A stabilized finite element method for incompressible viscous flows using a finite increment calculus formulation

Eugenio Oñate [*]

International Centre for Numerical Methods in Engineering, Universidad Politécnica de Cataluña, Gran Capitán s/n, 08034 Barcelona, Spain

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Abstract

A stabilized finite element formulation for incompressible viscous flows is derived. The starting point are the modified Navier–Stokes equations incorporating naturally the necessary stabilization terms via a finite increment calculus (FIC) procedure. Application of the standard finite element Galerkin method to the modified differential equations leads to a stabilized discrete system of equations overcoming the numerical instabilities emanating from the advective terms and those due to the lack of compatibility between approximate velocity and pressure fields. The FIC method also provides a natural explanation for the stabilization terms appearing in all equations for both the Navier–Stokes and the simpler Stokes equations. Transient solution schemes with enhanced stabilization properties are also proposed. Finally a procedure for computing the stabilization parameters is presented. © 2000 Elsevier Science S.A. All rights reserved.

1. Introduction

Finite element solution of the incompressible Navier–Stokes equations with the classical Galerkin method may suffer from numerical instabilities from two main sources. The first is due to the advective-diffusive character of the equations which induces oscillations for high values of the velocity. The second source has to do with the mixed character of the equations which limits the choice of finite element interpolations for the velocity and pressure fields [1].

Solutions of these two problems have been extensively sought in the last years. Compatible velocity–pressure interpolations satisfying the inf–sup condition emanating from the second problem above mentioned have been used [1,2]. In addition, the advective operator has been modified to include some "upwinding" effects [3–8]. Recent procedures based on Galerkin Least Square techniques [9,7] allow equal order interpolation for velocity and pressure by introducing a Laplacian of pressure term in the mass balance equation, while preserving the upwinding stabilization of the momentum equations. Similar effects can be obtained using Characteristic Galerkin methods [10,11], Variational Multiscale models [12,13] and analogous Residual-Free Bubbles techniques [14–16]. Most of these methods lack enough stability in the presence of sharp layers transversal to the velocity. This defficiency is usually corrected by adding new "shock capturing" stabilization terms to the already stabilized equations [17–20]. The computation of the stabilization parameters in all these methods is mostly based in "ad hoc" generalizations of the 1D linear

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advective-diffusive problem. Despite several recent attempts there still lacks a general methodology for evaluating the stabilization parameters for fluid flow problems in an objective and accurate manner.

This paper presents a different point view for deriving stabilized finite element methods for incompressible flow problems. The starting point are the stabilized form of the governing differential equations derived via a finite increment calculus (FIC) procedure. This technique presented in [21–25] is based on writting the momentum and mass balance equations over a domain of finite size and retaining higher order terms. These terms incorporate the ingredients for the necessary stabilization of any numerical solution *already at the differential equations level*. Application of the standard Galerkin formulation to the consistently modified differential equations leads to a stabilized system of discretized equations which overcomes *the two problems* above mentioned (i.e., the advective type instability and that due to lack of compatibility between the velocity and pressure fields). In addition, the modified differential equations can be used to derive a numerical scheme for iteratively computing the stabilization parameters in a sort of model adaptivity procedure [22–25].

The paper is structured as follows. In Section 2 the derivation of the stabilized modified differential equations for incompressible Navier–Stokes flows using the FIC method is presented. Details of the stabilized finite element formulation are then given. The case of simpler Stokes flows is considered next. Indeed the stabilized formulation obtained in this case should also be useful for solving the analogous incompressible elasticity problem. A three steps time marching solution scheme with enhanced stabilization properties is derived starting from the stabilization differential equations for the transient case. In the last part of the paper a procedure for computing the stabilization parameters is proposed.

2. Stabilized governing equations for incompressible flows

The stabilized governing equations for incompressible viscous flows are obtained by applying the standard conservation laws expressing balance of momentum and mass over a control domain. Assuming that the control domain has finite dimensions and representing the variation of mass and momentum over the domain using Taylor series expansions of one order higher than those used in the standard infinitesimal theory, the following expressions are found [21,22]:

Momentum

$$r_{m_i} - \frac{1}{2} h_{mj} \frac{\partial r_{m_i}}{\partial x_j} = 0 \quad \text{in } \Omega, \tag{1}$$

Mass balance

$$r_d - \frac{1}{2} h_{dj} \frac{\partial r_d}{\partial x_j} = 0 \quad \text{in } \Omega, \tag{2}$$

where for the steady state case

$$r_{m_i} = \rho \frac{\partial (u_i u_j)}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} - b_i, \tag{3}$$

$$r_d = \frac{\partial u_i}{\partial x_i} \tag{4}$$

with $i, j = 1, 2$ for a two dimensional flow.

In Eq. (3) ρ is the fluid density (here assumed to be constant), u_i is the velocity component in the ith direction, p the pressure, b_i the body forces and τ_{ij} the viscous stress components related to the velocity gradients through the fluid viscosity μ by

$$\tau_{ij} = 2\mu \left(\varepsilon_{ij} - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right) \tag{5a}$$

with

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$
(5b)

Einstein summation convention for repeated indexes in products and derivatives is used, i.e., $h_{dj}(\partial r_d/\partial x_j) = \sum_j h_{dj}(\partial r_d/\partial x_j)$.

Eqs. (1) and (2) are the *stabilized forms* of the governing differential equations for an incompressible flow. The terms underlined in (1) and (2) introduce naturally the necessary stabilization at the discretization level. The so called *characteristic length* vectors $\mathbf{h}_m$ and $\mathbf{h}_d$ are defined as (for 2D problems)

$$\mathbf{h}_m = \left\{\begin{array}{c} h_{m_1} \\ h_{m_2} \end{array}\right\}, \quad \mathbf{h}_d = \left\{\begin{array}{c} h_{d_1} \\ h_{d_2} \end{array}\right\},$$
(6)

where h_{m_1} and h_{m_2} are the dimensions of the finite control domain where balance of momentum is enforced. Similarly h_{d_1} and h_{d_2} represent the dimensions of the domain where mass conservation is expressed. The components of vectors $\mathbf{h}_m$ and $\mathbf{h}_d$ introduce the necessary stabilization along the streamline and transverse directions to the flow in the discrete problem.

The method to derive the modified differential Eqs. (1) and (2) incorporating the stabilization terms was termed in [22] FIC as a reference to the standard infinitesimal calculus techniques where the size of the domain where balance of mass and momentum is enforced is assumed to be negligible. Note that for $\mathbf{h}_m = \mathbf{h}_d \to 0$ the standard infinitesimal form of the momentum and mass balance equations is recovered [21,22].

