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# SPACE-TIME FLUID-STRUCTURE INTERACTION: FORMULATION AND DG(0) TIME DISCRETIZATION

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**Abstract.** This contribution is related to the key-note lecture 'Space-time fluid-structure interaction with adjoint-based methods for error estimation and optimization' given in the Minisymposium 'Innovative Methods for Fluid-Structure Interaction'. The main objective is two-fold. First, we design function spaces and a space-time variational-monolithic formulation of fluid-structure interaction in arbitrary Lagrangian-Eulerian coordinates. Second, we apply a Galerkin-time discretization using discontinuous finite elements of degree $r = 0$. Therein, the main emphasis is on the correct derivation of the jump terms and the integration of nonlinear time derivatives, as the latter arise due to the arbitrary Lagrangian-Eulerian transformation.

## 1 Introduction

Fluid-structure interaction (FSI) has been an important topic for at least two decades with numerical applications in applied mathematics, computational science, and engineering [9, 17, 19, 8, 4, 7, 28, 18, 41]. The objective of this work is two-fold. First, we derive a space-time fluid-structure interaction model based on arbitrary Lagrangian-Eulerian (ALE) coordinates [12, 23, 15]. For a general recent overview on space-time methods, we refer to [24]. Moreover, in [6] space-time Navier-Stokes formulations and in [3] space-time formulations of the elastic wave equation were derived. Early space-time formulations for fluid-structure interaction were proposed in [35, 34, 36, 32]. A space-time fluid-structure interaction formulation for temporal error control was utilized in [14]. Second, we apply a Galerkin finite element scheme in time using discontinuous trial and test functions. Therein, nonlinear time derivative terms due to the ALE transformation arise. Our main aim is to derive numerical approximations on how to deal with these terms.

The outline of this study is as follows: In Section 2 the equations in their natural coordinates, the ALE transformation, and the space-time fluid-structure interaction formulation are addressed. Next, in Section 3 a Galerkin space-time discretization is formally derived. In Section 4, a dG(0) realization in time is carried out with a focus on the nonlinear time derivative terms. Our work is briefly summarized in Section 5.

## 2 Fluid-structure interaction modeling

### 2.1 Notation

We denote by $\Omega := \Omega(t) \subset \mathbb{R}^d$, the domain of the FSI problem. The domain consists of two time-dependent subdomains $\Omega_f(t)$ and $\Omega_s(t)$. The FSI-interface between $\Omega_f(t)$ and $\Omega_s(t)$ is denoted by $\Gamma_i(t) = \overline{\partial\Omega_f}(t) \cap \overline{\partial\Omega_s}(t)$. The initial (or later reference) domains are denoted by $\widehat{\Omega}$, $\widehat{\Omega}_f$ and $\widehat{\Omega}_s$, respectively, with the interface $\widehat{\Gamma}_i = \overline{\partial\widehat{\Omega}_f} \cap \overline{\partial\widehat{\Omega}_s}$. Furthermore, we denote the outer boundary by $\partial\widehat{\Omega} = \widehat{\Gamma} = \widehat{\Gamma}_{\text{in}} \cup \widehat{\Gamma}_D \cup \widehat{\Gamma}_{\text{out}}$. To this end, we define the space-time cylinder $\widehat{Q} := \widehat{\Omega} \times I$.

### 2.2 Fluid flow and elastic solids in their natural coordinate systems

The isothermal, incompressible Navier-Stokes equations read [27, 33]: Find $v_f : \Omega_f \times I \to \mathbb{R}^d$ and $p_f : \Omega_f \times I \to \mathbb{R}$ such that

$$
\begin{aligned}
\rho_f \partial_t v_f + \rho_f v \cdot \nabla v - \nabla \cdot \sigma_f(v_f, p_f) &= 0 \quad \text{in } \Omega_f \times I, \\
\nabla \cdot v_f &= 0 \quad \text{in } \Omega_f \times I, \\
v_f^D = v_{in} \text{ on } \Gamma_{in} \times I, \qquad v_f = 0 \text{ on } \Gamma_D \times I, \qquad -p_f n_f + \rho_f \nu_f \nabla v \cdot n_f &= 0 \quad \text{on } \Gamma_{out} \times I, \\
v_f = h_f \text{ on } \Gamma_i \times I, \qquad v_f(0) &= v_0 \quad \text{in } \Omega_f \times \{0\},
\end{aligned}
$$

where the (symmetric) Cauchy stress is given by $\sigma_f(v_f, p_f) := -pI + \rho_f \nu_f (\nabla v + \nabla v^T)$, with the density $\rho_f$ and the kinematic viscosity $\nu_f$. Later in the FSI problem, the function $h_f$ will be given by the solid velocity $v_s$. The normal vector is denoted by $n_f$.

The equations for geometrically non-linear elastodynamics are given as follows: Find a displacement $\widehat{u}_s \colon \widehat{\Omega}_s \times I \to \mathbb{R}^d$ such that

$$
\begin{aligned}
\widehat{\rho}_s \partial_t^2 \widehat{u}_s - \widehat{\nabla} \cdot (\widehat{F}\widehat{\Sigma}) = 0 \quad \text{in } \widehat{\Omega}_s \times I, \\
\widehat{u}_s = 0 \text{ on } \widehat{\Gamma}_D \times I, \qquad \widehat{F}\widehat{\Sigma} \cdot \widehat{n}_s = \hat{h}_s \text{ on } \widehat{\Gamma}_i \times I, \\
\widehat{u}_s(0) = \widehat{u}_0 \text{ in } \widehat{\Omega}_s \times \{0\}, \qquad \widehat{v}_s(0) = \widehat{v}_0 \text{ in } \widehat{\Omega}_s \times \{0\}.
\end{aligned}
$$

The constitutive law is given by the second Piola-Kirchhoff stress tensor $\widehat{\Sigma} = \widehat{\Sigma}_s(\widehat{u}_s) = 2\mu\widehat{E} + \lambda tr(\widehat{E})I$, where the Green-Lagrange strain tensor is given by $\widehat{E} = \frac{1}{2}(\widehat{F}^T\widehat{F} - I)$. Here, $\mu$ and $\lambda$ are the Lamé coefficients for the solid and $\mu_v$ and $\lambda_v$ are constants in the solid damping term. The solid density is denoted by $\widehat{\rho}_s$. Later in FSI, the vector-valued function $\hat{h}_s$ will be given by the normal stress from the fluid problem. Furthermore, $\widehat{n}_s$ denotes the normal vector.

