left[\frac{6}{17}, 1\right)$;
  - *we have*
  - $\mathbf{IC}_{\lambda}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O4})) = 0.0514 \cdot [0.57, 0.635] + (0.2417 - 0.0514) \cdot [0.5, 0.5] + (0.5812 - 0.2417) \cdot [0.49, 0.555] + (1 - 0.5812) \cdot [0.47, 0.555] = [0.4729, 0.5486]$ for $\alpha \in \left[0, \frac{2}{13}\right)$;
  - $\mathbf{IC}_{\lambda}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O4})) = 0.0514 \cdot [0.57, 0.635] + (0.2475 - 0.0514) \cdot [0.49, 0.555] + (0.5812 - 0.2475) \cdot [0.5, 0.5] + (1 - 0.5812) \cdot [0.47, 0.555] = [0.4891, 0.5408]$ for $\alpha \in \left[\frac{2}{13}, \frac{6}{17}\right)$;
  - $\mathbf{IC}_{\lambda}^{\preccurlyeq_{\alpha,1}}(F(\mathrm{O4})) = 0.0514 \cdot [0.57, 0.635] + (0.2475 - 0.0514) \cdot [0.49, 0.555] + (0.5404 - 0.2475) \cdot [0.47, 0.555] + (1 - 0.5404) \cdot [0.5, 0.5] = [0.4929, 0.5338]$ for $\alpha \in \left[\frac{6}{17}, 1\right)$.

α	$\mathbf{IC}_{\lambda}^{\preccurlyeq_{\alpha,1}}(F(\text{O4}))$
$\left[0, \frac{2}{13}\right)$	[0.4729, 0.5486]
$\left[\frac{2}{13}, \frac{6}{17}\right)$	[0.4891, 0.5408]
$\left[\frac{6}{17}, 1\right)$	[0.4929, 0.5338]

Then,

$$\begin{aligned}\mathbf{\Phi}(\text{O4}) &= \int_0^1 \alpha \cdot \mathbf{IC}_{\lambda}^{\preccurlyeq_{\alpha,1}}(F(\text{O1}))\mathrm{d}\alpha \\ &= \int_0^{\frac{2}{13}} \alpha \cdot [0.4729, 0.5486]\mathrm{d}\alpha + \int_{\frac{2}{13}}^{\frac{6}{17}} \alpha \cdot [0.4891, 0.5408]\mathrm{d}\alpha + \int_{\frac{6}{17}}^{1} \alpha \cdot [0.4929, 0.5338]\mathrm{d}\alpha \\ &= [0.2410, 0.2674].\end{aligned}$$

Step 5. *By direct calculation, we have*

	O1	O2	O3	O4	Ranking
$\dfrac{\underline{\mathbf{\Phi}(\text{O})} + \overline{\mathbf{\Phi}(\text{O})}}{2}$	0.23515	0.2038	0.25485	0.2542	O3≻O4≻O1≻O2

This means that O3 is the best option.

Remark 5: *If we use only a certain order $\preccurlyeq_{\alpha,1}$ instead of the overall IV preference value of all orders $\preccurlyeq_{\alpha,1}$ ($\alpha \in [0, 1)$) to determine the performance of the options, the following Fig. 1 illustrates the variation of K_α-value for each option with respect to the variable $\alpha \in [0, 1)$. Observing from Fig. 2, we see that*

- *For $\alpha \in [0, 0.45)$, the ranking order of O1—O4 is O3≻O4≻O1≻O2 and the best option is O3;*
- *For $\alpha \in [0.45, 1)$, the ranking order of O1—O4 is O4≻O3≻O1≻O2 and the best option is O4.*

This means that the ranking depends on the selection of admissible order. Therefore, we use the overall IV preference value $\mathbf{\Phi}(\_)$ in Eq. (27) to evaluate the average of the global IV preference values of each option under all admissible orders $\preccurlyeq_{\alpha,1}$ ($\alpha \in [0, 1)$).