Eqs. (1) and (2) are complemented by the following boundary conditions [21,22].

Balance of momentum at the boundary Γ_t

$$n_j\tau_{ij} - t_i + \underline{\frac{1}{2}h_{m_j}n_j r_{m_i}} = 0 \quad \text{on } \Gamma_t,$$
(7)

where n_i is the ith component of the unit normal vector to the boundary and t_i are the prescribed tractions at the Neumann boundary Γ_t of the analysis domain Ω.

Prescribed velocity at the boundaries

$$u_t = u_t^p \quad \text{on } \Gamma_{u_t},$$
(8)

$$u_n - \underline{\frac{1}{2}h_{d_i}n_i r_d} = u_n^p \quad \text{on } \Gamma_{u_n}.$$
(9)

In Eq. (8) u_t and u_t^p denote the tangential velocity to the boundary and its prescribed value, respectively.

Eq. (9) expresses the balance of mass on an arbitrary domain next to the boundary. In Eq. (9) u_n and u_n^p denote the velocity normal to the boundary and its prescribed value, respectively. The value of u_n^p is zero on solid walls and stationary free surfaces.

Also in Eqs. (8) and (9) Γ_{u_t} and Γ_{u_n} are the parts of the boundary Γ of Ω where the tangential and normal velocities are prescribed, respectively. The Dirichlet boundary is defined as $\Gamma_u = \Gamma_{u_t} \cup \Gamma_{u_n}$.

The underlined terms in Eqs. (7) and (8) introduce the necessary stabilization at the boundaries in a form consistent with that of Eqs. (1) and (2). These terms are obtained by invoking balance of momentum and mass at a domain of finite size next to the boundary. Details of the derivation of Eqs. (1)–(8), can be found in [21,22] whereas the derivation of Eq. (9) is shown in Appendix A.

2.1. Alternative form of stabilized governing equations

Let us express the components of the characteristic vector $\mathbf{h}_d$ for the mass balance equation as

$$h_{d_i} = -2\rho\tau_{d_i}u_i,$$
(10)

where the τ_{d_i} parameters are termed "intrinsic times" per unit mass. The negative sign in Eq. (10) is necessary to introduce a positive stabilization in the mass balance equation at the discrete level as it will be shown later.

From simple differentiation rules we can write

$$u_i \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) = \frac{\partial}{\partial x_i} \left(u_i \frac{\partial u_j}{\partial x_j} \right) - \left(\frac{\partial u_k}{\partial x_k} \right)^2. \tag{11}$$

Substituting Eq. (10) into Eq. (2) and making use of Eqs. (11), (1) and (3) we can rewrite the mass balance equation (neglecting higher order terms) as

$$r_d - \tau_{di} \frac{\partial \hat{r}_{m_i}}{\partial x_i} = 0, \tag{12a}$$

where

$$\hat{r}_{m_i} = \rho u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} - b_i. \tag{12b}$$

Following a similar process, Eq. (9) expressing balance of mass at the boundary can be rewritten using Eqs. (1) and (10) as

$$u_n - \tau_{d_i} n_i \hat{r}_{m_i} = u_n^p \quad \text{on } \Gamma_{u_n}. \tag{13}$$

We summarize next for the sake of clarity the set of governing equations to be solved.
Momentum

$$r_{m_i} - \frac{1}{2} h_{m_j} \frac{\partial r_{m_i}}{\partial x_j} = 0 \quad \text{in } \Omega. \tag{14}$$

Mass balance

$$r_d - \tau_{d_i} \frac{\partial \hat{r}_{m_i}}{\partial x_i} = 0 \quad \text{in } \Omega. \tag{15}$$

Boundary conditions

$$n_j \sigma_{ij} - t_i + \frac{1}{2} h_{m_j} n_j r_{m_i} = 0 \quad \text{on } \Gamma_t, \tag{16}$$

$$u_t - u_t^p = 0 \quad \text{on } \Gamma_{u_t}, \tag{17}$$

$$u_n - \tau_{d_i} n_i \hat{r}_{m_i} - u_n^p = 0 \quad \text{on } \Gamma_{u_n}, \tag{18}$$

where r_{m_i} and $\hat{r}_{m_i}$ are defined in Eqs. (3) and (12b), respectively.

A similar form of the modified differential equations for momentum and mass balance (Eqs. (14) and (15)) has been recently proposed by Ilinca et al. [26]. They express the exact solution as sum of the numerical approximation and a perturbation. The modified equations are derived by expanding the original differential equations for momentum and mass balance in Taylor series and eliminating the perturbation terms. However, the boundary conditions remain unchanged and thus the stabilizing terms in Eqs. (16) and (18) are omitted in [26]. This leads to the appearance of additional boundary integrals in the Galerkin formulation. These terms vanish naturally if the full stabilized expressions (14)–(18) emanating from the FIC method are used as shown in Section 3.

3. Finite element formulation

Let us now introduce a standard finite element interpolation of the velocity and pressure fields written as

$$u_i \simeq \bar{u}_i = \sum_{j=1}^n N_j^u (\bar{u}_i)_j, \tag{19a}$$

$$p \simeq \bar{p} = \sum_{j=1}^{n} N_j^p \bar{p}_j, \tag{19b}$$

where N_i^u and N_i^p are the shape functions interpolating the velocity u_i and the pressure p within each element and $(\bar{\cdot})_j$ denotes nodal values [1]. The numerical solution residuals are now defined as

$$\bar{r}_{m_i} = r_{m_i}(\bar{u}_i, \bar{p}), \tag{20a}$$

$$\hat{\bar{r}}_{m_i} = \hat{r}_{m_i}(\bar{u}_i, \bar{p}), \tag{20b}$$

$$\bar{r}_d = r_d(\bar{u}_i). \tag{20c}$$

Let us next apply the standard weighted residual method to the discretized stabilized governing equations (14)–(18). This gives

Momentum

$$\int_{\Omega} v_k \left[\bar{r}_{m_i} - \frac{1}{2} h_{m_j} \frac{\partial \bar{r}_{m_i}}{\partial x_j} \right] d\Omega + \int_{\Gamma_t} \hat{v}_k \left[n_j \bar{\sigma}_{ij} - t_i + \frac{1}{2} h_{m_j} n_j \bar{r}_{m_i} \right] d\Gamma = 0. \tag{21}$$

Mass balance

$$\int_{\Omega} q \left[\bar{r}_d - \tau_{d_i} \frac{\partial \hat{\bar{r}}_{m_i}}{\partial x_i} \right] d\Omega + \int_{\Gamma_{u_n}} \hat{q} [\bar{u}_n - \tau_{d_i} n_i \hat{\bar{r}}_{m_i} - u_n^p] d\Gamma = 0. \tag{22}$$

In above v_k, $\hat{v}_k$, q and $\hat{q}$ are appropriate weighting functions and $(\bar{\cdot})$ denote approximate values. Note that in above equations exact satisfaction of the boundary condition on the tangential displacements (Eq. (17)) has been assumed. On the other hand, the condition on the normal displacement at the boundary (Eq. (18)) is imposed in a weak form via the second integral of Eq. (22).

