Enforcing interface flux continuity in enhanced XFEM: stability analysis

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XFEM is found to be an efficient approach for solving multiphase problems. The model problem reads as follows, find u taking values in $\Omega_1 \cup \Omega_2$ such that

$$(1a) \qquad \nabla\cdot(-\nu_1\nabla u) = f \qquad\qquad \text{in } \Omega_1$$

$$(1b) \qquad \nabla\cdot(-\nu_2\nabla u) = f \qquad\qquad \text{in } \Omega_2$$

$$(1c) \qquad -\nu\nabla u\cdot\mathbf{n} = g_N \qquad\qquad \text{on } \Gamma_N$$

$$(1d) \qquad u = u_D \qquad\qquad \text{on } \Gamma_D$$

$$(1e) \qquad \nu_1\nabla u|_{\Omega_1}\cdot\mathbf{n} = \nu_2\nabla u|_{\Omega_2}\cdot\mathbf{n} \qquad\qquad \text{on } \Gamma := \partial\Omega_1 \cap \partial\Omega_2$$

The level set representation of the phase domains allows having a grid independent of the location of the interface [2]. In order to introduce the necessary gradient discontinuities inside the elements crossed by the interface, XFEM uses the partition of the unity idea to enrich the discretization. In this context, a sensible choice for the enrichment is using a ridge function R defined as

$$R = \sum_{i=1}^{n_H} N_i|\phi_i| - \left|\sum_{i=1}^{n_H} N_i\phi_i\right|,$$

being N_i the shape functions and ϕ_i the nodal values of the level set, for $i = 1,\ldots,n_H$, see [1, 3]. Thus, the XFEM approximation reads

$$u_X = \sum_{i=1}^{n_H} N_i u_i + \sum_{j\in\mathcal{N}_a} R N_j a_j,$$

where the coefficients u_i for $i = 1,\ldots,n_H$ are the standard Finite Element nodal unknowns and a_j, $j \in \mathcal{N}_a$, stand for the enriched nodal coefficients.

XFEM provides a much better approximation of the multiphase solution, improving the quality the global quantity (energy like) that the variational form of the problem seeks minimizing. Nevertheless, when applied to diffusion problems in a multiphase setup with high diffusivity contrast, the XFEM strategy suffers from an inaccurate representation of the local fluxes in the vicinity of the interface. The XFEM enrichment improves the global quality of the solution but it is not properly enforcing any local feature to the fluxes. Thus, the resulting numerical fluxes in the vicinity of the interface are not realistic, in particular when the parametric contrast between the two phases is important. An additional restriction to the XFEM formulation is introduced, aiming at properly reproducing the features of the local fluxes in the transition zone. This restriction is implemented through Lagrange multipliers. The resulting enlarged variational problem reads find the XFEM approximation $u_X \in V_X$ and the (discrete) Lagrange multiplier $\lambda_H \in \tilde{V}_H$

such that

(2a) $\qquad a(u_X, w) + b(\lambda_H, w) = \ell(w) \qquad\qquad \forall w \in V_{X,0}$

(2b) $\qquad\qquad b(\mu, u_X) = 0 \qquad\qquad\qquad \forall \mu \in \tilde{V}_H$

being $a(\cdot, \cdot)$ the standard bilinear form representing the weak form of problem (1) and

(3) $$b(\mu, u) := \int_\Gamma (\nu_1 \nabla u|_{\Omega_1} - \nu_2 \nabla u|_{\Omega_2}) \cdot \mathbf{n}\, \mu\, d\Gamma.$$

Note that (2b) is the weak form of (1e) and it is the restriction aiming at improving the quality of the flux continuity and, consequently, the quality of the fluxes in the vicinity of the interface. Several examples are presented and the solutions obtained from (2) show a spectacular improvement of the quality of the fluxes with respect to the standard XFEM.
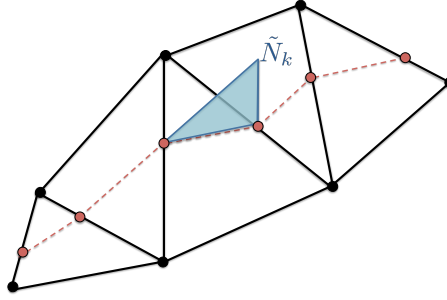

FIGURE 1. Illustration of the semi-hat functions of the Lagrange multipliers space, $\tilde{N}_k$.