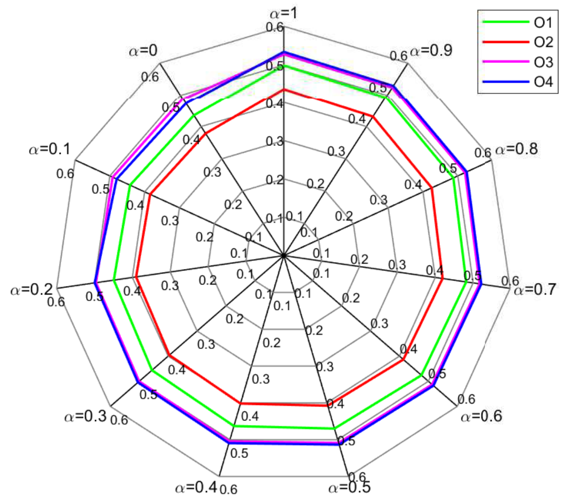


Figure 1: The K_α-values of $\mathbf{IC}_\lambda^{\preceq_{\alpha,1}}(F(\mathrm{O1}))$—$\mathbf{IC}_\lambda^{\preceq_{\alpha,1}}(F(\mathrm{O4}))$ for $\alpha \in [0, 1)$

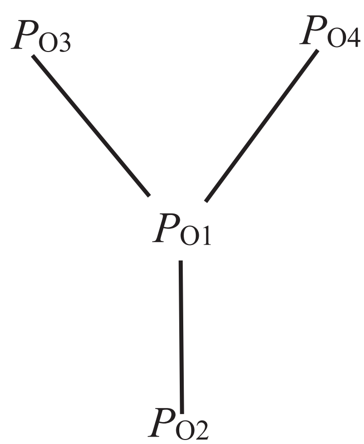


Figure 2: Hasse diagram of P_{O1}—P_{O4}

Comparative analysis

To further illustrate the effectiveness of the proposed algorithm, a comparative analysis is conducted by applying the [15] (Algorithm 1) and the CI based TOPSIS and TODIM methods in [36].

- If we use [15] (Algorithm 1) based on the fuzzy measure $\boldsymbol{\lambda}$ obtained in Step 3, we calculate the following evaluations for each option:

$$P_{O1} = [0.4349, 0.4985],$$

$$P_{O2} = [0.3784, 0.4368],$$

$$P_{O3} = [0.4877, 0.5276],$$

$$P_{O4} = [0.4876, 0.5338],$$

where $P_{Oi} = [\mathbf{C}_\lambda(\underline{F(\mathrm{Oi})}), \mathbf{C}_\lambda(\overline{F(\mathrm{Oi})})]$. Observing that $P_{O2} \preccurlyeq_{\mathbf{P}} P_{O1}$, $P_{O1} \preccurlyeq_{\mathbf{P}} P_{O3}$, and $P_{O1} \preccurlyeq_{\mathbf{P}} P_{O4}$, but P_{O3} and P_{O4} are incomparable, this is illustrated in the following Hasse diagram (see Fig. 2):

The admissible permutations of P_{O1}—P_{O4} are $\sigma_1 = (2, 1, 3, 4)$ and $\sigma_2 = (2, 1, 4, 3)$. And thus, we can calculate the votes for each option based on σ_1 and σ_2 as follows:

	V_{O1}	V_{O2}	V_{O3}	V_{O4}
σ_1	2	1	3	4
σ_2	2	1	4	3
	4	2	7	7

This means that the ranking order of O1—O4 is O4~O3≻O2≻O1.

- If we use CI based TOPSIS method in [36], we obtain the PIS and NIS as follows:

$$\mathrm{PIS} = \{s_1^+, s_2^+, s_3^+, s_4^+\} = \{[0.525, 0.600], [0.615, 0.690], [0.500, 0.555], [0.500, 0.545]\},$$
$$\mathrm{NIS} = \{s_1^-, s_2^-, s_3^-, s_4^-\} = \{[0.470, 0.500], [0.425, 0.500], [0.315, 0.390], [0.385, 0.445]\}.$$

Then, the Hausdorff distance d_H between each $[\underline{R}_{ij}, \overline{R}_{ij}]$ and the PIS and the NIS is calculated as follows:

d_H	s_1^+	s_2^+	s_3^+	s_4^+
O1	0.1	0.19	0.07	0.055
O2	0	0.19	0.185	0.155
O3	0.02	0	0.055	0.045
O4	0.055	0.055	0.01	0.045

d_H	s_1^-	s_2^-	s_3^-	s_4^-
O1	0.03	0	0.115	0.115
O2	0.1	0.75	0	0
O3	0.08	0.19	0.185	0.07
O4	0.055	0.145	0.175	0.115

