A hybrid adaptive method for initial-boundary value problems

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ABSTRACT

It is well-known that higher-order methods (as compared to lower order accurate methods) capture transient phenomena more efficiently since they allow for a considerable reduction in the degrees of freedom for a given error tolerance. In particular, high-order finite difference methods (HOFDMs) are ideally suited for problems of this type, cf. the pioneering paper by Kreiss and Oliger [5].

For long-time simulations, it is imperative to use finite difference approximations that do not allow growth in time if the PDE does not allow growth—a property termed *time stability* [3]. Achieving time-stable HOFDM has received considerable past attention. A robust and well-proven high-order finite difference methodology, for well-posed initial boundary value problems (IBVP), is to combine summation-by-parts (SBP) operators [4, 6] and either the simultaneous approximation term (SAT) method [1], or the projection method [7] to impose boundary conditions.

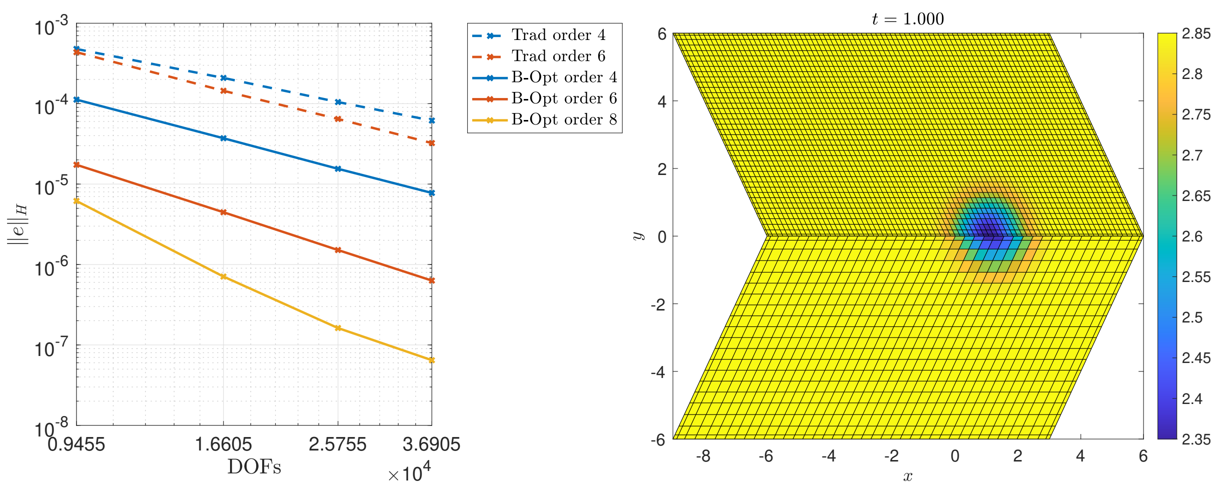

Figure 1: An example of FD-FD nonconforming multiblock coupling for the Euler equations, comparing traditional and B-Opt SBP. Left: Convergence properties. Right: The solution and non-conformal interface.

The SBP-SAT and SBP-Projection methods naturally extends to multi-block geometries. Thus, problems involving complex domains or non-smooth geometries are easily amenable to the approach. The SBP-SAT and SBP-Projection methods can also handle non-conformal interfaces, allowing adaptive grids. Traditional SBP FD operators found in the literature are essentially central finite difference stencils, defined on regular grids, closed at the boundaries with one-sided difference stencils. Traditional SBP operators however suffer from reduced accuracy close to the boundaries. To improve the accuracy, [6] introduced a type of boundary optimized (B-Opt) SBP operators, of orders up to twelve. Recently, we have extended the usage of B-Opt SBP operators to non-conformal interfaces. In Figure 1 the Euler equations are solved using the SBP-SAT method on a non-conformal interface. Here we compare the accuracy of the traditional and B-Opt SBP operators for orders 4,6 and 8.