The integrals in Eqs. (21) and (22) involve derivatives of the discretized residuals. These residuals are usually discontinuous across the element faces and hence the residual derivatives are not defined on element boundaries. This problem can be simply overcome by computing these derivatives in a distributional sense if the weighting functions and the stabilization parameters are assumed to be continuous as

$$\int_{\Omega} v_k h_{m_j} \frac{\partial \bar{r}_{m_i}}{\partial x_j} d\Omega = - \sum_e \int_{\Omega^e} \frac{\partial (h_{m_j} v_k)}{\partial x_j} \bar{r}_{m_i} d\Omega + \int_{\Gamma} v_k h_{m_j} \bar{r}_{m_i} n_j d\Gamma, \tag{23}$$

$$\int_{\Omega} q \tau_{d_i} \frac{\partial \hat{\bar{r}}_{m_i}}{\partial x_i} d\Omega = - \sum_e \int_{\Omega^e} \frac{\partial (\tau_{d_i} q)}{\partial x_i} \hat{\bar{r}}_{m_i} d\Omega + \int_{\Gamma} q \tau_{d_i} \hat{\bar{r}}_{m_i} n_i d\Gamma. \tag{24}$$

In above the sums extend over the element interiors Ω^e. A proof of above equalities is given in Appendix B and also in [26].

Substituting Eqs. (23) and (24) into Eqs. (21) and (22), respectively and choosing $\hat{v}_k = v_k$ with $v_k = 0$ on Γ_u and $\hat{q} = -q$ with $q = 0$ on Γ_t gives

$$\int_{\Omega} v_k \bar{r}_{m_i} d\Omega + \int_{\Gamma_t} v_k [n_j \bar{\sigma}_{ij} - t_i] d\Gamma + \sum_e \int_{\Omega^e} \frac{1}{2} \frac{\partial (h_{m_j} v_k)}{\partial x_j} \bar{r}_{m_i} d\Omega = 0, \tag{25}$$

$$\int_{\Omega} q \bar{r}_d d\Omega + \sum_e \int_{\Omega^e} \frac{\partial (\tau_{d_i} q)}{\partial x_i} \hat{\bar{r}}_{m_i} d\Omega - \int_{\Gamma_{u_n}} q(\bar{u}_n - u_n^p) d\Gamma = 0. \tag{26}$$

Note that the boundary Γ_{u_t} does not appear in Eq. (26) as the Dirichlet boundary for the mass balance equation coincides with Γ_{u_n}. Also, the last integral in Eq. (26) imposes the equality between the normal

velocity at the boundary and its prescribed value in a weak form. This is consistent with the original stabilized equations (9) or (18) derived from balance of mass at the boundary Γ_{u_n} (see Appendix A).

The integrals in the first term of Eq. (25) involving the derivatives of the pressure and the viscous stresses are treated in a distributional sense. This allows to use discontinuous pressure and stress fields across element interfaces. These integrals are computed as follows

$$\int_{\Omega} v_k \left[\frac{\partial \bar{p}}{\partial x_i} - \frac{\partial \bar{\tau}_{ij}}{\partial x_j} \right] \mathrm{d}\Omega = -\sum_e \int_{\Omega^e} \left[\frac{\partial v_k}{\partial x_i} \bar{p} - \frac{\partial v_k}{\partial x_j} \bar{\tau}_{ij} \right] \mathrm{d}\Omega + \int_{\Gamma} v_k(\bar{p}\delta_{ij} - \bar{\tau}_{ij})n_j \, \mathrm{d}\Gamma. \tag{27}$$

Substituting Eq. (27) into Eq. (25) noting that $\sigma_{ij} = \tau_{ij} - p\delta_{ij}$ and imposing $v_k = 0$ on Γ_u gives after simplification the stabilized integral form of the momentum equations as

$$\int_{\Omega} v_k \left(\rho \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_j} + \rho \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \right) \mathrm{d}\Omega + \sum_e \int_{\Omega^e} \left(-\frac{\partial v_k}{\partial x_i} \bar{p} + \frac{\partial v_k}{\partial x_j} \bar{\tau}_{ij} \right) \mathrm{d}\Omega$$

$$+ \sum_e \int_{\Omega^e} \frac{1}{2} \frac{\partial(h_{m_j} v_k)}{\partial x_j} \bar{r}_{m_i} \, \mathrm{d}\Omega - \int_{\Omega} v_k b_i \, \mathrm{d}\Omega - \int_{\Gamma_i} v_k t_i \, \mathrm{d}\Gamma = 0. \tag{28a}$$

A more convenient form of the mass balance equation is obtained integrating by parts the first integral of Eq. (26). This gives

$$-\int_{\Omega} \frac{\partial q}{\partial x_i} \bar{u}_i \, \mathrm{d}\Omega + \sum_e \int_{\Omega^e} \left[\frac{\partial(\tau_{d_i} q)}{\partial x_i} \right] \hat{\bar{r}}_{m_i} \, \mathrm{d}\Omega + \int_{\Gamma_{u_n}} q u_n^p \, \mathrm{d}\Gamma = 0. \tag{28b}$$

Note that the last integral of Eq. (28b) vanishes in rigid body and stationary free surface boundaries.

Eqs. (28a) and (28b) include all the terms emanating from the original stabilized differential equations. It is interesting to note that all boundary integrals involving stabilization terms have vanished. This is a direct consequence from using a consistent form of the stabilized differential equations for momentum and mass balance and the boundary conditions.