Based on the fuzzy measure $\boldsymbol{\lambda}$ obtained in Step 3, we obtain the separation measures of O1—O4 as follows:

$$\mathbb{S}_1^+ = 0.0865, \mathbb{S}_2^+ = 0.1117, \mathbb{S}_3^+ = 0.0248, \mathbb{S}_4^+ = 0.0366,$$
$$\mathbb{S}_1^- = 0.0561, \mathbb{S}_2^- = 0.0438, \mathbb{S}_3^- = 0.1132, \mathbb{S}_4^- = 0.1076.$$

Based on $\mathbf{CC}(\mathrm{Oi}) = \frac{\mathbb{S}_i^-}{\mathbb{S}_i^- + \mathbb{S}_i^+}$, we get the relative closeness coefficients **CC**(_) to the ideal solution for each option:

	O1	O2	O3	O4	Ranking
CC(_)	0.3934	0.2817	0.8203	0.7462	O3≻O4≻O1≻O2

This means that O3 is the best option.

- If we use CI based TODIM method in [36], for simplicity, we utilize the midpoint instead of each interval in Table 4 and obtain the following group preference relation matrix.

Table 4: The group preference relation matrix

	O1	O2	O3	O4
O1	0.5	0.4625	0.46	0.495
O2	0.5625	0.5	0.3525	0.415
O3	0.5475	0.6525	0.5	0.485
O4	0.5125	0.6025	0.5225	0.5

Then, based on the fuzzy measure λ obtained in Step 3, we calculate the dominance degree $\Phi(\mathrm{Oi}, \mathrm{Oj})$ of Oi over each option Oj for $\theta = 1$ as follows:

$\Phi(\mathrm{Oi}, \mathrm{Oj})$	O1	O2	O3	O4
O1	0	−0.1602	−0.2794	−0.2642
O2	−0.1980	0	−0.3346	−0.3270
O3	0.0772	0.1324	0	−0.0667
O4	0.1470	0.0896	−0.1109	0

By applying $\Phi(\mathrm{Oi}) = \sum_{j=1}^{4} \Phi(\mathrm{Oi}, \mathrm{Oj})$, we obtain the overall dominance degree Φ(_) of each option:

	O1	O2	O3	O4	Ranking
Φ(_)	−0.7038	−0.8596	0.1429	0.1257	O3≻O4≻O1≻O2

This means that O3 is the best option.

We summarize all the above results in Table 5.

Observing from Table 5, we can conclude that (1) [15] (Algorithm 1) cannot distinguish between O3 and O4; (2) The ranking obtained by our proposed algorithm is consistent with the results obtained by CI based TOPSIS method and CI based TODIM method in [36], indicating that our algorithm is effective and superior to [15] (Algorithm 1).

Table 5: Ranking order for different methods

Method	Ranking
[15] (Algorithm 1)	O3∼O4≻O1≻O2
CI based TOPSIS in [36]	O3≻O4≻O1≻O2
CI based TODIM in [36]	O3≻O4≻O1≻O2
Our proposed algorithm	O3≻O4≻O1≻O2

6 Conclusion

In this paper, we mainly focus on the construction of interval-valued OWA (IVOWA) operators and interval-valued Choquet integrals (IVCI). In theory, based on the complete lattice structure of $(L([0,1]), \preccurlyeq)$ under a generated admissible order $\preccurlyeq$, we construct the first monotonic IVOWA operator "$\mathsf{IIVOWA}_{\Omega}^{\preccurlyeq}$" and the first monotonic IVCI "$\mathbf{IC}_{\lambda}^{\preccurlyeq}$" under a fixed admissible order. This lattice-based construction fundamentally differs from existing interval-valued extensions, which mainly rely on formal or parallel translations of real-valued operators and generally fail to preserve monotonicity under admissible orders. We further prove that the proposed operators satisfy the axiomatic requirements of aggregation operators, including idempotency, monotonicity, boundedness, and consistency with their real-valued counterparts. In terms of application, we establish a multi-expert decision-making algorithm based on the proposed operators and the average of the global interval-valued preference of each alternative under all admissible orders $\preccurlyeq_{\alpha,1}$ $(\alpha \in [0,1))$. Compared with approaches relying on a single admissible order, this strategy reduces order-dependence and improves the robustness of the final ranking. The effectiveness of the proposed algorithm is illustrated through a broadband internet selection problem for students in the same dormitory.

In the future, our study will focus on the construction of monotonic interval-valued Sugeno integrals and other non-additive integrals. Moreover, the lattice-based construction method developed in this paper can be further extended to other fuzzy frameworks, such as intuitionistic fuzzy sets, picture fuzzy sets, hesitant fuzzy sets, probabilistic linguistic term sets, and spherical fuzzy sets, where existing OWA operators and Choquet integrals generally suffer from non-monotonicity issues.