#### 2.2.1 ALE: arbitrary Lagrangian-Eulerian

Usually, solid displacement is computed in the reference configuration $\widehat{\Omega}_s$. The transformation between $\widehat{\Omega}_s$ and $\Omega_s$ is realized by the mapping $\widehat{\mathcal{A}}_s$, which is determined naturally by the solid displacements (see for example [10]). In order to combine both coordinate systems, we employ the ALE approach [12, 23]. Therein, the flow equations are partially Eulerian and partially Lagrangian by a smooth transition. Away from the interface $\widehat{\Gamma}_i$ the flow remains in Eulerian coordinates. On the interface, a full Lagrangian setting is employed. This is realized by the ALE mapping $\widehat{\mathcal{A}}_f$. On the interface $\widehat{\Gamma}_i$, this transformation is given by taking the solid displacement $\hat{u}_f = \hat{u}_s$ such that we can define

$$\widehat{\mathcal{A}}_f(\hat{x})\big|_{\hat{\Gamma}_i} := \hat{x} + \hat{u}_s(\hat{x})\big|_{\hat{\Gamma}_i}. \tag{1}$$

On the outer boundary of the fluid domain, $\partial\widehat{\Omega}_f \setminus \widehat{\Gamma}_i$, there holds $\widehat{\mathcal{A}}_f = \mathrm{id}$. Inside $\widehat{\Omega}_f$, the transformation should be sufficiently smooth and regular. For some mathematical stability analyses, we refer to [15, 16]. Away from the interface $\widehat{\Gamma}_i$ the mapping can be extended arbitrarily in various fashions, for example harmonic, nonlinear harmonic/elastic [31], or biharmonic [38], to mention just a few cases. The biharmonic model reads (in strong formulation): Find a function $\hat{u}_f \colon \widehat{\Omega}_f \to \mathbb{R}^d$ such that

$$\widehat{\alpha}\widehat{\Delta}^2\hat{u} = 0, \quad \hat{u}_f = \hat{u}_s,\ \partial_n\hat{u}_f = \partial_n\hat{u}_s,\ \text{on } \hat{\Gamma}_i, \quad \hat{u}_f = \partial_n\hat{u}_f = 0 \text{ on } \partial\hat{\Omega}_f \setminus \hat{\Gamma}_i, \quad \widehat{\alpha} > 0.$$

In the following, we focus on a variational-monolithic description of the coupled problem [28, 41]. In variational-monolithic coupling, Dirichlet conditions are built into the function spaces and Neumann-type conditions arise naturally on the interface $\Gamma_i$.

To this end, we define a continuous variable $\hat{u}$ in $\hat{\Omega}$ defining the deformation in $\hat{\Omega}_s$ and supporting the transformation in $\hat{\Omega}_f$. Thus, we drop subscripts on $\widehat{u}$, and because the definition of $\widehat{\mathcal{A}}_f$ coincides with the previous definition of $\widehat{\mathcal{A}}_s$, we define in $\hat{\Omega}$:

$$\widehat{\mathcal{A}} := \mathrm{id} + \hat{u}, \quad \hat{F} := \widehat{\nabla}\widehat{\mathcal{A}} = I + \hat{\nabla}\hat{u}, \quad \hat{J} := \det(\hat{F}), \quad \widehat{w}_A := \partial_t\widehat{\mathcal{A}}. \tag{2}$$

In ALE-FSI, we deal with three coupling conditions: continuity of velocities, continuity of normal stresses, and continuity of displacements (geometrical coupling of phyiscal solids and fluid mesh motion). The first two conditions are of physical nature whereas the latter one has geometrical meaning. Mathematically, the first and third condition can be classified as (non-homogeneous) Dirichlet conditions and the second condition is a (non-homogeneous) Neumann condition. In mathematical notation, we have $\widehat{v}_f = \widehat{v}_s$ on $\widehat{\Gamma}_i$. The balance of the normal stresses on the interface is given by $\widehat{J}\widehat{\sigma}_f\widehat{F}^{-T}\widehat{n}_f + \widehat{F}\widehat{\Sigma}\widehat{n}_s = 0$ on $\widehat{\Gamma}_i$. Finally, we have the geometric coupling $\widehat{u}_f = \widehat{u}_s$ on $\widehat{\Gamma}_i$.

### 2.3 Function spaces

For the function spaces in the (fixed) reference domains $\widehat{\Omega}$, $\widehat{\Omega}_f$, $\widehat{\Omega}_s$, we start with the definition of the spaces for the spatial discretization. First we define $\widehat{V} := H^1(\widehat{\Omega})^d$. Next, in the fluid domain, we define further:

$$\begin{aligned}
\widehat{L}_f &:= L^2(\widehat{\Omega}_f), \qquad \widehat{L}_f^0 := L^2(\widehat{\Omega}_f)/\mathbb{R},\\
\widehat{V}_f^0 &:= \{\widehat{v}_f \in H^1(\widehat{\Omega}_f)^d : \widehat{v}_f = 0 \text{ on } \widehat{\Gamma}_{\mathrm{in}} \cup \widehat{\Gamma}_D\},\\
\widehat{V}_{f,\widehat{u}}^0 &:= \{\widehat{u}_f \in H^1(\widehat{\Omega}_f)^d : \widehat{u}_f = \widehat{u}_s \text{ on } \widehat{\Gamma}_i, \quad \widehat{u}_f = 0 \text{ on } \widehat{\Gamma}_{\mathrm{in}} \cup \widehat{\Gamma}_D \cup \widehat{\Gamma}_{\mathrm{out}}\},\\
\widehat{V}_{f,\widehat{u},\widehat{\Gamma}_i}^0 &:= \{\widehat{\psi}_f \in H^1(\widehat{\Omega}_f)^d : \widehat{\psi}_f = 0 \text{ on } \widehat{\Gamma}_i \cup \widehat{\Gamma}_{\mathrm{in}} \cup \widehat{\Gamma}_D \cup \widehat{\Gamma}_{\mathrm{out}}\}.
\end{aligned}$$