Eqs. (28a) and (28b) can be simplified by neglecting the change of the stabilization parameters h_{m_i} and τ_{d_i} within an element. This approximation, typically used in standard stabilized finite element methods [1–11], gives after rearranging some terms

Momentum

$$\int_{\Omega} v_k \left(\rho \bar{u}_i \frac{\partial \bar{u}_j}{\partial x_j} + \rho \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} \right) + \sum_e \int_{\Omega^e} \left(-\frac{\partial v_k}{\partial x_i} \bar{p} + \frac{\partial v_k}{\partial x_j} \bar{\tau}_{ij} \right) \mathrm{d}\Omega + \sum_e \int_{\Omega^e} \left[\frac{\partial v_k}{\partial x_j} \rho \frac{\bar{u}_i h_{m_j}}{2} \frac{\partial \bar{u}_r}{\partial x_r} \right] \mathrm{d}\Omega$$

$$+ \sum_e \int_{\Omega^e} \frac{h_{m_j}}{2} \frac{\partial v_k}{\partial x_j} \left[\rho \bar{u}_r \frac{\partial \bar{u}_i}{\partial x_r} + \frac{\partial \bar{p}}{\partial x_i} - \frac{\partial \bar{\tau}_{ir}}{\partial x_r} - b_i \right] \mathrm{d}\Omega - \int_{\Omega} v_k b_i \, \mathrm{d}\Omega - \int_{\Gamma_t} v_k t_i \, \mathrm{d}\Gamma = 0. \tag{29a}$$

Mass balance

$$-\int_{\Omega} \frac{\partial q}{\partial x_i} \bar{u}_i \, \mathrm{d}\Omega + \sum_e \int_{\Omega^e} \tau_{d_i} \frac{\partial q}{\partial x_i} \frac{\partial \bar{p}}{\partial x_i} \, \mathrm{d}\Omega + \sum_e \int_{\Omega^e} \tau_{d_i} \frac{\partial q}{\partial x_i} g_i \, \mathrm{d}\Omega + \int_{\Gamma_{u_n}} q u_n^p \, \mathrm{d}\Gamma = 0, \tag{29b}$$

where

$$g_i = \rho \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{\tau}_{ij}}{\partial x_j} - b_i. \tag{30}$$

Eqs. (28a) and (28b) or Eqs. (29a) and (29b) lead to the set of stabilized discretized equations for the velocity and pressure variables. Indeed for $v_k = N_k^u$ and $q = N_j^p$ the stabilized discrete Galerkin variational form is recovered.

Eqs. (29a) and (29b) incorporate terms traditionally encountered in standard stabilized formulations using the FEM. Thus, if vector $\mathbf{h}_m$ is assumed to be aligned with the velocity, i.e., if $h_{m_j} = 2\tau_m u_j$, where τ_m is

an intrinsic time parameter, we find that the fourth integral in Eq. (29a) is identical to that usually introduced in Streamline Upwind Petrov Galerkin (SUPG) [3–8] and Characteristic Galerkin (CG) [10,11] methods in order to remedy the instabilities due to the advection operator. The expression given by Eq. (29a) is more general and it allows to define vector $\mathbf{h}_m$ in a more appropiate manner to account for both streamline and transverse stabilization effects. Also by using the more general form of Eqs. (28a) and (28b) a non uniform (continuous) distribution of vector $\mathbf{h}_m$ can be taken into account in a consistent manner.

The third integral in Eq. (29a) has a form very similar to that found in the Galerkin Least Square (GLS) method [9,7].

Note also that the divergence of the velocity term has been kept within the first integral of the momentum equation (29a). This term is usually neglected in standard SUPG and GLS approaches. In this case the presence of this term ensures consistency of the derivation. The computational relevance of this term should be verified in numerical tests.

The second and third integrals in Eq. (29b) are typically found when using GLS methods [9,7]. Note the appearance of a Laplacian of pressure term in the second integral of Eq. (29b) of the type

$$\int_{\Omega^e} \tau_{d_i} \frac{\partial q}{\partial x_i} \frac{\partial \bar{p}}{\partial x_i} \, \mathrm{d}\Omega.$$

This term has the well known property of introducing the necessary stabilization in the incompressibility equation ensuring a correct solution in the incompressible limit, while allowing the use of equal order approximations for velocity and pressure [1,6,7,9].

In summary, the original stabilized differential equations for a viscous fluid are the basis for deriving, using a standard finite element Galerkin approach, a general stabilized discrete system of equations which incorporates the best features of the best known stabilized methods for simultaneously correcting both the possible oscillations induced by high convection effects and incompatible velocity–pressure fields.

4. Stokes flow

The stabilized formulation previously presented is applicable to the whole range of viscous flows. In particular, it can be used for non viscous Euler type flows and for highly viscous flows where convective effects are negligible (Stokes flow). Indeed the situations of zero viscosity and zero convection effects are clearly non physical and they represent two limit cases of some particular fluids found in nature. It is interesting however to study the particular case of a pure viscous incompressible flow where the effect of convection is simply neglected in the momentum equations. This assumption is typically used for modelling the deformation of metals and plastics during forming situations and also in the study of some creep problems [1,27]. The additional interest of this type of flow model is the analogy of the governing equations with those of incompressible elasticity [1,27]. Indeed, the stabilized formulation for the flow problem will be directly applicable to the analogous incompressible elastic problem.

The stabilized form of the momentum equations for a pure Stokes flow are simply obtained by neglecting the convective terms in Eqs. (1) and (3). The resulting equations can be written as

$$\frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} - b_i - \frac{1}{2} h_{m_j} \frac{\partial}{\partial x_j} \left[\frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ir}}{\partial x_r} - b_i \right] = 0. \tag{31}$$

The next step is to express the volumetric strain rate in terms of the pressure from the momentum equations. As the convective terms are now zero the volumetric strain rate is introduced into Eq. (31) by means of the constitutive equation. Thus, substituting Eqs. (5a) and (5b) into Eq. (31) gives

$$\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j} (2\mu\varepsilon_{ij}) + \frac{\partial}{\partial x_i} \left(\frac{2}{3}\mu \frac{\partial u_k}{\partial x_k} \right) - b_i - \frac{1}{2} h_{m_j} \frac{\partial}{\partial x_j} \left[\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_r} (2\mu\varepsilon_{ir}) + \frac{\partial}{\partial x_i} \left(\frac{2}{3}\mu \frac{\partial u_k}{\partial x_k} \right) - b_i \right] = 0. \tag{32}$$

Let us assume now the viscosity μ to be constant. Eq. (32) allows to obtain the derivatives of the volumetric strain rate as

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) = -\frac{3}{2\mu}\left[\frac{\partial p}{\partial x_i} - 2\mu\frac{\partial \varepsilon_{ij}}{\partial x_j} - b_i - \frac{1}{2}h_{m_j}\frac{\partial}{\partial x_j}\left(\frac{\partial p}{\partial x_i} - 2\mu\frac{\partial \varepsilon_{ir}}{\partial x_r} + \frac{2}{3}\mu\frac{\partial^2 u_r}{\partial x_i\partial x_r} - b_i\right)\right]. \tag{33}$$