The problem of choosing the proper Lagrange multiplier space introduces a classical dilemma: if $\tilde{V}_H$ is too *small* the restriction is not properly enforced and if it is too *large* the resulting method may be unstable. After some numerical tests, the option selected corresponds to the semi hat functions along the interface, as illustrated in figure 1. In this case, the dimension of $\tilde{V}_H$ is twice the number of elements crossed by the interface.

The mathematical proof of the stability of the numerical scheme requires checking if the LBB condition (also known as inf-sup condition) is fulfilled for the elected spaces and bilinear restriction. We propose a novel approach to prove this proposition by introducing an equivalent form of the theorem and two auxiliary lemmas.

Recall that the well-known LBB compatibility condition, is sufficient to guarantee the stability of the formulation. In other words, the formulation is stable if

it exists $k > 0$ such that

$$(4) \qquad \inf_{\mu \in \tilde{V}_H} \sup_{w \in V_X} \frac{b(\mu, w)}{||\mu|| \, ||w||} \geq k$$

The LBB condition is equivalent to the following

Proposition: $\exists \alpha > 0$ such that $\forall \mu \in \tilde{V}_H$, $\exists v \in V_X$ verifying

$$(5a) \qquad \llbracket \nu \nabla v \cdot \mathbf{n} \rrbracket = \mu$$

$$(5b) \qquad ||v||_{V_X} \leq \alpha ||\mu||_{\tilde{V}_H}$$

The equivalence is straightforwardly shown by considering that

$$b(\mu, w) = \int_\Gamma \mu^2 d\Gamma = ||\mu|| \text{ and } \frac{b(\mu, v)}{||\mu|| \, ||v||} = \frac{||\mu||}{||v||}$$

Thus, since $||v|| \leq \alpha ||\mu||$, taking $k = 1/\alpha$ the LBB condition follows.

The latter proposition is reduced to proof the two following lemmas.

Lemma 1 (local version of the proposition, restricted to one element):
Let Ω^k be one linear triangular element crossed by the interface Γ. The restriction of Γ to Ω^k is denoted Γ^k. The nodes of Ω^k are denoted P_1, P_2 and P_3, choosing the order such that P_1 and P_2 are on the same side of the interface. As classically done in XFEM, we assume that $\exists \epsilon > 0$ such that $|\Gamma^k| > \epsilon$. The restrictions of the functional spaces V_X and $\tilde{V}_H$ to Ω^k and Γ^k are denoted V_X^k and $\tilde{V}_H^k$, with respective norms $||v||_{V_X^k}^2 = \int_{\Omega^k} v^2 d\Omega$ and $||\mu||_{\tilde{V}_H^k}^2 = \int_{\Gamma^k} \mu^2 d\Gamma$. The standard FE shape function corresponding to the node P_1 is denoted N_1, and the ridge function R.

Then, $\exists \alpha > 0$ such that $\forall \mu \in \tilde{V}_H^k$, $\exists v \in \text{span}\{N_1, RN_1\} \subset V_X^k$ (i.e. describing v with the d.o.f. corresponding to P_1 only) verifying

$$(6a) \qquad \llbracket \nu \nabla v \cdot \mathbf{n} \rrbracket = \mu$$

$$(6b) \qquad ||v||_{V_X^k} \leq \alpha ||\mu||_{\tilde{V}_H^k}$$

Lemma 2 (controlled propagation of the norm along the interface elements strip): Let Ω^k and Ω^{k+1} be two contiguous elements crossed by the interface. Let us denote P_1 and P_3 the common nodes to Ω^k and Ω^{k+1}, being P_2 the third node in Ω^k. P_1 is selected such that it is on the same side of the interface as P_2. The third node in Ω^{k+1} is denoted as P_4. Then, $\exists \beta > 0$ such that, for any v defined by the d.o.f. of Ω^k, $v \in \text{span}\{N_i, RN_i\}$, $i = 1, 2, 3$ it holds that $||v||_{V_X^{k+1}} \leq \beta ||v||_{V_X^k}$.

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