In the solid domain, we use

$$\widehat{L}_s := L^2(\widehat{\Omega}_s)^d, \qquad \widehat{V}_s^0 := \{\widehat{u}_s \in H^1(\widehat{\Omega}_s)^d : \widehat{u}_s = 0 \text{ on } \widehat{\Gamma}_D\}.$$

As trial and test spaces for a space-time model, we define

$$\begin{aligned}\widehat{X} = \{U = (\widehat{v}, \widehat{u}_f, \widehat{u}_s, \widehat{w}, \widehat{p}_f) |\ \widehat{v} \in L^2(I, \{\widehat{v}^D + \widehat{V}^0\}),\ \partial_t \widehat{v} \in L^2(I, H^{-1}(\widehat{\Omega})^d),\\ \widehat{u}_f \in L^2(I, \{\widehat{u}_f^D + \widehat{V}^0_{f,\widehat{u}}\}),\ \partial_t \widehat{u}_f \in L^2(I, H^{-1}(\widehat{\Omega}_f)^d),\ \widehat{u}_s \in L^2(I, \{\widehat{u}_s^D + \widehat{V}_s^0\}),\\ \partial_t \widehat{u}_s \in L^2(I, H^{-1}(\widehat{\Omega}_s)^d),\ \widehat{w} \in L^2(I, \widehat{V}),\ \widehat{p}_f \in L^2(I, \widehat{L}_f^0)\}\end{aligned}$$

and

$$\begin{aligned}\widehat{X}^0 = \{U = (\widehat{v}, \widehat{u}_f, \widehat{u}_s, \widehat{w}, \widehat{p}_f) |\ \widehat{v} \in L^2(I, \widehat{V}^0),\ \partial_t \widehat{v} \in L^2(I, H^{-1}(\widehat{\Omega})^d),\ \widehat{u}_f \in L^2(I, \widehat{V}^0_{f,\widehat{u},\widehat{\Gamma}_i}),\\ \partial_t \widehat{u}_f \in L^2(I, H^{-1}(\widehat{\Omega}_f)^d),\ \widehat{u}_s \in L^2(I, \widehat{V}_s^0),\ \partial_t \widehat{u}_s \in L^2(I, H^{-1}(\widehat{\Omega}_s)^d),\\ \widehat{w} \in L^2(I, \widehat{V}),\ \widehat{p}_f \in L^2(I, \widehat{L}_f^0)\}.\end{aligned}$$

### 2.4 A space-time fluid-structure interaction model

**Proposition 2.1** (Variational-monolithic space-time ALE-FSI in $\widehat{\Omega}$)**.** *Find a global vector-valued velocity, vector-valued displacements, additional displacements (due to the splitting of the biharmonic mesh motion model into two second-order equations) and a scalar-valued fluid pressure, i.e.,* $\widehat{U} := (\widehat{v}, \widehat{u}_f, \widehat{u}_s, \widehat{w}, \widehat{p}_f) \in \widehat{X}$ *such that*

$$\textit{Fluid/solid momentum}\begin{cases}\int_I \Big( (\widehat{J}\widehat{\rho}_f \partial_t \widehat{v}, \widehat{\psi}^v)_{\widehat{\Omega}_f} + (\widehat{\rho}_f \widehat{J}(\widehat{F}^{-1}(\widehat{v} - \widehat{w}_A) \cdot \widehat{\nabla})\widehat{v}), \widehat{\psi}^v)_{\widehat{\Omega}_f} + (\widehat{J}\widehat{\sigma}_f \widehat{F}^{-T}, \widehat{\nabla}\widehat{\psi}^v)_{\widehat{\Omega}_f} \\ + \langle \widehat{\rho}_f \nu_f \widehat{J}(\widehat{F}^{-T}\widehat{\nabla}\widehat{v}^T \widehat{n}_f)\widehat{F}^{-T}, \widehat{\psi}^v\rangle_{\widehat{\Gamma}_{out}} + (\widehat{\rho}_s \partial_t \widehat{v}, \widehat{\psi}^v)_{\widehat{\Omega}_s} + (\widehat{F}\widehat{\Sigma}, \widehat{\nabla}\widehat{\psi}^v)_{\widehat{\Omega}_s}\Big)\,\mathrm{d}t \\ + (\widehat{J}(\widehat{v}(0) - \widehat{v}_0), \widehat{\psi}^v(0))_{\widehat{\Omega}_f} + (\widehat{v}(0) - \widehat{v}_0, \widehat{\psi}^v(0))_{\widehat{\Omega}_s} = 0\end{cases}$$

$$\textit{Fluid mesh motion (biharmonic/split)}\begin{cases}\int_I \Big( (\widehat{\alpha}\widehat{\nabla}\widehat{w}|_{\widehat{\Omega}_f}, \widehat{\nabla}\widehat{\psi}^u)_{\widehat{\Omega}_f}\Big)\,\mathrm{d}t & = 0\\ \int_I \Big( (\widehat{\alpha}\widehat{w}, \widehat{\psi}^w)_{\widehat{\Omega}} - (\widehat{\alpha}\widehat{\nabla}\widehat{u}_{f,s}, \widehat{\nabla}\widehat{\psi}^w)_{\widehat{\Omega}}\Big)\,\mathrm{d}t & = 0\end{cases}$$

$$\textit{Solid momentum, 2nd eq.}\Big\{\int_I \Big(\widehat{\rho}_s(\partial_t \widehat{u}_s - \widehat{v}|_{\widehat{\Omega}_s}, \widehat{\psi}^u_s)_{\widehat{\Omega}_s}\Big)\,\mathrm{d}t + (\widehat{u}_s(0) - \widehat{u}_{s,0}, \widehat{\psi}^u_s(0)) = 0$$

$$\textit{Fluid mass conservation}\Big\{\int_I \Big( (\widehat{\mathrm{div}}\,(\widehat{J}\widehat{F}^{-1}\widehat{v}), \widehat{\psi}^p_f)_{\widehat{\Omega}_f}\Big)\,\mathrm{d}t = 0$$