Eq. (33) can be written in a more compact form as

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right) = -\frac{3}{2\mu}\left[r_i - \frac{h_{m_j}}{2}\frac{\partial \hat{r}_i}{\partial x_j}\right] \tag{34}$$

with

$$r_i = \frac{\partial p}{\partial x_i} - 2\mu\frac{\partial \varepsilon_{ij}}{\partial x_j} - b_i, \tag{35}$$

$$\hat{r}_i = \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} - b_i. \tag{36}$$

Substituting Eq. (34) into the stabilized form of the mass balance equation given by Eq. (2) yields

$$\frac{\partial u_k}{\partial x_k} + \frac{3h_{d_i}}{4\mu}\left[r_i - \frac{h_{m_j}}{2}\frac{\partial \hat{r}_i}{\partial x_j}\right] = 0. \tag{37}$$

The weak form of the momentum and mass balance equations is obtained following a similar procedure as for the Navier–Stokes case explained in Section 3. This gives, after finite element discretization, the following system of equations:

Momentum

$$\sum_e \int_{\Omega^e}\left[-\frac{\partial v_k}{\partial x_i}\bar{p} + \frac{\partial v_k}{\partial x_j}\bar{\tau}_{ij}\right]d\Omega - \sum_e \int_{\Omega^e}\frac{h_{m_j}}{2}\frac{\partial v_k}{\partial x_j}\hat{\bar{r}}_i\,d\Omega - \int_\Omega v_k b_i\,d\Omega - \int_{\Gamma_t} v_k t_i\,d\Gamma = 0. \tag{38}$$

Mass balance

$$-\int_\Omega \frac{\partial q}{\partial x_k}\bar{u}_k\,d\Omega + \sum_e \int_{\Omega^e}\frac{\partial q}{\partial x_j}\left(\frac{3h_{d_i}h_{m_j}}{8\mu}\right)\hat{\bar{r}}_i\,d\Omega + \sum_e \int_{\Omega^e}q\frac{3h_{d_i}}{4\mu}\bar{r}_i\,d\Omega + \int_{\Gamma_{u_n}}qu_n^p\,d\Gamma = 0, \tag{39}$$

where as usual $(\bar{\cdot})$ denotes approximate finite element values. Note that in the derivation of Eqs. (38) and (39) a uniform distribution of the stabilization parameters within each element has been assumed. In addition, the residual $\hat{r}_i$ has been assumed to vanish on the Dirichlet boundary.

For the sake of clarity Eq. (39) is written in the following expanded form

$$-\int_\Omega \frac{\partial q}{\partial x_k}\bar{u}_k\,d\Omega + \sum_e \int_{\Omega^e}\frac{\partial q}{\partial x_j}\left(\frac{3h_{d_i}h_{m_j}}{8\mu}\right)\frac{\partial \bar{p}}{\partial x_i}\,d\Omega - \sum_e \int_{\Omega^e}\frac{\partial q}{\partial x_j}\frac{3h_{d_i}h_{m_j}}{8\mu}\left(\frac{\partial \bar{\tau}_{ik}}{\partial x_k} + b_i\right)d\Omega$$

$$+\sum_e \int_{\Omega^e}q\frac{3h_{d_i}}{4\mu}\bar{r}_i\,d\Omega + \int_{\Gamma_{u_n}}qu_n^p\,d\Gamma = 0. \tag{40}$$

Eqs. (38) and (40) provide the set of stabilized algebraic equations for computing the velocity and pressure fields after substitution of the viscous stresses and the pressure in terms of the nodal displacements and nodal pressures using Eqs. (5a) and (5b), and (19b). Indeed the pressure term in the mass balance equation allows to use equal order interpolations for velocities and pressure.

It is interesting to point out again that the stabilized discrete form provided by Eqs. (38) and (40) for the Stokes flow problem is also directly applicable to the analogous incompressible elasticity problem using equal order interpolations for displacements and pressure.

Remark 1. *The standard Laplacian of pressure form is recovered in Eq. (40) if $h_{d_1} h_{m_2} + h_{d_2} h_{m_1} = 0$ (for 2D problems). Otherwise, the term involving the cross derivatives of the pressure remains within the second integral of Eq. (40). The effect of this term should be validated in numerical tests.*

Remark 2. *Eqs. (29b) and (40) differ essentially in the method chosen to substitute the term involving the derivative of the divergence of the velocity field in the stabilized mass balance equations (see Eq. (2)). Thus, Eq. (29b) was derived making use of the convective operator and the momentum equations through the identity expressed by Eq. (11). Conversely, in the derivation of Eq. (40) use has been made of the constitutive equation to express the derivatives of the velocity divergence field in terms of rest of terms from the momentum equations (see (Eq. (33)).*

5. The transient case

The stabilization formulation above presented is naturally extended to the transient case. The stabilized form of the momentum and mass balance equations are written now as [22,25]

Momentum

$$\left(r_{m_i} - \frac{h_{m_j}}{2} \frac{\partial r_{m_i}}{\partial x_j} \right) - \frac{\delta}{2} \frac{\partial}{\partial t} \left(r_{m_i} - \frac{h_{m_j}}{2} \frac{\partial r_{m_i}}{\partial x_j} \right) = 0. \tag{41}$$

Mass balance

$$\left(r_d - \frac{h_{d_j}}{2} \frac{\partial r_d}{\partial x_j} \right) - \frac{\delta}{2} \frac{\partial}{\partial t} \left(r_d - \frac{h_{d_j}}{2} \frac{\partial r_d}{\partial x_j} \right) = 0. \tag{42}$$

In above δ is a time stabilization parameter. Transient effects are also included in the term r_{m_i} given by

$$r_{m_i} = \rho \left(\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} \right) + \frac{\partial p}{\partial x_i} - \frac{\partial \tau_{ij}}{\partial x_j} - b_i. \tag{43}$$

Eqs. (41) and (42) are obtained by expressing the balance of momentum and mass in space-time domains of finite dimensions $[\mathbf{h}_m \times \delta]$ and $[\mathbf{h}_d \times \delta]$, respectively. Details of the derivation can be found in [25].

Eqs. (41) and (42) can be used to derive a number of stabilized numerical schemes for the transient solution of the Navier–Stokes equations.

5.1. Three steps splitting scheme

It is interesting to derive a splitting algorithm starting with the new stabilized equations. For the sake of clarity the time stabilization terms involving δ will be neglected in Eqs. (41) and (42). Also the stabilized mass balance equation will be written in the more convenient form given by Eq. (15).