The SBP-SAT and SBP-Projection methods also allow for a hybrid coupling of different schemes having SBP property. Other examples of discrete operators with the SBP property include spectral collocation, finite volume methods, and continuous Galerkin FE. In a recent paper [2] a hybrid SBP-SAT method to couple HOFDM and FEM in a nonconforming multiblock fashion is presented. The proposed technique results in a time-stable, and accurate discretization. Our most recent results indicate that the less well-known SBP-Projection method has some important advantages as compared to the now relatively mature SBP-SAT method. One of the more obvious advantages with the SBP-Projection method is that it exactly mimics the stability properties of the underlying well-posed IBVP, without tuning of parameters. The SBP-Projection method only requires the discrete operators to have a SBP property. In the present study we will show how the SBP-Projection method can be employed to ensure time-stability and efficiency in the framework of well-posed IBVP. As a proof of concept we will show results from various wave dominated problems, including the Navier-Stokes equations, the elastic wave equation, the acoustic wave equation, and flexural-gravity waves in ice-covered oceans. Some novel results of FD-FD, as well as hybrid FE-FD coupling using the SBP-Projection method on non-conformal interfaces will also be presented.

In Figure 2 the advection equation is solved using the hybrid SBP-SAT method to couple a sixth-order traditional FD scheme with a FE scheme (see [2] for details) on a non-conformal interface.

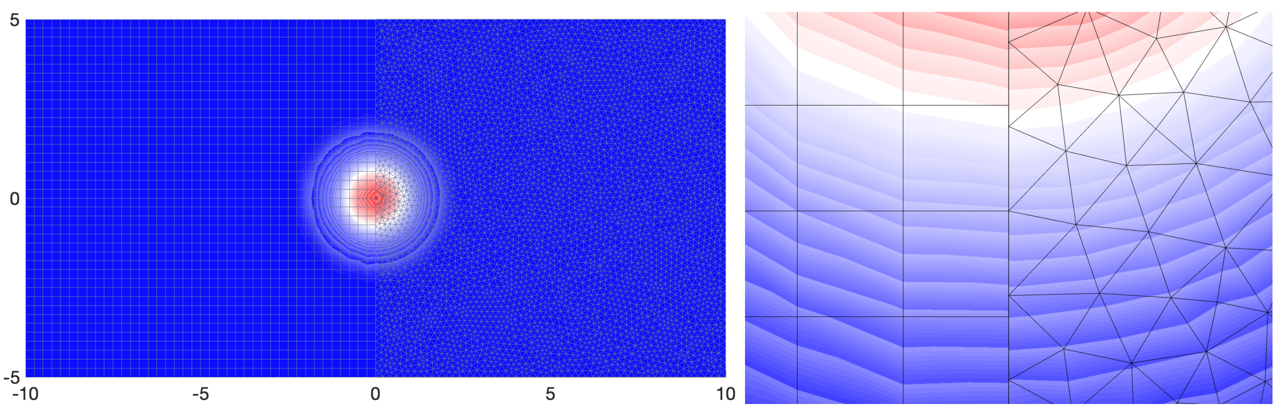

Figure 2: An example of FD-FE nonconforming multiblock coupling for advection equation using a sixth-order FD scheme on the left domain and a FE scheme on the right domain. A close up in the right subfigure.

REFERENCES

[1] M. H. Carpenter, D. Gottlieb, and S. Abarbanel. Time-stable boundary conditions for finite-difference schemes solving hyperbolic systems: Methodology and application to high-order compact schemes. *J. Comput. Phys.*, 111(2):220–236, 1994.

[2] Tuan Anh Dao, Ken Mattsson, and Murtazo Nazarov. Energy stable and accurate coupling of finite element methods and finite difference methods. *Journal of Computational Physics*, 449:110791, 2022.

[3] Bertil Gustafsson. *High Order Difference Methods for Time Dependent PDE*. 2007.

[4] H.-O. Kreiss and G. Scherer. Finite element and finite difference methods for hyperbolic partial differential equations. In Carl de Boor, editor, *Mathematical Aspects of Finite Elements in Partial Differential Equations*, pages 195 – 212. Academic Press, 1974.

[5] Heinz-Otto Kreiss and Joseph Oliger. Comparison of accurate methods for the integration of hyperbolic equations. *Tellus*, XXIV:199–215, 1972.

[6] Ken Mattsson, Martin Almquist, and Edwin van der Weide. Boundary optimized diagonal-norm SBP operators. *Journal of Computational Physics*, 2018.

[7] Ken Mattsson and Pelle Olsson. An improved projection method. *Journal of Computational Physics*, 372:349 – 372, 2018.