*for all* $\widehat{\Psi} = (\widehat{\psi}^v, \widehat{\psi}^u_f, \widehat{\psi}^u_s, \widehat{\psi}^w, \widehat{\psi}^p_f) \in \widehat{X}^0$*. In short, the above problem reads: Find* $\widehat{U} \in \widehat{X}$ *such that*

$$\widehat{A}(\widehat{U})(\widehat{\Psi}) = 0 \quad \forall \widehat{\Psi} \in \widehat{X}^0$$

*where the FSI equations are combined in the semi-linear form* $\widehat{A}(\widehat{U})(\widehat{\Psi})$.

We write this result in an equivalent form by distinguishing time derivative terms, initial conditions and all remaining terms:

**Proposition 2.2.** *Find* $\widehat{U} := (\widehat{v}, \widehat{u}_f, \widehat{u}_s, \widehat{w}, \widehat{p}_f) \in \widehat{X}$ *such that*

$$\begin{aligned}\int_I (\widehat{J}\widehat{\rho}_f \partial_t \widehat{v}, \widehat{\psi}^v)_{\widehat{\Omega}_f}\,\mathrm{d}t + \int_I (\widehat{\rho}_s \partial_t \widehat{v}, \widehat{\psi}^v)_{\widehat{\Omega}_s}\,\mathrm{d}t + \int_I (\widehat{\rho}_s \partial_t \widehat{u}_s, \widehat{\psi}^u_s)_{\widehat{\Omega}_s}\,\mathrm{d}t + \int_I \widehat{A}_{notimeder}(\widehat{U})(\widehat{\Psi})\,\mathrm{d}t\\ + (\widehat{J}\widehat{\rho}_f(\widehat{v}(0) - \widehat{v}_0), \widehat{\psi}^v(0))_{\widehat{\Omega}_f} + \widehat{\rho}_s(\widehat{v}(0) - \widehat{v}_0, \widehat{\psi}^v(0))_{\widehat{\Omega}_s} + \widehat{\rho}_s(\widehat{u}_s(0) - \widehat{u}_{s,0}, \widehat{\psi}^u_s(0))\end{aligned}$$

*where* $\widehat{A}_{notimeder}(\widehat{U})(\widehat{\Psi})$ *(here notimeder stands for 'no time derivatives') contains all terms from Proposition 2.1 that are not initial conditions and contain no time derivatives. This result is the starting point for a discontinuous Galerkin finite element discretizatin in time.*

## 3 Galerkin space-time discretization

### 3.1 Temporal discretization

For a discontinuous Galerkin finite element discretization in time, let

$$\bar{I} = \{0\} \cup I_1 \cup \ldots \cup I_M$$

be a partition of the closed time interval $\bar{I} = [0, T]$, where $T > 0$ is the end time value. Therein, we have left–open subintervals $I_m := (t_{m-1}, t_m]$ and the time step size, i.e., temporal discretization parameter, $k_m := t_m - t_{m-1}$ for $m = 1, \ldots, M$. The time points (i.e., temporal edges in the FEM context) are $0 = t_0 < \ldots < t_m < \ldots t_M = T$. As usual in the finite element method, we need three ingredients, namely geometric elements (here intervals), simple functions (polynomials), and a set of degrees of freedom. Let $r \in \mathbb{N}_0$ be the temporal polynomial degree. We define the semi-discrete space

$$\tilde{X}_k^r := \{\widehat{U}_k \in \widehat{X} |\, U_k|_{I_m} \in P_r(I_m, \widehat{X}),\ \widehat{U}_k(0) \in L^2(\widehat{\Omega})\},$$

where $k$ stands for the temporal discretization parameter, indicating from now on that we work with the semi-discrete spaces $\tilde{X}_k^r$ and semi-discrete solutions $\widehat{U}_k$. The polynomial space is defined as

$$P_r(I_m, \widehat{X}) = \left\{ \sum_{i=0}^{r} a_i t^i |\, a_i \in \widehat{X} \right\}.$$

**Definition 3.1** ($dG(r)$ method)**.** *In the space $\tilde{X}_k^r$ the functions can have jumps at the time points $t_m$ for $m = 1, \ldots, M$, yielding a discontinuous Galerkin method of degree $r$, denoted by $dG(r)$. This space is used as trial and test space. Having at least discontinuous test functions, allows for decoupling in time and setting up a sequential procedure in time. By evaluating the arising integrals with quadrature formulae yields (depending on which quadrature formula is used) schemes that coincide or are similar to well-known finite difference schemes.*

For setting up the $dG(r)$ method, we need to account for the jumps and introduce further notation for $\widehat{U}_k \in \tilde{X}_k^r$:

$$\widehat{U}_{k,m}^{\pm} := \lim_{s \to 0} \widehat{U}_k(t_m \pm s), \quad [\widehat{U}_k]_m := \widehat{U}_{k,m}^{+} - \widehat{U}_{k,m}^{-}.$$

#### 3.1.1 dG(r) discretization of an ODE model problem

To be able to discretize FSI in time with the dG(r) method, we first consider a prototype ODE model problem with discontinuous in time density $\rho(t) \geq 0$, and $\mu(t) \in \mathbb{R}$, which reads: Find $u : I \to \mathbb{R}$ such that

$$\rho(t)\partial_t u(t) + \mu(t) u(t) = f(t) \quad \text{in } I, \qquad u(0) = u_0.$$