A time marching solution scheme for Eq. (41) can be written as (for $\delta = 0$)

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\rho} \left[\rho \frac{\partial (u_i u_j)^n}{\partial x_j} + \frac{\partial p^{n+1}}{\partial x_i} - \frac{\partial \tau_{ij}^n}{\partial x_j} - b_i^n - \left(\frac{h_{m_j}}{2} \frac{\partial r_{m_i}}{\partial x_j} \right)^n \right]. \tag{44}$$

The analogy of Eq. (44) with that found using the so called characteristic integration schemes [10,11] is clear if vector $\mathbf{h}_m$ is chosen aligned with the velocity field, i.e., $\mathbf{h}_m = \tau \mathbf{u}$ where τ is an intrinsic time parameter. Indeed the arbitrary form of vector $\mathbf{h}_m$ in Eq. (44) provides a more general procedure where the components of vector $\mathbf{h}_m$ can be freely chosen.

A semi-implicit time splitting or "fractional step" [10,11] algorithm can now be obtained as follows. Eq. (44) is split as

$$u_i^* = u_i^n - \frac{\Delta t}{\rho} \left[\rho \frac{\partial (u_i u_j)}{\partial x_j} - \frac{\partial \tau_{ij}}{\partial x_j} - b_i - \frac{h_{m_j}}{2} \frac{\partial r_{m_i}}{\partial x_j} \right]^n, \tag{45}$$

$$u_i^{n+1} = u_i^* - \frac{\Delta t}{\rho} \frac{\partial p^{n+1}}{\partial x_i}. \tag{46}$$

Note that the sum of Eqs. (45) and (46) gives the original form of Eq. (44). Substituting Eq. (45) into Eq. (15) gives

$$r_d^* - \Delta t \frac{\partial^2 p^{n+1}}{\partial x_i \partial x_i} - \tau_{d_i} \left[\frac{\partial \hat{r}_{m_i}}{\partial x_i} \right]^{n+1} = 0, \tag{47}$$

where

$$r_d^* = \frac{\partial u_i^*}{\partial x_i}, \tag{48}$$

$$\left[\frac{\partial \hat{r}_{m_i}}{\partial x_i} \right]^{n+1} = \frac{\partial}{\partial x_i} \left[\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) - \frac{\partial \tau_{ij}}{\partial x_j} - b_i \right]^n - \frac{\partial^2 p^{n+1}}{\partial x_i \partial x_i}. \tag{49}$$

The solution steps are the following:

Step 1: Solve explicitly for the so called "fractional" velocities u_i^* [10,11] using Eq. (45).

Step 2: Compute the pressure field p^{n+1} by solving the equation for the Laplacian of pressure derived from Eq. (47). Note that this equation has the following form

$$\Delta t \frac{\partial^2 p^{n+1}}{\partial x_i \partial x_i} + \tau_{d_i} \frac{\partial^2 p^{n+1}}{\partial x_i \partial x_i} = r_d^* - \tau_{d_i} \frac{\partial}{\partial x_i} \left[\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) - \frac{\partial \tau_{ij}}{\partial x_j} - b_i \right]^n. \tag{50}$$

Clearly for $\tau_{d_i} = \tau$ above equation simplifies to

$$(\Delta t + \tau) \Delta p^{n+1} = \hat{r}_d^*, \tag{51}$$

where Δ is the Laplacian operator and

$$\hat{r}_d^* = r_d^* - \tau \frac{\partial}{\partial x_i} \left[\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) - \frac{\partial \tau_{ij}}{\partial x_j} - b_i \right]^n. \tag{52}$$

Step 3: Compute the velocities u_i^{n+1} by using Eq. (46).

Eq. (51) differs slightly from the form typically used in fractional step schemes where the term involving τ does not appear [10,11]. This term, however, is essential to preserve the stability of the mixed formulation for problems where very small time increments Δt are needed due to the stability requirements of the time integration algorithm chosen.

Obviously, other forms of above three steps transient solution scheme involving the implicit computation of u_i^{n+1} are also possible.

Extension of these transient solution methods to the simpler Stokes problem are straightforward. The same schemes can be applied to derive enhanced algorithms for transient non linear structural dynamic problems allowing equal order interpolation for velocities and pressure as described in [28].

6. Computation of the stabilization parameters

Accurate evaluation of the stabilization parameters is one of the crucial issues in stabilized methods. Most of existing methods use expressions which are direct extensions of the values obtained for the simplest 1D case. It is also usual to accept the so called SUPG assumption, i.e., to admit that vector $\mathbf{h}_m$ has the direction of the velocity field. This restriction leads to instabilities when sharp layers transversal to the velocity direction are present. This additional defficiency is then corrected by adding a "shock capturing" (SC) stabilization term [17–20].

Let us first assume for simplicity that the stabilization parameters for the mass balance equations are the same than those for the momentum equations. This implies

$$h_{m_i} = h_{d_i}. \tag{53}$$

The problem remains now finding the value of the characteristic length vectors h_{m_i}. Indeed, the components of $\mathbf{h}_m$ can introduce the necessary stabilization along the streamline and transversal directions to the flow.

Excellent results have been obtained in [29] using linear triangles and tetrahedra with the following value for h_{m_i}

$$\mathbf{h}_m = h_s \frac{\mathbf{u}}{|\mathbf{u}|} + h_c \frac{\nabla|\mathbf{u}|}{|\nabla|\mathbf{u}||}, \tag{54}$$

where h_s and h_c are the "streamline" and "shock capturing" contributions given by

$$h_s = \max(\mathbf{l}_j^T \mathbf{u}/|\mathbf{u}|),$$
$$h_c = \max\left(\mathbf{l}_j^T \frac{\nabla|\mathbf{u}|}{|\nabla|\mathbf{u}||}\right), \quad j = 1, n_s, \tag{55}$$

where $\mathbf{l}_j$ are the vectors defining the element sides ($n_s = 3$ for triangles and $n_s = 6$ for tetrahedra).

An alternative method for computing $\mathbf{h}_m$ in a more consistent manner is explained in the next section.