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Appendix A Basic Order-Theoretic Concepts

Definition A1 ([37]): *(**Poset**) A partial order $\preccurlyeq$ on the set L is a binary relation on L, which satisfies the following conditions: for $x, y, z \in L$,*

(P1) $x \preccurlyeq x$ (reflexivity);

(P2) If $x \preccurlyeq y$ and $y \preccurlyeq x$, then $x = y$ (antisymmetry);

(P3) If $x \preccurlyeq y$ and $y \preccurlyeq z$, then $x \preccurlyeq z$ (transitivity).

If $x \preccurlyeq y$ and $x \neq y$, we write $x \prec y$, and say that x "is less than" y. A set L with a partial order $\preccurlyeq$ is called a *partially ordered set* (poset) and denoted by $(L, \preccurlyeq)$.

Definition A2 ([37]): *(**Chain**) A partial order $\preccurlyeq$ on the set L is called a linear order if it satisfies the following condition (P4):*

(P4) For any $x, y \in L$, either $x \preccurlyeq y$ or $y \preccurlyeq x$ (comparability).

A set L with a linear order $\preccurlyeq$ is called a *chain*.

Definition A3 ([37]): *(**Lattice**) A lattice is a poset satisfying that every pair of two elements have a greatest lower bound and a smallest upper bound. A lattice is bounded if it has a top element and a bottom element.*

Definition A4 ([37]): *A lattice $(L, \preccurlyeq)$ is complete if every subset of L has a greatest lower bound and a smallest upper bound under the partial order $\preccurlyeq$.*

Appendix B IV Fuzzy Preference Relation Matrices in Example 5

Table A1: The IV fuzzy preference relation matrix $\mathsf{R}_{IV}^{(\mathsf{US}_1)}$

		O1	O2	O3	O4
$\mathsf{R}_{IV}^{(\mathsf{US}_1)} =$	O1	[0.5, 0.5]	[0.3, 0.4]	[0.4, 0.5]	[0.45, 0.5]
	O2	[0.6, 0.7]	[0.5, 0.5]	[0.3, 0.4]	[0.4, 0.45]
	O3	[0.5, 0.6]	[0.6, 0.7]	[0.5, 0.5]	[0.4, 0.5]
	O4	[0.5, 0.55]	[0.55, 0.6]	[0.5, 0.6]	[0.5, 0.5]

Table A2: The IV fuzzy preference relation matrix $\mathsf{R}_{IV}^{(\mathsf{US}_2)}$

		O1	O2	O3	O4
$\mathsf{R}_{IV}^{(\mathsf{US}_2)} =$	O1	[0.5, 0.5]	[0.4, 0.45]	[0.45, 0.5]	[0.4, 0.5]
	O2	[0.55, 0.6]	[0.5, 0.5]	[0.3, 0.4]	[0.45, 0.5]
	O3	[0.5, 0.55]	[0.6, 0.7]	[0.5, 0.5]	[0.45, 0.5]
	O4	[0.5, 0.6]	[0.5, 0.55]	[0.5, 0.55]	[0.5, 0.5]

Table A3: The IV fuzzy preference relation matrix $R_{IV}^{(US_3)}$

		O1	O2	O3	O4
$R_{IV}^{(US_3)} =$	O1	[0.5, 0.5]	[0.55, 0.6]	[0.4, 0.45]	[0.5, 0.6]
	O2	[0.4, 0.45]	[0.5, 0.5]	[0.35, 0.4]	[0.3, 0.4]
	O3	[0.55, 0.6]	[0.6, 0.65]	[0.5, 0.5]	[0.5, 0.55]
	O4	[0.4, 0.5]	[0.6, 0.7]	[0.45, 0.5]	[0.5, 0.5]

Table A4: The IV fuzzy preference relation matrix $R_{IV}^{(US_4)}$

		O1	O2	O3	O4
$R_{IV}^{(US_4)} =$	O1	[0.5, 0.5]	[0.4, 0.5]	[0.45, 0.5]	[0.45, 0.55]
	O2	[0.5, 0.6]	[0.5, 0.5]	[0.3, 0.35]	[0.35, 0.4]
	O3	[0.5, 0.55]	[0.65, 0.7]	[0.5, 0.5]	[0.45, 0.5]
	O4	[0.45, 0.55]	[0.6, 0.65]	[0.5, 0.55]	[0.5, 0.5]