Its variational formulation is given by: Find $u \in V := \{v \in L^2(I) |\, \rho \partial_t v \in L^2(I)\}$ such that

$$a(u, \varphi) = l(\varphi) \quad \forall \varphi \in V$$

with

$$a(u,\varphi) := \sum_{m=1}^{M} \int_{I_m} (\rho\partial_t u + \mu u)\varphi \,\mathrm{d}t + \rho_0^- u_0^- \varphi_0^-, \qquad l(\varphi) := \sum_{m=1}^{M} \int_{I_m} f\varphi \,\mathrm{d}t + \rho_0 u_0 \varphi_0^-.$$

In what follows, we apply the discontinuous Galerkin method for advection-reaction problems, see also [13] and [11, Chap.2]. Through application of the product rule to $\partial_t(\rho u)$ and then integrating by parts in time, on the energy level for $\varphi = u$, we obtain

$$\begin{aligned}
\int_{I_m} \rho\partial_t u u \,\mathrm{d}t &= \int_{I_m} \partial_t(\rho u) u \,\mathrm{d}t - \int_{I_m} \partial_t \rho u^2 \,\mathrm{d}t \\
&= -\int_{I_m} \rho u \partial_t u \,\mathrm{d}t + \Big(\rho_m^-(u_m^-)^2 - \rho_{m-1}^+(u_{m-1}^+)^2\Big) - \int_{I_m} \partial_t\rho u^2 \,\mathrm{d}t \\
\Rightarrow \int_{I_m} \rho\partial_t u u \,\mathrm{d}t &= -\frac{1}{2}\int_{I_m} \partial_t\rho u^2 \,\mathrm{d}t + \frac{1}{2}\Big(\rho_m^-(u_m^-)^2 - \rho_{m-1}^+(u_{m-1}^+)^2\Big)
\end{aligned}$$

By a straightforward calculation, it can be shown that

$$a(u,u) = \sum_{m=1}^{M} \int_{I_m} \Big(\mu - \frac{1}{2}\partial_t\rho\Big) u^2 \,\mathrm{d}t + \frac{1}{2}\sum_{m=1}^{M} \Big(\rho_m^-(u_m^-)^2 - \rho_{m-1}^+(u_{m-1}^+)^2\Big) + \rho_0^-(u_0^-)^2.$$

Or equivalently:

$$a(u,u) = \sum_{m=1}^{M} \int_{I_m} \Big(\mu - \frac{1}{2}\partial_t\rho\Big) u^2 \,\mathrm{d}t - \frac{1}{2}\sum_{m=0}^{M-1} [\rho u^2]_m + \frac{1}{2}(\rho_0^-(u_0^-)^2 + \rho_M^-(u_M^-)^2).$$

To show the coercivity of the bilinear form, we assume that $\mu - \frac{1}{2}\partial_t\rho \geq \lambda_0 > 0$ and now we only need to deal with the jump terms. By inserting a zero and using the third binomial theorem in the last step, we get

$$\begin{aligned}
\rho_m^-(u_m^-)^2 - \rho_m^+(u_m^+)^2 &= \rho_m^-(u_m^-)^2 - \rho_m^-(u_m^+)^2 + \rho_m^-(u_m^+)^2 - \rho_m^+(u_m^+)^2 \\
&= -\rho_m^-[u^2]_m - [\rho]_m(u_m^+)^2 \\
&= -2\rho_m^-[u]_m\Big(\frac{u_m^+ + u_m^-}{2}\Big) - [\rho]_m(u_m^+)^2.
\end{aligned}$$

Using the average of $u$ at $t_m$ defined by $\{u\}_m := \frac{1}{2}\big(u_m^+ + u_m^-\big)$ and rewriting the two last terms from before, a coercive bilinear form is given by

$$\tilde{a}(u,\varphi) := \sum_{m=1}^{M} \int_{I_m} (\rho\partial_t u + \mu u)\varphi \,\mathrm{d}t + \sum_{m=0}^{M-1} \rho_m^-[u]_m\{\varphi\}_m + \frac{1}{2}\sum_{m=0}^{M-1} [\rho]_m u_m^+ \varphi_m^+ + \rho_0^- u_0^- \varphi_0^-.$$

In order to get stability of the scheme above, we apply upwinding to the centered flux $\rho_m^-\{\varphi\}_m$. Hence, we get the additional term $\frac{1}{2}\sum_{m=0}^{M-1} \rho_m^-[u]_m[\varphi]_m$ and the modified bilinear form $\hat{a}$ reads

$$\begin{aligned}
\hat{a}(u,\varphi) &:= \sum_{m=1}^{M} \int_{I_m} (\rho\partial_t u + \mu u)\varphi \,\mathrm{d}t + \sum_{m=0}^{M-1} \rho_m^-[u]_m\bigg(\{\varphi\}_m + \frac{1}{2}[\varphi]_m\bigg) + \frac{1}{2}\sum_{m=0}^{M-1} [\rho]_m u_m^+ \varphi_m^+ + \rho_0^- u_0^- \varphi_0^- \\
&= \sum_{m=1}^{M} \int_{I_m} (\rho\partial_t u + \mu u)\varphi \,\mathrm{d}t + \sum_{m=0}^{M-1} \rho_m^-[u]_m\varphi_m^+ + \frac{1}{2}\sum_{m=0}^{M-1} [\rho]_m u_m^+ \varphi_m^+ + \rho_0^- u_0^- \varphi_0^-.
\end{aligned}$$

Consequently, the nonlinear time derivative terms yield additional jump terms.