6.1. Computation of the stabilization parameters via a diminishing residual procedure

The idea of this technique first presented in [21,22] and tested in [23–25] for advective-diffusive problems is the following. Let us assume that a finite element solution for the velocity and pressure fields has been found for a given mesh. The residual of the momentum equation corresponding to this particular solution is

$${}^1\bar{r}_{m_i} = \bar{r}_{m_i} - \frac{1}{2}h_{m_j}\frac{\partial \bar{r}_{m_i}}{\partial x_j}. \tag{56}$$

The average residual over an element can be defined as

$${}^1\bar{r}_{m_i}^{(e)} = \frac{1}{\Omega^{(e)}} \int_{\Omega^{(e)}}^{1} {}^1\bar{r}_{m_i} \, d\Omega. \tag{57}$$

Let us assume now that an enhanced numerical solution has been found for the same mesh and the same approximation (i.e., neither the number of elements nor the element type have been changed). This enhanced solution could be based, for instance, in a superconvergent recovery of derivatives [30,31].

The element residual for the enhanced solution is denoted ${}^2\bar{r}_{m_i}^{(e)}$. As the element residuals must tend to zero, the following condition must be satisfied

$${}^1\bar{r}_{m_i}^{(e)} - {}^2\bar{r}_{m_i}^{(e)} \geqslant 0. \tag{58}$$

Above equation applies for ${}^1\bar{r}_{m_i}^{(e)} > 0$. Clearly for ${}^1\bar{r}_{m_i}^{(e)} < 0$ the inequality in Eq. (58) should be changed to $\leqslant 0$.

Eq. (58) provides a system of equations which unknowns are the characteristic length parameters. Substituting Eq. (56) into (58) and applying the identity condition in Eq. (58) gives

$$\mathbf{h}_m^{(e)} = \mathbf{A}^{-1}\mathbf{f} \tag{59}$$

with

$$A_{ij} = 2\left[\frac{{}^2\partial \bar{r}_{m_i}^{(e)}}{\partial x_j} - \frac{{}^1\partial \bar{r}_{m_i}^{(e)}}{\partial x_j}\right], \tag{60}$$

$$f_i = {}^2\bar{r}_{m_i}^{(e)} - {}^1\bar{r}_{m_i}^{(e)}. \tag{61}$$

The following "adaptive" algorithm can be proposed for obtaining a stabilized solution:
1. Solve for numerical values of velocities and pressure for an initial value $\mathbf{h}_m^{(e)} = \mathbf{h}_o^{(e)}$. Compute ${}^1\bar{r}_{m_i}^{(e)}$.
2. Evaluate the enhanced velocity and pressure fields. Compute ${}^2\bar{r}_{m_i}^{(e)}$.
3. Compute the updated value of $\mathbf{h}_m^{(e)}$ using Eq. (59).
4. Repeat (1)–(3) until a stable solution is found.

Above strategy can be naturally incorporated into a transient solution scheme where the value of $\mathbf{h}_m^{(e)}$ is updated after the solution for each time step has been found.

The assumption $\mathbf{h}_d = \mathbf{h}_m$ can be relaxed and an independent value of the characteristic length vector $\mathbf{h}_d$ for the mass balance equation can be found following a similar approach as described for computing $\mathbf{h}_m$. Further details can be found in [24,25] where this technique has been successfully tested for steady state and transient advective-diffusive problems.

7. Concluding remarks

The objective of the paper was to derive a stabilized formulation for finite element analysis of incompressible viscous flow problems. It has been shown that the stabilized governing equations obtained via the so called FIC procedure presented in [21,22] are the basis for deriving stabilized finite element schemes for both steady state and transient situations. Moreover, the final stabilized forms obtained in all cases remedy the two main problems associated with the numerical solution of viscous flows, i.e., the lack of stability induced by high convective terms and the oscillations caused by the choice of incompatible velocity–pressure fields.

The FIC method provides a natural explanation for the stabilization terms appearing in all equations, many of which have been heuristically proposed by different authors. It is interesting to note that the method extends naturally to Stokes flow problems and it also allows to derive time marching solution schemes with enhanced stabilization properties.

Extension of the FIC procedure to derive stabilized finite schemes for compressible flow problems are possible following the lines presented in [21,22].

Future work remains to verify the efficiency of the "adaptive type" method proposed to compute the stabilization parameters. It is also envisaged that the FIC method could serve for deriving numerical schemes for stabilized solution of high Reynolds flows where the characteristic length parameters could naturally incorporate the stabilization properties credited to the eddy viscosity in turbulent flow models.

Acknowledgements

Appendix A. Balance of mass next to a boundary segment

Let us consider the balance of mass in the triangular domain of Fig. 1 next to a boundary segment Γ. From simple observation we can write

$$\rho[u_C h_y + v_B h_x] = \rho l u_n^p, \tag{A.1}$$

where u_C and v_B are the horizontal and vertical velocities at the mid points of the triangle sides and u_n^p is the normal velocity at the boundary. Obviously $u_n^p = 0$ at a solid boundary or a stationary free surface.

The velocities at points B and C are expressed in terms of those at point A using a Taylor series approximation as

$$
\begin{aligned}
u_C &= u_A - \frac{h_x}{2} \left. \frac{\partial u}{\partial x} \right|_A, \\
v_B &= v_A - \frac{h_y}{2} \left. \frac{\partial v}{\partial y} \right|_A.
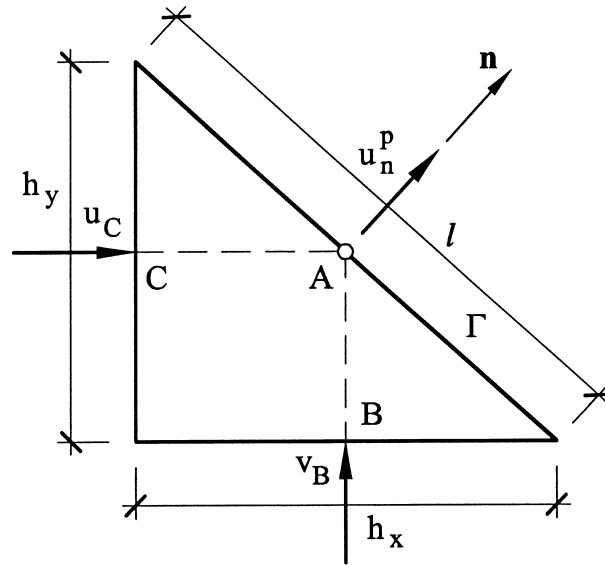\end{aligned}
\tag{A.2}
$$

Fig. 1. Balance domain next to a boundary segment.

Substituting Eq. (A.1) into Eq. (A.2) and denoting generically $u_A = u$ and $v_A = v$ gives

$$(un_x + vn_y) - \frac{1}{2}(h_x n_x + h_y n_y)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = u_n^p. \tag{A.3}$$

In the derivation of Eq. (A.3) use of the identities $n_x = h_y/l$ and $n_y = h_x/l$ has been made where n_x, n_y are the components of the unit normal vector $\mathbf{n}$ (see Fig. 1).

Eq. (A.3) can be further simplified to the form of Eq. (9)

$$u_n - \frac{1}{2}\mathbf{h}_d^T \mathbf{n} r_d = u_n^p \tag{A.4}$$

with

$$\begin{aligned}
&u_n = u_x n_x + u_y n_y, \\
&\mathbf{h}_d = [h_x, h_y]^T, \quad \mathbf{n} = [n_x, n_y]^T, \\
&r_d = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.
\end{aligned} \tag{A.5}$$

Appendix B. Computation of integrals in a distributional sense

The following proof is based on the ideas presented in [26]. Consider for instance the computation of

$$\int_\Omega w_j \frac{\partial r}{\partial x_j}\, d\Omega, \tag{B.1}$$

where w_j is a continuous function and r is assumed to be a function defined at element level and which is discontinuous across the element faces. Therefore the derivatives $\partial r/\partial x_j$ are not defined along element sides.

Integral (B.1) can be computed in a distributional sense as

$$\int_\Omega w_j \frac{\partial r}{\partial x_j}\, d\Omega = \sum_e \int_{\Omega^e} w_j \frac{\partial r}{\partial x_j}\, d\Omega + \sum_f \int_{\Gamma^f} w_j [\![r]\!] n_j\, d\Gamma. \tag{B.2}$$

The first term on the righthand side represents the sum of integrals over the element interiors. The second term accounts for the jump of the discontinuous function r across element faces. This term contains

the sum of integrals computed on all faces shared by two elements. Counters e and f run respectively on the number of elements and faces in the mesh.

Fig. 2 illustrates the case of two triangular elements A and B sharing a common side Γ. For each element we consider the outward normal vector to any given side Γ. Since the jump of the discontinuous functions is computed in the normal directions, the sign of the normal makes no difference in the result of (B.2) and the jump for the case illustrated here is

$$[\![r]\!] = (r)_B - (r)_A \tag{B.3}$$

where $(r)_B$ and $(r)_A$ are the values of r on Γ obtained from elements B and A, respectively. If the sign of the normal vector changes, the sign of the jump will also change so that the sign of the product $[\![r]\!]n_j$ will remain unchanged.

The integral of the jump along Γ can be obtained form the contributions of each element sharing the side Γ

$$\int_\Gamma w_j [\![r]\!] n_j \, \mathrm{d}\Gamma = \int_{\Gamma_B} w_j r_B n_j \, \mathrm{d}\Gamma - \int_{\Gamma_A} w_j r_A n_j \, \mathrm{d}\Gamma. \tag{B.4}$$

Here Γ_B denotes Γ in element A, while Γ_A represents the same face viewed from element B. From Fig. 2 we see that $\mathbf{n}_A = \mathbf{n}$ and $\mathbf{n}_B = -\mathbf{n}$. Therefore (B.4) can be rewritten as

$$\int_\Gamma w_j [\![r]\!] n_j \, \mathrm{d}\Gamma = - \int_{\Gamma_B} w_j r_B n_{B_j} \, \mathrm{d}\Gamma - \int_{\Gamma_A} w_j r_A n_{A_j} \, \mathrm{d}\Gamma. \tag{B.5}$$

Consequently the integral of the jump on Γ can be decomposed into two integrals, each one involving values from only one of the two adjacent elements. The sum of such integrals for all element faces can be expressed in terms of integrals over the element boundaries as

$$\sum_f \int_{\Gamma^f} w_j [\![r]\!] n_j \, \mathrm{d}\Gamma = - \sum_e \int_{\Gamma^e} w_j r n_j \, \mathrm{d}\Gamma + \int_\Gamma w_j r n_j \, \mathrm{d}\Gamma, \tag{B.6}$$

where Γ^e represents the three sides of element e, while Γ represents the set of element sides lying on the boundary of Ω. The second integral on the righthand side of (B.6) appears because the jump terms are computed only between two elements and not on the boundary Γ. This term will cancel out contributions from boundary edges in the first term of the righthand side.

Substituting (B.6) into (B.2) yields

$$\int_\Omega w_j \frac{\partial r}{\partial x_j} \, \mathrm{d}\Omega = \sum_e \left[\int_{\Omega^e} w_j \frac{\partial r}{\partial x_j} \, \mathrm{d}\Omega - \int_{\Gamma^e} w_j r n_j \, \mathrm{d}\Gamma \right] + \int_\Gamma w_j r n_j \, \mathrm{d}\Gamma. \tag{B.7}$$
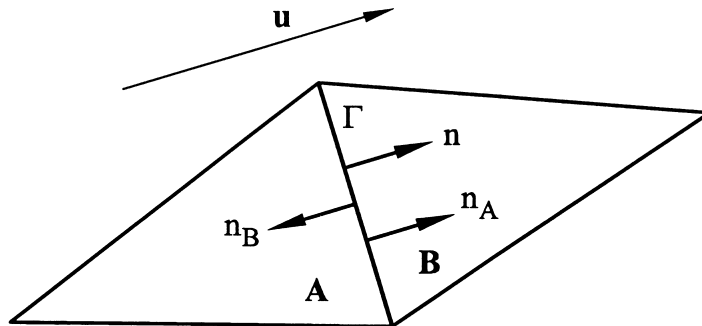

Fig. 2. Interface for computing the jump between two elements.

The last step consists in integrating by parts the integrals over the element interiors, i.e.,

$$\int_{\Omega^e} w_j \frac{\partial r}{\partial x_j} \, \mathrm{d}\Omega = - \int_{\Omega^e} \frac{\partial w_j}{\partial x_j} r \, \mathrm{d}\Omega + \int_{\Gamma^e} w_j r n_j \, \mathrm{d}\Gamma. \tag{B.8}$$

Substituting (B.8) into (B.7) gives finally

$$\int_{\Omega} w_j \frac{\partial r}{\partial x_j} \, \mathrm{d}\Omega = - \sum_e \int_{\Omega^e} \frac{\partial w_j}{\partial x_j} r \, \mathrm{d}\Omega + \int_{\Gamma} w_j r n_j \, \mathrm{d}\Gamma.$$

This coincides with Eqs. (23) and (24) for $w_j = v_k h_{m_j}$ and $r = \bar{r}_{m_i}$ in Eq. (23) and $w_j = q\tau_{d_j}$ and $r = \hat{\bar{r}}_{m_i}$ in Eq. (24).

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