### 3.1.2 dG(r) semi-discretization of FSI

For fluid-structure by applying the concepts of temporal discretization from the previous ODE model problem, we obtain

**Proposition 3.1** ($dG(r)$ semi-discretization of FSI)**.** *Find $U_k \in \tilde{X}_k^r$ such that*

$$\begin{aligned}
&\sum_{m=1}^{M} \int_{I_m} (\widehat{J}\widehat{\rho}_f \partial_t \widehat{v}_k, \widehat{\psi}^v)_{\widehat{\Omega}_f} + (\widehat{\rho}_s \partial_t \widehat{v}_k, \widehat{\psi}^v)_{\widehat{\Omega}_s} + (\widehat{\rho}_s \partial_t \widehat{u}_k, \widehat{\psi}^u)_{\widehat{\Omega}_s} \,\mathrm{d}t + \sum_{m=1}^{M} \int_{I_m} \widehat{A}_{notimeder}(\widehat{U}_k)(\widehat{\Psi}) \,\mathrm{d}t \\
&+ \sum_{m=0}^{M-1} (\widehat{J}_m^- \widehat{\rho}_f [\widehat{v}_k]_m, \widehat{\psi}_m^{v,+})_{\widehat{\Omega}_f} + \frac{1}{2} (\widehat{\rho}_f [J]_m \widehat{v}_{k,m}^+, \widehat{\psi}_m^{v,+})_{\widehat{\Omega}_f} + (\widehat{\rho}_s [\widehat{v}_k]_m, \widehat{\psi}_m^{v,+})_{\widehat{\Omega}_s} + (\widehat{\rho}_s [\widehat{u}_k]_m, \widehat{\psi}_m^{u,+})_{\widehat{\Omega}_s} \\
&+ (\widehat{J}_0^- \widehat{\rho}_f \widehat{v}_{k,0}^-, \widehat{\psi}_0^{v,-})_{\widehat{\Omega}_f} + (\widehat{\rho}_s \widehat{v}_{k,0}^-, \widehat{\psi}_0^{v,-})_{\widehat{\Omega}_s} + (\widehat{\rho}_s \widehat{u}_{k,0}^-, \widehat{\psi}_0^{u,-})_{\widehat{\Omega}_s} \\
&= (\widehat{J}_0 \widehat{\rho}_f \widehat{v}_0, \widehat{\psi}_0^{v,-})_{\widehat{\Omega}_f} + (\widehat{\rho}_s \widehat{v}_0, \widehat{\psi}_0^{v,-})_{\widehat{\Omega}_s} + (\widehat{\rho}_s \widehat{u}_0, \widehat{\psi}_0^{u,-})_{\widehat{\Omega}_s}
\end{aligned}$$

*for all $\widehat{\Psi} \in \tilde{X}_k^r$ and where $\widehat{A}_{notimeder}(\widehat{U})(\widehat{\Psi})$ is defined as in Proposition 2.2 and where $\widehat{J}_0 = \widehat{J}(\widehat{u}_0)$.*

**Remark 3.2** ($dG(0)$ vs. backward Euler, $\theta = 1$)**.** *For $r = 0$, we deal with the $dG(0)$ scheme, first order in time, which is a variant of the backward Euler scheme (see below) for $\theta = 1$.*

**Remark 3.3** ($dG(1)$)**.** *For $r = 1$, we deal with the $dG(1)$ scheme, which is second order in time and has order three (superconvergence) for some problems; see e.g., [5].*

**Remark 3.4** ($cG(1)$ vs. Crank-Nicolson, $\theta = 0.5$)**.** *When the trial space consists of continuous functions (see definitions for parabolic problems, for instance in [30]), and the test space as above is $\tilde{X}_k^r$, then we obtain a variant of the Crank-Nicolson scheme. Moreover, the shifted Crank-Nicolson scheme and its Galerkin counterpart were investigated in [21].*

**Remark 3.5** (Galerkin representations of the Fractional-Step-Theta scheme)**.** *Recently, Galerkin representations of the Fractional-Step-Theta scheme (see also below) were derived in [25, 26] and also later utilized by ourselves [14].*

## 3.2 Full $cG(s)dG(r)$ space-time discretization

We now apply the spatial discretization with $cG(s)$. Having three globally defined variables $(\widehat{v}, \widehat{u}, \widehat{p})$, and respecting the discrete inf-sup condition for the Navier-Stokes part, e.g., [20], for instance with Taylor-Hood elements, we employ on each element $K \in \mathcal{T}_h^m$

$$\widehat{v}_h|_K \in Q_2^c, \quad \widehat{u}_h|_K \in Q_2^c, \quad \widehat{w}_h|_K \in Q_2^c, \quad \widehat{p}_h|_K \in Q_1^c$$

thus $s = 2$, which means $Q_s^c$ for $\widehat{v}, \widehat{u}$ and the additional displacements $\widehat{w}$ and $Q_{s-1}^c$ for $\widehat{p}$. The fully discretized space-time formulation then reads as

**Proposition 3.2** ($cG(s)dG(r)$ full discretization of FSI). *Find $U_{kh} \in \tilde{X}_{k,h}^{r,s}$ such that*

$$\begin{aligned}
&\sum_{m=1}^{M} \int_{I_m} (\widehat{J}_{kh}\widehat{\rho}_f \partial_t \widehat{v}_{kh}, \widehat{\psi}_{kh}^v)_{\widehat{\Omega}_f} + (\widehat{\rho}_s \partial_t \widehat{v}_{kh}, \widehat{\psi}_{kh}^v)_{\widehat{\Omega}_s} + (\widehat{\rho}_s \partial_t \widehat{u}_{kh}, \widehat{\psi}_{kh}^u)_{\widehat{\Omega}_s} \,\mathrm{d}t \\
&+ \sum_{m=1}^{M} \int_{I_m} \widehat{A}_{notimeder}(\widehat{U}_{kh})(\widehat{\Psi}_{kh}) \,\mathrm{d}t \\
&+ \sum_{m=0}^{M-1} (\widehat{J}_{kh,m}^{-}\widehat{\rho}_f [\widehat{v}_{kh}]_m, \widehat{\psi}_{kh,m}^{v,+})_{\widehat{\Omega}_f} + \frac{1}{2}(\widehat{\rho}_f [J_{kh}]_m \widehat{v}_{kh,m}^{+}, \widehat{\psi}_{kh,m}^{v,+})_{\widehat{\Omega}_f} \\
&\qquad + (\widehat{\rho}_s [\widehat{v}_{kh}]_m, \widehat{\psi}_{kh,m}^{v,+})_{\widehat{\Omega}_s} + (\widehat{\rho}_s [\widehat{u}_{kh}]_m, \widehat{\psi}_{kh,m}^{u,+})_{\widehat{\Omega}_s} \\
&+ (\widehat{J}_{kh,0}^{-}\widehat{\rho}_f \widehat{v}_{kh,0}^{-}, \widehat{\psi}_{kh,0}^{v,-})_{\widehat{\Omega}_f} + (\widehat{\rho}_s \widehat{v}_{kh,0}^{-}, \widehat{\psi}_{kh,0}^{v,-})_{\widehat{\Omega}_s} + (\widehat{\rho}_s \widehat{u}_{kh,0}^{-}, \widehat{\psi}_{kh,0}^{u,-})_{\widehat{\Omega}_s} \\
&= (\widehat{J}_0 \widehat{\rho}_f \widehat{v}_0, \widehat{\psi}_{kh,0}^{v,-})_{\widehat{\Omega}_f} + (\widehat{\rho}_s \widehat{v}_0, \widehat{\psi}_{kh,0}^{v,-})_{\widehat{\Omega}_s} + (\widehat{\rho}_s \widehat{u}_0, \widehat{\psi}_{kh,0}^{u,-})_{\widehat{\Omega}_s}
\end{aligned}$$

*for all $\widehat{\Psi}_{kh} \in \tilde{X}_{k,h}^{r,s}$ and where $\widehat{A}_{notimeder}(\widehat{U}_{kh})(\widehat{\Psi}_{kh})$ is defined as in Proposition 2.2 and where $\widehat{J}_0 = \widehat{J}(\widehat{u}_0)$.*

## 4 Time-stepping scheme based on a dG(0) discretization

We now apply a dG(0) discretization by choosing $r = 0$. To this end, we integrate time derivative terms and apply appropriate quadrature rules to the other terms. We notice that for many FSI problems, the $dG(0)$ scheme is numerically too much damping due to being strongly $A$-stable. For instance, the FSI benchmarks 2 and 3 [22] yield physical oscillations, which are (wrongly) damped out when using strongly $A$-stable schemes (such as backward Euler, i.e., $dG(0)$) or too large time steps; see discussions in [29, 40] or simply take the open-source codes[1] from [39] and set $\theta = 1.0$ and run FSI-2 or FSI-3. Therefore, a $cG(1)$ in time scheme (variant of Crank-Nicolson) or $dG(1)$ should be used in practice for such problems. For the FSI-1 benchmark, however, the $dG(0)$ scheme will work fine.

At $I_m$ we have

$$\begin{aligned}
&\int_{I_m} (\widehat{J}_{kh}\widehat{\rho}_f \partial_t \widehat{v}_{kh}, \widehat{\psi}_{kh}^v)_{\widehat{\Omega}_f} + (\widehat{\rho}_s \partial_t \widehat{v}_{kh}, \widehat{\psi}_{kh}^v)_{\widehat{\Omega}_s} + (\widehat{\rho}_s \partial_t \widehat{u}_{kh}, \widehat{\psi}_{kh}^u)_{\widehat{\Omega}_s} \,\mathrm{d}t + \int_{I_m} \widehat{A}_{notimeder}(\widehat{U}_{kh})(\widehat{\Psi}_{kh}) \,\mathrm{d}t \\
&+ (\widehat{J}_{kh,m-1}^{-}\widehat{\rho}_f [\widehat{v}_{kh}]_{m-1}, \widehat{\psi}_{kh,m-1}^{v,+})_{\widehat{\Omega}_f} + \frac{1}{2}(\widehat{\rho}_f [J_{kh}]_{m-1} \widehat{v}_{kh,m-1}^{+}, \widehat{\psi}_{kh,m-1}^{v,+})_{\widehat{\Omega}_f} \\
&+ (\widehat{\rho}_s [\widehat{v}_{kh}]_{m-1}, \widehat{\psi}_{kh,m-1}^{v,+})_{\widehat{\Omega}_s} + (\widehat{\rho}_s [\widehat{u}_{kh}]_{m-1}, \widehat{\psi}_{kh,m-1}^{u,+})_{\widehat{\Omega}_s} = 0.
\end{aligned}$$

We approximate $\widehat{J}_{kh} = \widehat{J}_{kh,m} := \widehat{J}_{kh,m}^{-}$ in an implicit fashion and obtain

$$\begin{aligned}
&\int_{I_m} (\widehat{J}_{kh,m}^{-}\widehat{\rho}_f \partial_t \widehat{v}_{kh}, \widehat{\psi}_{kh}^v)_{\widehat{\Omega}_f} + (\widehat{\rho}_s \partial_t \widehat{v}_{kh}, \widehat{\psi}_{kh}^v)_{\widehat{\Omega}_s} + (\widehat{\rho}_s \partial_t \widehat{u}_{kh}, \widehat{\psi}_{kh}^u)_{\widehat{\Omega}_s} \,\mathrm{d}t + \int_{I_m} \widehat{A}_{notimeder}(\widehat{U}_{kh})(\widehat{\Psi}_{kh}) \,\mathrm{d}t \\
&+ (\widehat{J}_{kh,m-1}^{-}\widehat{\rho}_f [\widehat{v}_{kh}]_{m-1}, \widehat{\psi}_{kh,m-1}^{v,+})_{\widehat{\Omega}_f} + \frac{1}{2}(\widehat{\rho}_f [J_{kh}]_{m-1} \widehat{v}_{kh,m-1}^{+}, \widehat{\psi}_{kh,m-1}^{v,+})_{\widehat{\Omega}_f} \\
&+ (\widehat{\rho}_s [\widehat{v}_{kh}]_{m-1}, \widehat{\psi}_{kh,m-1}^{v,+})_{\widehat{\Omega}_s} + (\widehat{\rho}_s [\widehat{u}_{kh}]_{m-1}, \widehat{\psi}_{kh,m-1}^{u,+})_{\widehat{\Omega}_s} = 0.
\end{aligned}$$

[1] `https://github.com/tommeswick/fsi`.

Now we employ constant-in-time test functions, integrate in time and use the right sided box rule

$$
\begin{aligned}
&(\widehat{J}^-_{kh,m}\widehat{\rho}_f(\widehat{v}^-_{kh,m} - \widehat{v}^+_{kh,m-1}), \widehat{\psi}^v_h)_{\widehat{\Omega}_f} + (\widehat{\rho}_s(\widehat{v}^-_{kh,m} - \widehat{v}^+_{kh,m-1}), \widehat{\psi}^v_h)_{\widehat{\Omega}_s} + (\widehat{\rho}_s(\widehat{u}^-_{kh,m} - \widehat{u}^+_{kh,m-1}), \widehat{\psi}^u_h)_{\widehat{\Omega}_s} \\
&+ k_m \widehat{A}_{notimeder}(\widehat{U}^-_{kh,m})(\widehat{\Psi}_h) \\
&+ (\widehat{J}^-_{kh,m-1}\widehat{\rho}_f(\widehat{v}^+_{kh,m-1} - \widehat{v}^-_{kh,m-1}), \widehat{\psi}^v_h)_{\widehat{\Omega}_f} + \frac{1}{2}(\widehat{\rho}_f(J^+_{kh,m-1} - J^-_{kh,m-1})\widehat{v}^+_{kh,m-1}, \widehat{\psi}^v_h)_{\widehat{\Omega}_f} \\
&+ (\widehat{\rho}_s(\widehat{v}^+_{kh,m-1} - \widehat{v}^-_{kh,m-1}), \widehat{\psi}^v_h)_{\widehat{\Omega}_s} + (\widehat{\rho}_s(\widehat{u}^+_{kh,m-1} - \widehat{u}^-_{kh,m-1}), \widehat{\psi}^u_h)_{\widehat{\Omega}_s} \\
&= 0.
\end{aligned}
$$

Setting $\widehat{v}_{kh,m} := \widehat{v}^-_{kh,m}$, $\widehat{u}_{kh,m} := \widehat{u}^-_{kh,m}$ and $\widehat{p}_{kh,m} := \widehat{p}^-_{kh,m}$ some terms cancel out, and we get for $m = 1, \ldots, M$ the time stepping scheme

$$
\begin{aligned}
&\frac{1}{2}((\widehat{J}_{kh,m} + \widehat{J}_{kh,m-1})\widehat{\rho}_f\widehat{v}_{kh,m}, \widehat{\psi}^v_h)_{\widehat{\Omega}_f} - (\widehat{J}_{kh,m-1}\widehat{\rho}_f\widehat{v}_{kh,m-1}, \widehat{\psi}^v_h)_{\widehat{\Omega}_f} \\
&+ (\widehat{\rho}_s(\widehat{v}_{kh,m} - \widehat{v}_{kh,m-1}), \widehat{\psi}^v_h)_{\widehat{\Omega}_s} + (\widehat{\rho}_s(\widehat{u}_{kh,m} - \widehat{u}_{kh,m-1}), \widehat{\psi}^u_h)_{\widehat{\Omega}_s} \\
&+ k_m \widehat{A}_{notimeder}(\widehat{U}_{kh,m})(\widehat{\Psi}_h) = 0.
\end{aligned}
$$

This scheme is exactly what we wanted, and now we remark that $\widehat{J}_{kh,m} := \widehat{J}_{kh,m}(\widehat{u})$ from which we see more clearly (indeed we are hiding all other nonlinear terms in $\widehat{A}_{notimeder}(\widehat{U}_{kh,m})(\widehat{\Psi}_h)$) that we deal with a nonlinear system. Thus, we employ Newton's method. To this end, we define

$$
\begin{aligned}
g(\widehat{U}_{kh,m})(\widehat{\Psi}_{kh}) := &\frac{1}{2}((\widehat{J}_{kh,m} + \widehat{J}_{kh,m-1})\widehat{\rho}_f\widehat{v}_{kh,m}, \widehat{\psi}^v_h)_{\widehat{\Omega}_f} - (\widehat{J}_{kh,m-1}\widehat{\rho}_f\widehat{v}_{kh,m-1}, \widehat{\psi}^v_h)_{\widehat{\Omega}_f} \\
&+ (\widehat{\rho}_s(\widehat{v}_{kh,m} - \widehat{v}_{kh,m-1}), \widehat{\psi}^v_h)_{\widehat{\Omega}_s} + (\widehat{\rho}_s(\widehat{u}_{kh,m} - \widehat{u}_{kh,m-1}), \widehat{\psi}^u_h)_{\widehat{\Omega}_s} + k_m \widehat{A}_{notimeder}(\widehat{U}_{kh,m})(\widehat{\Psi}_h)
\end{aligned}
$$

and solve at $t_m$: Find $\widehat{U}_{kh,m} \in \widehat{V}^{s,m}_h$ (respectively some non-homogeneous Dirichlet extensions) such that

$$
g(\widehat{U}_{kh,m})(\Psi_{kh}) = 0 \quad \forall \Psi_{kh} \in \widehat{V}^{s,m}_h.
$$

Here, we can then apply the defect-correction version of Newton's method. For details on Newton's method, derivation of the directional derivatives and corresponding implementations, we refer the reader to [37] and [39] including open-source codes based on the finite element package deal.II [2, 1].

## 5 Conclusions

In this contribution, our focus was on a space-time formulation of fluid-structure interaction in arbitrary Lagrangian-Eulerian coordinates and a dG(0) time discretization. In the latter, a crucial point is the integration of nonlinear time derivative terms. To understand the procedure, we utilized a prototype ODE model problem to carry out integration by parts in time and extended the outcome to our fluid-structure interaction problem. Having such a space-time variational-monolithic fluid-structure interaction formulation at hand, we can derive a consistent adjoint equation, which can be utilized for gradient-based optimization and goal-oriented a posteriori error estimation with the dual-weighted residual method.

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